Research Article

Exploration of Solutions for an Exponential Diophantine Equation $p^x + (p+1)^y = z^2$

P. Sandhya[1] & V. Pandichelvi[2]

[1]Assistant Professor, Department of Mathematics,SRM Trichy Arts and Science College
Affiliated to Bharathidasan University, Trichy,India, Email: sandhyaprasad2684@gmail.com
[2]Assistant Professor, PG & Research Department of Mathematics, UrumuDhanalakshmi College
Affiliated to Bharathidasan University, Trichy, Email: mvpmahesh2017@gmail.com

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Abstract: In this text, the exclusive exponential Diophantine equation $p^x + (p + 1)^y = z^2$such that the sum of integer powers x and y of two consecutive prime numbers engrosses a square is examined or estimating enormous integer solutions by exploiting the fundamental notion of Mathematics and the speculation of divisibility or all possibilities of x + y = 1, 2, 3, 4..

Keywords: exponential Diophantine equation; integer solutions

1. INTRODUCTION

The study of an exponential Diophantine equations has stimulated the curiosity of plentiful Mathematicians since ancient times as can be seen from [2-6, 9].BanyatSroysang [7] showed that $7^x+ 8^y= z^2$ has a unique non-negative integer solution (x, y, z) as (0,1,3) in 2013 and he proposed an open problem where x, y and z are non-negative integers and p is a positive odd prime number. In 2014, Suvarnamani. A [8] proved that$p^x + (p + 1)^y= z^2$has a unique solution (p, x, y, z) = (3, 1, 0, 2) and was disproved by Nechemia Burshtein [1] by few examples. In this text, the list of infinite numbers of integer solutions of the equation$p^x + (p + 1)^y= z^2$ where p is a prime number by using the basic concept of Mathematics and the theory of divisibility.

2. APPROACHOF RECEIVINGINTEGERSOLUTIONS

The approach of search out an integer solution to the equation under contemplation is proved by the following theorem.

Theorem:

If p is any prime and x, y and z are integers persuading the condition that x + y = 1, 2, 3, 4,then all feasible integer solutions to the exponential Diophantine equation $p^x + (p + 1)^y= z^2$ are given by (p, x, y, z) = {(2,0,1,2), (3,1,0,2), (3,2,2,5)}when p = 2, 3and (p, x, y, z) = $(4n^2 + 4n – 1, 0, 1,2n + 1)$ where $n \in N$ for p > 3.

Proof:

The equation for performing solutions in integer is taken as

$$p^x + (p + 1)^y = z^2 \qquad (1)$$

All doable predilection ofthe supposition$x + y = 1,2,3,4$ arecarried out by eight cases for assessing solutions in integers.

Case 1: x = 0, y = 1

Equation (1)toexplore solutionsinintegertrimsdown by

$$p + 2 = z^2 \qquad (2)$$

If p = 2, then z = 2. Hence, theoneandonly oneinteger solution iscommunicated as (p, x, y, z) = (2,0,1,2).
If p is an odd prime, then p + 2 is an odd number.
Thismeansthat $z^2$is anoddnumber and consequently z is also an odd number.
Ifz = 1, then p + 2 =1which isimpossible.
Asaresult, $z \geq 3$.

$$\text{Describe } z = 2n + 1,\ n \in N \qquad (3)$$

The square of the selection of z in (3) can be characterized by $z^2 = 4n^2 + 4n + 1,\ n \in N$. In sight of (2), the promising value of anodd prime complied with the specified equation is distinguished by $p = 4n^2 + 4n – 1,\ n \in N$
Hence, the enormous solutions to (1) is (p, x, y, z) = $(4n^2 + 4n – 1, 0, 1, 2n + 1)$ where $n \in N$

Case 2: x = 1, y = 0

The inventive equation (1) is diminished as

$$p + 1 = z^2 \quad (4)$$

If p =2, then z^2= 3 which is not possible for integer value of z.
If p is an odd prime, then p + 1 is an even number which can be articulated by
$p + 1 = 2n, n \in N$
Match up the above equation with (4), 2n is a perfect square only if $n = m^2$ where $m \in N$.
Thus, $p = (2m)^2 – 1$.
If m = 1, then p = 3. Therefore, the solution belongs to the set Z of integers is (p, x, y, z) = (3,1,0,2).
If $m \neq 1$, then p = (2m – 1) (2m + 1)
If p divides (2m – 1), then 2m – 1 = ap and as a consequence2m + 1 = ap + 2.
Thus, p = ap(ap + 2) and leads to the ensuing equation 1 = a (a + 2).
But the above equation is not true for any integer value of a.
Ifp divides(2m + 1), then 2m + 1 = bp and from now2m – 1 = bp – 2.
Therefore, p = bp(bp – 2) andconsequently 1 = b (b + 2) which is not factual for any integer options for b.
Hence, in this case there exists a unique solution to (1) given by (p, x, y, z) = (3,1,0,2)

Case 3: x = 1, y = 1
The creative equation (1) is adjust by

$$2p + 1 = z^2$$

Since $z^2$is an odd number for all selections of*p*, it follows that
$z^2 \equiv 1 \pmod 4$
$\Rightarrow 2p + 1 \equiv 1 \pmod 4$
$\Rightarrow 2p \equiv 0 \pmod 4$
Capture that 2p = 4kwhich means that p = 2k for some positive integer k.
This declaration is possible only when k = 1.
Then p = 2, and 2p + 1 =5 which is not a perfect square of an integer.
Hence, in this case there is no integer solution to the presupposed equation.

