Turkish Journal of Computer and Mathematics Education Vol.12 No. 5 (2021), 1406-1413

Research Article

Factorisation and Labeling in Hypergraphs

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Article History: Received: 11 January 2021; Accepted: 27 February 2021; Published online: 5 April 2021

Abstract: Graphs have lots of applications in various domains. They support only pair wise relationships. Hypergraphs does more than graphs. In graph theory, a graph where an edge can join any number of vertices is called as the hyper graph. The corresponding edges are called as hyper edges. The integers used for assignment of labels to the edges and vertices or to only vertices of a graph or to only the edges is called as the graph labeling in this paper we study about factorization and labeling in hyper graphs with the hyper graphs obtained from graphs.

AMS Classification: 05C90, 05C99.

Keywords: Hyper Graph, Graph Labeling.

1. Introduction

Graphs have applications both in theory and practice. Graphs are discrete objects used to describe pair wise relation between the objects. A well known generalization of graphs commonly called as Hyper graphs was introduced in the 1960s [3]. They are known to have numerous applications in several fields of computer science, machine learning, game theory, indexing of databases, SAT problem, data mining, and optimization. Generality of hyper graphs over graphs is that graphs are for the fixed parameters whereas hyper graph can store more information. [2,3]

Hyper graphs become a natural modeling of collaboration networks and various other situations as they preserve the multi-adic relationships. They are useful in modeling but the limitations of using hyper graphs was discussed by Xavier Onrard. [1] A factor F of graph G is an r factor if the degree of each vertex in F is r. The most dealt degree factors are those in which r equals 1 that is each component is a single edge. Bichitra Kalita has discussed about different types of factorization in complete graphs of some particular forms. He developed the algorithm for the solution of TSP. He discussed theoretical investigations related to 3-factors, 2-factors and 1 factors and cited experimental results. There is a vast study of work on factors and factorization and this topic has much in common with other areas of study in graph theory. For example factorization significantly overlaps the topic of edge coloring. [5,6]

Graph labeling or a graph valuation is a function that maps the vertex elements and edge elements of a graph to positive integers. Labeling (or valuation) of a graph is a map that carries graph elements (vertices and edges) to numbers (usually positive integers). If the set is defined with respect to vertices then it is said to be vertex labeling and if defined with respect to edges, then it is edge labeling A total labeling is a labeling in which the domain consist of both vertex set and edge set. There are many types of graph labeling like magic labeling, anti magic labeling, graceful labeling. A weighted graph is a graph in which the labels on the edges are elements of an ordered set, which are real numbers. The author is motivated to study about factorization and labeling with the work done by different authors on graph factorization and graph labeling. [4]

2. Preliminaries and New Definitions

In this section we recall the definition of labeling in graphs and we introduce new definitions about hyper graphs

2.1) Graph Labeling

If vertices and edges of a graph G are assigned natural numbers from 1 to n, where n is the number of vertices and edges then such a mapping $f : V \cup E \rightarrow \{1,2,3,.....|V \cup E|\}$ is called as graph labeling. Graph is said to be a labeled graph according to Rosa.

2.2) Main Hyper Edge

An edge which contains all the edges of the hyper graph is called as the main hyper edge or the outer hyper edge.

2.3) Factors Hyper Graphs

If a hyper graph is split into hyper edge suh graphs whose join is the original hyper graph then those sub graphs are called as hyper edge factors. A hyper graph can be split into n hyper edge sub graphs which are called as n edge factor hyper graphs. If n equals 1 then we define them as 1 edge factor graphs. If n equals two then we define them as two edge factor hyper graphs or two edge hyper graphs.

2.4) 1 Factor in a Hyper Graph

1 factor of a hyper graph G is defined as the graph containing only one vertex.

2.5) 2 Factor in a Hyper Graph

2-Factor of a hyper graph is a sub-graph of a graph G which contains two vertices and only one edge.

2.6) 3 Factor in a Hyper Graph

3 Factor of a hyper graph is a sub graph of a graph G which contains 3 vertices and 2 edges and one hyper edge.

3. Results

In this section we discuss the labeling in path hyper graphs and cycle hyper graphs obtained from graphs.

