On t-Neighbourhoods in Trigonometric Topological Spaces

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Abstract: In this paper we introduce a new type of neighbourhoods, namely, t-neighbourhoods in trigonometric topological spaces and study their basic properties. Also, we discuss the relationship between neighbourhoods and t-neighbourhoods. Further, we give the necessary condition for t-neighbourhoods in trigonometric topological spaces.

Keywords: t-open; t-closed; t-neighbourhood

1. Introduction

In this paper, we introduce t-neighbourhoods in Trigonometric topological spaces. These spaces are based on Sine and Cosine topologies. In a bitopological space we have consider two different topologies but in a trigonometric topological space the two topologies are derived from one topology. From this, we observe that trigonometric topological space is different from bitopological space.

Section 2 deals with the preliminary concepts. In section 3, t-neighbourhoods are introduced and study their basic properties.

2. Preliminaries

Throughout this paper X denotes a set having elements from \([0, \frac{\pi}{2}]\). If \((X, \tau)\) is a topological space, then for any subset A of X, \(X \setminus A\) denotes the complement of A in X. The following definitions are very useful in the subsequent sections.

**Definition:** 2.1 [2] Let X be any non-empty set having elements from \([0, \frac{\pi}{2}]\) and \(\tau\) be the topology on X. Let SinX be the set consisting of the Sine values of the corresponding elements of X. Define a function \(f_s: X \rightarrow \text{SinX}\) by \(f_s(x) = \sin x\). Then \(f_s\) is a bijective function. This implies, \(f_s(\phi) = \phi\) and \(f_s(X) = \text{SinX}\). That is, \(\text{Sin} \phi = \phi\).

Let \(\tau_s\) be the set consisting of the images (under \(f_s\)) of the corresponding elements of \(\tau\). Then \(\tau_s\) form a topology on \(\text{SinX}\). This topology is called a Sine topology (briefly, Sin-topology) of X. The space \((\text{SinX}, \tau_s)\) is said to be a Sine topological space corresponding to X.

The elements of \(\tau_s\) are called Sin-open sets. The complement of Sin-open sets is said to be Sin-closed. The set of all Sin-closed subsets of SinX is denoted by \(\tau_s^c\).

**Definition:** 2.2 [2] Let X be any non-empty set having elements from \([0, \frac{\pi}{2}]\) and \(\tau\) be the topology on X. Let CosX be the set consisting of the Cosine values of the corresponding elements of X. Define a function \(f_c: X \rightarrow \text{CosX}\) by \(f_c(x) = \cos x\). Then \(f_c\) is bijective. Also, \(f_c(\phi) = \phi\) and \(f_c(X) = \text{CosX}\). This implies, \(\text{Cos} \phi = \phi\).

Let \(\tau_{cs}\) be the set consisting of the images (under \(f_c\)) of the corresponding elements of \(\tau\). Then \(\tau_{cs}\) form a topology on CosX. This topology is called Cosine topology (briefly, Cos-topology) of X. The pair \((\text{CosX}, \tau_{cs})\) is called the Cosine topological space corresponding to X. The elements of \(\tau_{cs}\) are called Cos-open sets. The complement of the Cos-open set is said to be Cos-closed. The set of all Cos-closed subsets of Cos X is denoted by \(\tau_{cs}^c\).

**Definition:** 2.3 [2] Let X be a non-empty set having elements from \([0, \frac{\pi}{2}]\). Define \(T_s(X)\) by \(T_s(X) = \text{SinX} \cap \text{CosX}\) and \(T_l(X)\) by \(T_l(X) = \text{SinX} \cup \text{CosX}\).
Definition 2.4 [2] Let X be a non-empty set having elements from $[0, \frac{\pi}{2}]$ and $\tau$ be the topology on X. We define a set $\mathcal{T} = \{\emptyset, U \cup V \cup T_0(X) : U \in \tau_X \text{ and } V \in \tau_T\}$. Then $\mathcal{T}$ form a topology on $T_0(X)$. This topology is called trigonometric topology on $T_0(X)$. The pair $(T_0(X), \mathcal{T})$ is called a trigonometric topological space. The elements of $\mathcal{T}$ are called trigonometric open sets (briefly, t-open sets). The complement of a trigonometric open set is said to be a trigonometric closed (briefly, t-closed) set. The set of all trigonometric closed sets is denoted by $\mathcal{T}^c$.