Case 4 : x = 1, y = 2
The resourceful equation (1) is reconstructed as
$P^2 + 3p = z^2 – 1$
$\Rightarrow p (p + 3) = (z – 1) (z + 1) \quad (5)$
If p | (z – 1), then z – 1 = kp, and z + 1 = kp + 2.Executions of these two equations in (5) go along with the subsequent quadratic equation in k
$pk^2 + 2k – (p + 3) = p$,
which consent the value of $k = (-1 \pm\sqrt{(p(p +3) + 1)}) / p$. It is deeply monitored that no prime number p provides an integer value for k.
An alternative vision of p|(z + 1) reveals that z + 1 = lp and z – 1 = lp – 2
By make use of these two equations in (5) espouse the second degree equation in l as
$pl^2 – 2l – (p + 3) = 0$
which yields $l = (1\pm\sqrt{(1- p(p +3) 1)}) / p$.
The above value ofl is a complex number or any prime p.
Hence, the ultimate result is no integer solutions to the most wanted equation (1).

Case 5 :x = 2, y = 1
The quick-witted equation (1) is restructured as$p^2 + p = z^2 – 1$
$\Rightarrow p(p + 1) = (z – 1)(z + 1) \quad (6)$
It follows from equation (6) that p must divide any one of the values (z – 1) or (z + 1)
If p|(z – 1), then z – 1 = mp and z + 1 = mp + 2for some integer m.
Then, (6) make available with the value ofp as
$p = (1 – 2m) / (m^2 – 1) \quad (7)$

Accordingly, one can easily notice that the right-hand side of (7) can never be a prime number for $m \in Z$.

This circumstance offers that p does not divide $(z - 1)$.
If $p|(z + 1)$, then $z + 1 = np$ and $z - 1 = np - 2$ for some integer n. Then, (6) endow with the value ofpas
$p = (1 + 2n) / (n^2 - 1)$ (8)
None of the value of $n \in Z$ in the right-hand side of (8) supplies the prime number establish that p does not divide$(z + 1)$
Hence, this case does not grant an integer solution for (1).

Case 6 :x = 1, y = 3
For these choices ofx and y, the well-groomed equation (1) be converted into
$(p + 1)^3 + p = z^2$ (9)
If $p = 2$, $z^2 = 29$ which make sure that z cannot be an integer.If p is any odd prime, then p takes any one of the forms $4N + 1$ or $4 N + 3$.
If $p = 4N + 1$ and the perception thatz^2must be odd reduces (9) to
$64N^3 + 96N^2 + 52N + 9 = (2T - 1)^2$
$\Rightarrow 16N^3 + 24N^2 + 13N + 8 = T (T - 1)$ (10)
It is perceived that none of the values ofN ensure that the left hand side of (10) as the product of two consecutive integers.
Similarly, the chance of $p = 4N + 3$, and the discernment thatz^2 is an odd integer reduces (9) to
$64N^3 + 192N^2 + 196N + 67 = (2T - 1)^2$
$\Rightarrow 2(32N^3 + 96N^2 + 94N + 33)= 4T (T - 1)$
The above equality does not hold since the left hand side is a twice an odd number and the right hand side is a multiple of 4.
Hence, in this case there does not exist an integer solution.

Case 7: x = 2, y = 2
These preferences of x and y altered the well-designed equation (1) into
$p^2 + (p + 1)^2 = z^2$
$\Rightarrow 2p (p + 1) + 1 = z^2$ (11)
Sincez^2 is an odd number, $z^2 \equiv 1 \pmod 4$
Then, $2p (p + 1) \equiv 0 \pmod 4$
Hence, either p or $p + 1$ is a multiple of 2.
If p is a multiple of 2, then p must be 2.
Implementation of this value of p in (7) furnishes $z^2 = 13$ which does not enable as an integer for z.
If $p + 1$ is a multiple of 2, then$p + 1 = 2A$, for some $A \in Z$.
The only odd prime satisfying all the above conditions is 3 and the corresponding value of $z = 3$
Consequently, the only integer solution to (1) is
$(p, x, y, z) = (3, 2, 2, 5)$

Case 8: x = 3, y = 1
The original equation (1) can be written as
$p^3 + p + 1 = z^2$
If $p = 2$, then $z^2 = 11$ which cannot acquiesce an integer for z.
Also, $z^2 \equiv 1 \pmod 4$ and $p(p^2 + 1) \equiv 0 \pmod 4$ which implies that either $4|p$ or $4|(p^2 + 1)$
For the reason that p is an odd prime, 4 does not divide pand so $p^2 + 1 = 4n$
This is not possible since p can take either of the form$4N + 1$, $N \geq 1$ or $4N + 3$, $N \geq 0$.
Hence, the conclusion of this case is there cannot discover an integer solution to (1).

3. CONCLUSION

In this text, the special exponential Diophantine equation $p^x + (p + 1)^y = z^2$ where p is a prime number and x, y and z are integers is studied by developing the fundamental concept of Mathematics and the conjecture of divisibility for all possibilities of x + y = 1, 2, 3, 4. In this manner, one can find an integersolutions by using the property ofcongruence and other thoughts of Number theory.

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