3.1) Path Hyper Graphs

Consider a path P_2 with 2 vertices and 1 edge which is also the hyper edge. Let the vertices be labeled as natural numbers 1 and 2 and the edge as label 3. The labeling of the path or the hyper path P_2 is

$$f(v_i) = i,\ 1 \le i \le 2$$
$$f(e_j) = j + 2,\ i = 1$$

Consider a path P_3 with 3 vertices. Let the vertices be labeled as natural numbers 1,2 and 3 and the edges with 2 vertices will be $3C_2$ that is 3 with labels 4,5,6 and the hyper edge with the label 7 The labeling of the path P_3 is

$$f(v_i) = i,\ 1 \le i \le 3$$
$$f(e_j) = j + 3,\ j = 1,2,3$$

Consider a path P_4 with 4 vertices. Let the vertices be labeled as natural numbers 1,2, 3 and 4. The number of edges with 2 vertices will be $4C_2$ that is 6 with the labels 5 to 10 and the hyper edges of 3 vertices with labels 11,12,13 and the outer hyper edge of all vertices with label 14. The labeling of the path P_4 is

$$f(v_i) = i,\ 1 \le i \le 4$$
$$f(e_j) = j + 4 = p,\ j = 1,2,3,4,5,6$$
$$f(e_k) = p + 10,\ p = 1,2,3$$

Consider a path P_5 with 5 vertices labeled as 1,2,3,4,5. The number of edges with 2 vertices will be $5C_2$ that is 10 with the labels 6 to 15 and the hyper edges of 3 vertices with labels 11,12,13,14,15,16 and the hyper edge with 4 vertices with labels 17,18,19,20 and the outer hyper edge with all vertices and edges with label 21. The labeling of the path P_5 is

$$f(v_i) = i,\ 1 \le i \le 5$$
$$f(e_j) = j + 5 = p,\ j = 1\ to\ 10$$
$$f(e_k) = \max(p) + 1 + k, k = 1,2,3,4$$

In general if there is a path with n vertices with labels 1,2,3,…n. The number of edges with 2 vertices will be C(n,2) with the labels starting with n+1 onwards. Then the labeling in the hyper graph is as follows

$$f(v_i) = i,\ 1 \le i \le n$$
$$f(e_j) = j + n = p,\ j = 1,2,3,...$$
$$f(e_k) = \max(p) + k = r, k = 1,2,3,.......$$
$$..................$$
$$f(el) = \max(s) + 1$$

3.2) Cycle Hyper Graphs

Consider a path C_3 with 3 vertices. Let the vertices be labeled as natural numbers 1,2 and 3 and the edges with 2 vertices will be $3C_2$ that is 3 with labels 4,5,6 and the hyper edges with the labels 7,8,9 and 10. The labeling of the cycle hyper graph C_3 is

$$f(v_i) = i,\ 1 \le i \le 3$$
$$f(e_j) = j + 3 = p,\ j = 1,2,3$$
$$f(e_k) = \max(p) + k, k = 1,2,3,4$$

Consider a path C_4 with 4 vertices. Let the vertices be labeled as natural numbers 1,2,3 and 4 and the edges with 2 vertices will be $4C_2$ that is 6 with labels 5,6,7,8,9,10 and the hyper edges of 3 vertices are $4C_3$ that is 4 with the labels 11,12,13,14. The label of the outermost hyper edge is 15. The labeling of the cycle hyper graph C_4 is

$$f(v_i) = i,\ 1 \le i \le 4$$
$$f(e_j) = j + 4 = p,\ j = 1,2,3,4,5,6$$
$$f(e_k) = \max(p) + k, k = 1,2,3,4,5$$

Consider a path C_5 with 5 vertices. Let the vertices be labeled as natural numbers 1,2,3,4 and 5 and the edges with 2 vertices will be $5C_2$ that is 10 with labels 6,7,8,9,10,11,12,13,14,15 and the hyper edges with 3 vertices are $5C_3$ that is 10 in number and with the .labels 16 to 25 and the hyper edge with 4 vertices are $5C_4$ that is 5 in number with the labels 26 to 30. The label assigned to the outer hyper edge is 31. The labeling of the cycle hyper graph C_5 is

$$f(v_i) = i,\ 1 \le i \le 5$$
$$f(e_j) = j + 5 = p,\ j = 1\ to\ 10$$
$$f(e_k) = \max(p) + k = q, k = 1\ to\ 10$$
$$f(e_l) = \max(q) + l = m, l = 1,2,3,4,5$$
$$f(e_l) = \max(m) + 1$$

