Dynamics of the Stage Structured Population Model, Predator Accompanied by Michaelis Menten Holling Type Functional Responseand Delay and Prey Taking Refuge

N. Mohana Sorubha Sundari[a] and Dr. M. Valliathal[b]

[a]
Assistant Professor, Department of Mathematics, Chikkaiah Naicker College,
Erode-638 004.
[b]Assistant Professor, Department of Mathematics, Chikkaiah Naicker College, Erode-638 004.

Abstract: The current work considers predator prey system, prey taking refuge, predator reckoned with time delay and Michaelis Menten Holling type II response function undergoing two stages: juvenile and mature. From the characteristic equation, we derive conditions for the local stability of the system at the equilibrium points. Also, at the coexistence equilibrium point, the system is analyzed for the occurrence of Hopf bifurcation. Lyapunov function provides sufficient conditions for the global stability of the system. Numerical simulations are given to support the theory.

1. Introduction

An Ecosystem is composed of a group of species living in the same scenario. Interactions among several living organisms occur at different levels in the ecosystem which is vital for them to survive and thrive with their own species and these interactions play a major role in shaping the food web. One such interaction is between predator and prey. In fact, it is a complex system that involves many varieties of predator and preys and also involves multiple components like habitat, resource needs, individual and group behavior, population growth etc.,.Researchers, from Volterra (1920) attempted to study the qualitative features of such population dynamics by developing suitable mathematical models. Refinements of each model are under way by gradually integrating the factors (such as stage structure, refuge, time delay, harvest etc.) that affects the qualitative behaviors of the modelsand also by eluding the limitations in the developed models.

The functional response which is generally categorized as prey dependent, predator dependent and ratio dependent is one of the key components that determines the population model. Ratio dependent functional response is used when the predators search and share the food and also when the prey to predator abundance ratio defines the per capita growth rate of the predator. This model is studied extensively. It is well known that many species undergo different stages in their life span, especially immature and mature stage which in turn decides the size of their population at different periods of time. Therefore, stage structured mathematical model grabbed the scholars' attention and they started to research its impact on the dynamics of the system. Refuge is one of the predominant characters exhibited by the preys to avoid being food for the predators which is inevitable for them to survive and it also reduces the likelihood of mating that is reflected in their densities. Predator prey system can be better modeled using delay differential equations because it reveals many perplexing phenomena, making the model considered more rational.

Kar in 2006 studied the predator prey mechanism with Holling type II functional response by implementing prey refuge and harvest. Using harvesting efforts as control parameter, he arrived at conditions to hold the system in a required state [1]. Persistence theory on infinite dimensional system has been utilized in [2,3] to show that the delayed stage structure model is permanent if both prey and predator species coexists. Yang et al. in 2008 showed that the predator can survive even if the growth rate is negative for some period due to certain reasons in a periodic Holling type IV stage structured predator prey system [4].

Ratio dependent Holling Tanner model has been discussed by Liang and Pan in 2007. Through change of variables the model has been transformed into Lienard equation to obtain the uniqueness of limit cycle [5]. Saha and Chakrabarti [6] worked on the Holling Tanner delay model and proved that the system is permanent under certain conditions. By using blow up technique the qualitative behavior at (0,0) has been explored. Classical ratio dependent model with Allee effect has been discussedand interesting phenomena such as cusp point, separatrix curve were observed [7]. Holling type II functional response coupled with modified Leslie Gower and prey refuge [8] has been studied with harvesting [9] and adequate conditions were obtained for the system to be stable both locally and globally. Applying Sotomayor's theorem, local bifurcation was studied along with Hopf bifurcation [10].

Xu and Ma in 2008 investigated ratio dependent predator prey model with gestation lag for the predator. Sufficient conditions for Hopf bifurcation and global stability (using comparison arguments) are obtained [11].Yu et al.introduced Michaelis Menten harvesting which is nonlinear in nature in a single species stage structured model and studied its qualitative behavior [12]. Prey, predator and top predator with functional response Michaelis Menten kind and two unequal delays is attempted by Dai et al. in 2014. They used normal form method and center manifold theorem to evaluate the properties of periodic solution [13]. Research have also been

performed in Volterra type functional response, linear functional response, bilinear functional response [14-16] incorporated with stage structure and harvesting.

Mortoja et al. proposed a model with stage structure in both prey and predator with prey flaunting anti predator activity and group defense mechanism with different type of Holling type response function. Hopf bifurcation is analysed by considering transition rate as bifurcation parameter[17]. Using MATCONT software bifurcation and stability of a ratio-based model with noncontant predator harvesting rate were computed. Their analysis revealed that the system exhibits various form of bifurcation including bifurcations of Fold, Cusp and Bogdanov Takens [18].