3. t-neighbourhoods

In this section we study about t-neighbourhoods in Trigonometric topological spaces. Throughout this section $T_0(X)$ denotes the trigonometric topological space with trigonometric topology $\mathcal{T}$.

Definition 3.1 Let $T_0(X)$ be a trigonometric topological space. A subset N of $T_0(X)$ is said to be a t-neighbourhood (briefly, t-nbd) of $y \in T_0(X)$ if there exists a t-open set M such that $y \in M \subseteq N$.

Definition 3.2 Let $T_0(X)$ be a trigonometric topological space. A subset N of $T_0(X)$ is said to be a t-neighbourhood (briefly, t-nbd) of a subset A of $T_0(X)$ if there exists a t-open set M such that $A \subseteq M \subseteq N$.

Example 3.3 Let $X = \{\frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{3}\}$ with topology $\tau = \{\emptyset, \{\frac{\pi}{6}\}, \{\frac{\pi}{2}\}, \{\frac{\pi}{3}, \frac{\pi}{2}\}, X\}$. Then $T_0(X) = \{\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, 1, 0\}$ and $\mathcal{T} = \{\emptyset, T_0(X), \{\frac{\sqrt{3}}{2}, \{\frac{\sqrt{2}}{2}\}, \{\frac{1}{2}\}, \{1\}, \emptyset\}, \{\emptyset\}, \{1\}, \{\frac{\sqrt{2}}{2}\}, \{\frac{\sqrt{3}}{2}\}, \{\frac{1}{2}\}, \emptyset\}$. Let $N = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$. Then N is a t-nbd of $\frac{\sqrt{3}}{2}$.

Proposition 3.4 Let $T_0(X)$ be a trigonometric topological space. If N is a proper subset of $T_0(X)$, then N is not a t-nbd of any point of $T_0(X)$.

Proof: Assume that N is a proper subset of $T_0(X)$. Suppose that N is a t-nbd of $y \in T_0(X)$. Then there exists a t-open set M such that $y \in M \subseteq N$. This implies, M is a proper subset of $T_0(X)$. This contradicts the fact that every t-open set containing $T_0(X)$. Therefore, N is not a t-nbd of any point of $T_0(X)$.

Definition 3.5 Let $T_0(X)$ be a trigonometric topological space and N be a subset of X. Define the set $N^*$ by $N^* = \text{Sin NUCos NUT}_0(X)$. Then $N^*$ is a subset of $T_0(X)$.

Proposition 3.6 Let $T_0(X)$ be a trigonometric topological spaces and N,M be a subset of X. Then
1. If N is open in X, then $N^*$ is t-open in $T_0(X)$,
2. If $N \subseteq M$, then $N^* \subseteq M^*$.

Proof: The proof follows directly from the definition.

Proposition 3.7 Let $T_0(X)$ be a trigonometric topological space. If N is a neighbourhood of x, then $N^*$ is a t-nbd of Sin x and Cos x.

Proof: Assume that N is a neighbourhood of x. Then there exists an open set M such that $x \in M \subseteq N$. This implies, Sin xSin M\subseteq Sin N and Cos xCos M\subseteq Cos N. This implies, Sin xSin MUCos MUT_0(X)\subseteq Sin NUCos NUT_0(X) and Cos xSin MUCos MUT_0(X)\subseteq Sin NUCos NUT_0(X). That is, Sin x\subseteq N^* and Cos x\subseteq N^*. Since M is open in X, we have $M^*$ is t-open. Therefore, $N^*$ is a t-nbd of Sin x and Cos x.

Proposition 3.8 Let $T_0(X)$ be a trigonometric topological space. If N is a neighbourhood of any point $x \in X$, then $N^*$ is a t-nbd of every point of $T_0(X)$. 

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Proof: Assume that the subset N of X is a neighbourhoods of x. Then N* is a t-nbd of Sin x and Cos x. Then by Proposition 3.7, N* contains T_i(X). Therefore, for each y∈T_i(X), we have y∈T_i(X)⊆N*. Hence N* is a t-nbd of every point of T_i(X).