Consider a path C_n with n vertices. Let the vertices be labeled as natural numbers 1 to n and the edges with 2 vertices will be nC_2 that is with labels A = {n+1,……} and the hyper edges with 3 vertices are nC_3 in number that is with the labels belonging to the set B={max(A)+1,…..} and with the .labels and the hyper edge with 4 vertices are nC_4 in number with the labels belonging to the set C={max(B)+1,…..} The label assigned to the outer hyper edge is the largest positive integer. The labeling of the cycle hyper graph C_n is

$$f(v_i) = i,\ 1 \le i \le n$$
$$f(e_j) = j + n = p,\ j = 1,2,3,....$$
$$f(e_k) = \max(p) + k = q, k = 1,2,3,...$$
$$f(e_l) = \max(q) + l = m, l = 1,2,3,4,5,...$$
....
$$f(e_l) = \max(s) + 1$$

3.3) Labeling in Star Hyper Graphs Obtained from Star Graphs

Consider a star $K_{1,2}$ with 3 vertices and 2 edges. The vertices are labeled as 1,2,3 and the edges named as 4,5 and the hyper edge with the label 6. The labeling of the star graph $K_{1,2}$ is

$$f(v_i) = i,\ 1 \le i \le 3$$
$$f(e_j) = j + 3,\ j = 1,2,3$$

Consider a star $K_{1,3}$ with 4 vertices and 3 edges. The vertices are labeled as 1,2,3,4 and the edges named as 5,6,7 and the hyper edge with the label 8. The labeling of the star graph $K_{1,3}$ is

$$f(v_i) = i,\ 1 \le i \le 4$$
$$f(e_j) = j + 4 = p,\ j = 1,2,3,4$$

Consider a star $K_{1,4}$ with 5 vertices and 4 edges. The vertices are labeled as 1,2,3,4,5 and the edges named as 5,6,7,8 and the hyper edge with the label 9. The labeling of the star graph $K_{1,4}$ is

$$f(v_i) = i,\ 1 \le i \le 5$$
$$f(e_j) = j + 5 = p,\ j = 1,2,3,4$$

Consider a star $K_{1,n}$ with n vertices and n-1 edges. The vertices are labeled as 1 to n and the edges labeled from n+1 onwards and the outer hyper edge with the label as maximum positive integer. The labeling of the star graph $K_{1,4}$ is

$$f(v_i) = i,\ 1 \le i \le n$$
$$f(e_j) = j + n = p,\ j = 1,2,3,4,....$$

3.4) Labeling in Complete Hypergraphs Obtained from Graphs

Consider the complete graph with 3 vertices. It has 3 edges.

Let the vertices be labeled as natural numbers 1, 2 and 3 and the edges with 2 vertices will be $3C_2$ that is 3 with labels 4,5,6 and the hyper edges with the labels 7,8,9 and 10.
The labeling in complete graph taken as the hyper graph is defined as follows:

$$f(v_i) = i,\ 1 \le i \le 3$$
$$f(e_j) = j + 3 = p,\ j = 1,2,3$$
$$f(e_k) = \max(p) + k, k = 1,2,3,4$$

Consider the complete graph with 4 vertices which has only 6 edges.

Let the vertices be labeled as natural numbers 1,2 3 and 4 and the edges with 2 vertices will be $4C_2$ that is 6 with labels 5,6,7,8,9,10 and the hyper edges with 3 vertices are $4C_3 = 4$ with the labels 11,12,13 and 14 and the outer hyper edge with the label 15.
The labeling in complete graph taken as the hyper graph is defined as follows:

$$f(v_i) = i,\ 1 \le i \le 4$$
$$f(e_j) = j + 4 = p,\ j = 1,2,3,4,5,6$$

$$f(e_k) = \max(p) + k, k = 1,2,3,4$$

Consider the complete graph with n vertices which has $\frac{n(n-1)}{2}$ edges.

Let the vertices be labeled as natural numbers 1 to n and the edges with 2 vertices are nC_2 in number that is with labels A = {n+1,……} and the hyper edges with 3 vertices are nC_3 in number that is with the labels belonging to the set B={max(A)+1,…..} and the hyper edge with 4 vertices are nC_4 in number with the labels belonging to the set C={max(B)+1,…..} The label assigned to the outer hyper edge is the largest positive integer. The labeling of the complete graph K_n is

$$f(v_i) = i,\ 1 \le i \le n$$
$$f(e_j) = j + n = p,\ j = 1,2,3,....$$
$$f(e_k) = \max(p) + k = q, k = 1,2,3,...$$
$$f(e_l) = \max(q) + l = m, l = 1,2,3,4,5,...$$
$$f(e_l) = \max(m) + r, r = 1,2,3,4,5,...$$

3.5) Factor Hyper Graph Obtained from Particular forms of Complete Hyper Graph

In this section we discuss about the n factor graphs, n = 1,2,3 which are taken as hyper graphs obtained from hyper graphs of complete graphs.

3.5.1. Theorem The number of 3 factor graphs obtained from hyper graphs of complete graphs $K_4, K_{10}, ...that\ is\ K_{6m-2}$ is 2m-1.

Proof: We use the principle of mathematical induction to discuss the proof. The number of vertices in complete hyper graphs of complete graphs are same as the complete graphs we consider and the number of edges of complete hyper graphs of complete graphs are same as that of complete graphs. For m = 1, we have the complete hyper graph K_4 has 4 vertices and 6 edges. The complete hyper graph itself is the 3 regular subgraph, that is the 3 factor graph. Let us assume that the statement is true for m = k. That is the complete hyper graph for m = k will have same number of end vertices and same number of edges with respect to end vertices. For m equals k+1, the complete hyper graph will have 6k-4 number of end vertices and $18k^2+21k$ number of edges with respect to end vertices. We find that whenever m is greater than or equal to 1, k+1 is greater than or equal to 1 means k is greater than or equal to zero. These hyper graphs will have 2m+1 number of 3 factor graphs. Hence the proof.

3.5.2. Theorem The hyper graph obtained from complete graphs of the form K_{6p} will have 2p-1 number of 3 factor graphs and one 2 factor graphs by considering only with respect to the end vertices.

Proof: We use the principle of mathematical induction to discuss the proof. For p equals 1, the hyper graph K_6 has one 3 factor graph and one 2 factor graph. Let us assume that the theorem is true for p= k. The complete hyper graph of the complete graph K_{6p} for p equals k, has 6k vertices and $18k^2$-3k edges, has 2k-1 number of 3 factor graphs and one 2 factor graph. For p equals k+1, the complete graph K_{6p+6} has 6 vertices and $18k^2+33k+15$ edges will have 2p+1 number of 3 factor graphs and one 2 factor graph. We find that whenever p is greater than or equal to 1, k+1 is greater than or equal to 1 means k is greater than or equal to zero. By Mathematical Induction, the complete hyper graphs of complete graphs will have 2p-1 number of 3 factor graphs and only one 2 factor graph.

3.5.3. Theorem The complete hyper graphs obtained from complete graphs K_{6p+2} for p≥1 has 2p number of 3 factor graphs and one 1 factor graph by considering only with respect to the end vertices.

Proof: We use the principle of mathematical induction to discuss the proof. For p = 1, the complete hyper graph K_6 of the complete graph has 6 vertices and 15 edges. Let us assume that the theorem is true for p = k. The complete hyper graph K_{6k} with 6k+2 vertices and $18k^2+9k+1$ edges, will have 2k-1 number of 3 factor graphs and one 2 factor graph. For p = k+1, the complete hyper graph K_{6k+8} with 6k+8 vertices and $18k^2+45k+28$ will have 2k+2 number of 3 factor graphs and one 2 factor graph. It is clear that whenever p is greater than or equal to 1, k+1 is greater than or equal to 1 means k is greater than or equal to zero. The theorem is true for m equals k+1.

By the principle of mathematical induction the complete hyper graphs will have 2p number of 3 factor graphs and one 1 factor graph.

4. Labeling in Special Hyper Graphs and its Sub Graphs

4.1) Fan Hyper Graph

The fan hyper graph Fn for $n \geq 2$ is obtained by joining all vertices of the hyper graph that is a path of n vertices with middle vertices to a further vertex called as the middle vertex and 2n-1 edges. $F_n = P_n + K_1$.
Let the vertices be labeled as natural numbers 1 to n and the edges with 2 vertices will be nC_2 in number at is with labels A = {n+1,……} and the hyper edges with 3 vertices are nC_3 in number that with the labels belonging to the set B={max(A)+1,…..} and with the .labels and the hyper edge with 4 vertices are nC_4 in number with the labels belonging to the set C={max(B)+1,…..} The label assigned to the outer hyper edge is the largest positive integer. The labeling of the complete graph K_n is

$$f(v_i) = i,\ 1 \leq i \leq n$$
$$f(e_j) = j + n = p,\ j = 1,2,3,....$$
$$f(e_k) = \max(p) + k = q, k = 1,2,3,...$$
$$f(e_l) = \max(q) + l = m, l = 1,2,3,4,5,...$$
$$f(e_l) = \max(m) + r = s, s = 1,2,3,4,5,...$$
$$...........$$
$$f(e_s) = \max(s) + 1$$

If the sub graph is one triangle then the labeling of that triangle is same as the cycle with 3 vertices and if the sub graphs are two triangles with 4 vertices and 5 edges, then the labeling of that sub graph follows the above rule.