Proposition: 3.9 Let T_a(X) be a trigonometric topological space. Then T_i(X) is a t-nbd of each of its points.
Proof: For each point x∈T_i(X), there exists a t-open set T_i(X) such that x∈T_i(X)⊆T_i(X). Therefore, T_i(X) is a t-nbd of each of its points.

Proposition: 3.10 Let T_a(X) be a trigonometric topological space. Then N is a t-open set if and only if N is a t-nbd of each of its points.
Proof: Assume that N is t-open. Let y∈N. Then N is a t-open set and y∈N⊂N. This implies, N is a t-nbd of y. Since y∈N is arbitrary, we have N is a t-nbd of each of its points. Conversely, assume that N is a t-nbd of each of its points. Then for each point y_i of N, there exists a t-open set M_i such that y_i∈M_i ⊆N. This implies, N is the union of M_i. Therefore, N is t-open.

Remark: 3.11 If N is a t-nbd of some of its points, then N need not be a t-open set. For example, Consider X= {x, y, z} with τ={φ, X}. Then T_a(X)={0, 1/2, 1}. Let N={1/2, 1} be a subset of T_a(X). Then N is a t-nbd of 1/2. But it is not a t-open set.

Proposition: 3.12 Let T_a(X) be a trigonometric topological space. If A is a t-closed subset of T_a(X) and y∈A, then there exists a t-nbd N of y such that N∩A=φ.
Proof: Let A be a t-closed set and y∉A. Let N=T_a(X)\A. Then N is a t-open set containing y. Since every t-open set is a t-nbd of each of its points, we have N is a t-nbd of y. Also, N∩A=φ.

Definition: 3.13 Let T_a(X) be a trigonometric topological space and y∈T_a(X). The set of all t-nbd of y is called the t-nbd system at y and is denoted by t-N(y).

Proposition: 3.14 Let T_a(X) be a trigonometric topological space and y∈T_a(X). Then t-N(y)≠φ for every y∈T_a(X).
Proof: Since T_a(X) is the t-open set, we have T_a(X) is the t-nbd of each of its points. Therefore, for every point y of T_a(X), t-N(y)≠φ.

Proposition: 3.15 Let T_a(X) be a trigonometric topological space and y∈T_a(X). If N∈t-N(y), then y∈N.
Proof: If N∈t-N(y), then N is a t-nbd of y. This implies, y∈N.

Proposition: 3.16 Let T_a(X) be a trigonometric topological space and y∈T_a(X). If N∈t-N(y) and N⊆M, then M∈t-N(y).
Proof: Assume that N∈t-N(y) and N⊆M. Then N is a t-nbd of y. Therefore, there exists a t-open set W such that y∈W⊆M. This implies, M is a t-nbd of y. Hence M∈t-N(y).

Proposition: 3.17 Let T_a(X) be a trigonometric topological space and y∈T_a(X). If N∈t-N(y) and M∈t-N(y), then N∪M, N∩M∈t-N(y).
Proof: Assume that N∈t-N(y) and M∈t-N(y). Then there exist t-open sets A and B such that y∈A⊆N and y∈B⊆M. This implies, y∈A∩B∈N and y∈A∪B∈N∪M. Since A and B are t-open, we have A∩B and A∪B are t-open. Therefore, N∪M and N∩M are t-nbd of y. Hence N∪M, N∩M∈t-N(y).
Proposition: 3.18 Let $T_d(X)$ be a trigonometric topological space and $y \in T_d(X)$. If $N \in t-N(y)$, then there exists $M \in t-N(y)$ such that $M \subseteq N$ and $M \in t-N(x)$ for every $x \in M$.

Proof: Assume that $N \in t-N(y)$. Then there exists a $t$-open set $M$ such that $y \in M \subseteq N$. Since $M$ is $t$-open, we have $M$ is a $t$-nbd of each of its points. Therefore, $M \in t-N(y)$ and $M \in t-N(x)$ for every $x \in M$.

4. Conclusion:
In this paper we have introduced $t$-neighbourhoods in Trigonometric Topological Spaces and studied some of their basic properties.

References