4.2) Friendship Hyper Graph

A friendship hyper graph is the hyper graph which consist of n triangles, which are hyper graphs with a common vertex.
Let the vertices be labeled as natural numbers 1 to n and the edges with 2 vertices will be nC_2 that is with labels A = {n+1,……} and the hyper edges with 3 vertices are nC_3 in number that is with the labels belonging to the set B={max(A)+1,…..} and the hyper edge with 4 vertices are nC_4 in number with the labels belonging to the set C={max(B)+1,…..} The label assigned to the outer hyper edge is the largest positive integer. The labeling of the friendship hyper graph is

$$f(v_i) = i,\ 1 \leq i \leq n$$
$$f(e_j) = j + n = p,\ j = 1,2,3,....$$
$$f(e_k) = \max(p) + k = q, k = 1,2,3,...$$
$$f(e_l) = \max(q) + l = m, l = 1,2,3,4,5,...$$
$$f(e_l) = \max(m) + r = s, s = 1,2,3,4,5,...$$
$$...........$$
$$f(e_s) = \max(s) + 1$$

If the sub graph is one triangle then the labeling of that triangle is same as the cycle with 3 vertices and if the sub graphs are two triangles with 4 vertices and 6 edges, then the labeling of that sub graph follows the above rule. The labeling procedure for sub graphs of the friendship hyper graph.

4.3) Wheel Hyper Graph

The wheel hyper graph is defined as the join of the complete graph with one edge with the cycle of n vertices that is C_n. The middle vertex is the intersection of a vertex from K_1 with every vertex of the cycle C_n .

Let the vertices be labeled as natural numbers 1 to n and the edges with 2 vertices will be nC_2 that is with labels A = {n+1,……} and the hyper edges with 3 vertices are nC_3 in number that is with the labels belonging to the set B={max(A)+1,…..} and with the .labels and the hyper edge with 4 vertices are nC_4 in number with the labels belonging to the set C={max(B)+1,…..} The label assigned to the outer hyper edge is the largest positive integer. The labeling of the complete graph K_n is

$$f(v_i) = i,\ 1 \le i \le n$$
$$f(e_j) = j + n = p,\ j = 1, 2,3,....$$
$$f(e_k) = \max(p) + k = q, k = 1,2,3,...$$
$$f(e_l) = \max(q) + l = m, l = 1,2,3,4,5,...$$
$$f(e_l) = \max(m) + r = s, s = 1,2,3,4,5,...$$
....
$$f(e_s) = \max(s) + 1$$

If the sub graph is one triangle then the labeling of that triangle is same as the cycle with 3 vertices and if the sub graphs are join of two triangles with 4 vertices and 4 edges, then the labeling of that sub graph follows the above rule. The labeling procedure for sub graphs of the friendship hyper graph.

4.4) Helm Hyper Graph

The helm hyper graph is constructed from a wheel hyper graph by attaching a pendant vertex at each vertex of the n cycle.

Let the vertices be labeled as natural numbers 1 to n and the edges with 2 vertices will be nC_2 that is with labels A = {n+1,……} and the hyper edges with 3 vertices are nC_3 in number that is with the labels belonging to the set B={max(A)+1,…..} and with the .labels and the hyper edge with 4 vertices are nC_4 in number with the labels belonging to the set C={max(B)+1,…..} The label assigned to the outer hyper edge is the largest positive integer. The labeling of the complete graph K_n is

$$f(v_i) = i,\ 1 \le i \le n$$
$$f(e_j) = j + n = p,\ j = 1, 2,3,....$$
$$f(e_k) = \max(p) + k = q, k = 1,2,3,...$$
$$f(e_l) = \max(q) + l = m, l = 1,2,3,4,5,...$$
$$f(e_l) = \max(m) + r = s, s = 1,2,3,4,5,...$$
....
$$f(e_s) = \max(s) + 1$$

If the sub graph is one triangle with 3 vertices and 5 edges and if the sub graph is two triangles with 5 vertices and 5 edges, then the labeling of the same follows the above rule. The labeling procedure for sub graphs of the friendship hyper graph.

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