Jost et al. in 1999 [19] studied the behavior of the predator prey system particularly at the origin with ratio dependent functional response.

$$\frac{dN}{dt} = R\left(1 - \frac{N}{S}\right) - \frac{SN}{P + SN}P$$

$$\frac{dP}{dt} = \frac{SN}{P + SN}P - QP$$

Here N and P are prey abundance and predator abundance. They showed that the system exhibits different behaviors for different parameter values at the point (0, 0) and at this point the system has well defined dynamics. The above system has been consideredwith self-diffusion, cross diffusion and prey taking refuge for its spatial patterns [20].

$$\frac{\partial u}{\partial t} = d_{11}\Delta u + d_{12}\Delta v + u(1 - u) - \frac{bu(1 - M)v}{u(1 - M) + v}$$

$$\frac{\partial v}{\partial t} = d_{21}\Delta u + d_{22}\Delta v - dv + \frac{ebu(1 - M)v}{u(1 - M) + v}$$

Triggered by the research work of Xiao et al. [21], and keeping the basic model from Jost [19] and Sambath [20] we come up with a following population model. Here the predator is divided into two subgroups as juvenile and mature. Only the mature predator is capable of hunting and the immature predators depend solely on their elders for their food. Time delay due to gestation of mature predator and Michaelis Menten Holling type II functional response have been considered. The model takes the form

$$\dot{u}(t) = Ru(t)\left(1 - \frac{u(t)}{K}\right) - \frac{Au(t)(1 - \lambda)v_2(t)}{u(1 - \lambda) + K_1 v_2(t)} \tag{1}$$

$$\dot{v_1}(t) = -Dv_1(t) - d_1 v_1(t) + \frac{BA(1 - \lambda)u(t - \tau)v_2(t - \tau)}{u(t - \tau)(1 - \lambda) + K_1 v_2(t - \tau)}$$
$$\tag{2}$$

$$\dot{v_2}(t) = Dv_1(t) - d_2 v_2(t) \tag{3}$$

with initial conditions

$$u(\theta) = \phi_1(\theta), v_1(\theta) = \phi_2(\theta), v_2(\theta) = \phi_3(\theta) \tag{4}$$

$$\phi_1(\theta) \geq 0, \phi_2(\theta) \geq 0, \phi_3(\theta) \geq 0, \theta \in [-\tau, 0),$$

$$\phi_1(0) > 0, \phi_2(0) > 0, \phi_3(0) > 0,$$

where $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0), \mathbb{R}_+^3)$, the continuous functions in Banach space mapping the interval $[-\tau, 0)$ into $\mathbb{R}_+^3 = \{x_1, x_2, x_3 : x_i \geq 0, i = 1, 2, 3\}$.

Here $u(t), v_1(t), v_2(t)$ denotes size of prey, juvenile and adult predator respectively. R, K, K_1 represents the prey growth rate, environmental carrying capacity, predators benefit rate from cofeeding.λ is the proportion of prey taking refuge and $\lambda \in [0,1)$; A, B, d_1, d_2 are the capture rate, conversion coefficiency, death rate of immature and mature predators respectively. τ is the time lag due to gestation of mature predators.

This paper is sectioned as follows. In section 2 we show that the model system is bounded. We find equilibrium points and conditions for local stability at these equilibrium points. Hopf bifurcation at the positive equilibrium point is discussed in section 3. Permanence and global stability of the system at the coexistence equilibrium point are investigated in section 4. Numerical simulations are given in section 5. Section 6 is the conclusion.

2.POSITIVITY, BOUNDEDNESS, EQUILIBRIA AND LOCAL STABILITY

2.1. Positivity

The proposed system (1-3) can be written in the matrix form as $\dot{X} = P(X)$
where $X = (u, v_1, v_2)^T$ and

$$P(X) = \begin{bmatrix} P_1(X) \\ P_2(X) \\ P_3(X) \end{bmatrix} = \begin{bmatrix} Ru(t)\left(1 - \dfrac{u(t)}{K}\right) - \dfrac{Au(t)(1-\lambda)v_2(t)}{u(1-\lambda) + K_1 v_2(t)} \\ -Dv_1(t) - d_1\, v_1(t) + \dfrac{BA(1-\lambda)u(t-\tau)v_2(t-\tau)}{u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)} \\ Dv_1(t) - d_2\, v_2(t) \end{bmatrix}$$

It can be easily verified that $P_i(X)/_{X_{i=0}} \geq 0, X \in \mathbb{R}_+^3$ for i = 1, 2, 3. Due to Nagumo's theorem any solution of (1-3) with initial conditions $X(0) = X_0 \epsilon \mathbb{R}_+^3$ always lies in $\mathbb{R}_+^3$ for all $t \geq 0$. (ie) it remains positive throughout the region for all finite time.

2.2. Boundedness

Theorem 2.1: All the solutions of the system (1-3) with initial conditions (4) are uniformly bounded for all $t \geq 0$.

Proof: Let

$$\mathbb{V}(t) = Bu(t - \tau) + v_1(t) + v_2(t) \tag{5}$$

Taking the time derivative of the above along the nonnegative solution of (1-3),

$$\dot{\mathbb{V}}(t) = B\dot{u}(t - \tau) + \dot{v}_1(t) + \dot{v}_2(t)$$

$$\dot{\mathbb{V}}(t) = BRu(t - \tau)\left[1 - \frac{u(t-\tau)}{K}\right] - \frac{A(1-\lambda)u(t-\tau)v_2(t-\tau)}{u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)} - Dv_1(t) - d_1\, v_1(t)$$

$$+ \frac{BA(1-\lambda)u(t-\tau)v_2(t-\tau)}{u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)} + Dv_1(t) - d_2\, v_2(t)$$

$$= BRu(t - \tau)\left[1 - \frac{u(t-\tau)}{K}\right] - d_1\, v_1(t) - d_2\, v_2(t)$$

Let $S = \min\{d_1, d_2\}$

$$\dot{\mathbb{V}}(t) + S\mathbb{V}(t) = BRu(t - \tau)\left[1 - \frac{u(t-\tau)}{K}\right] - d_1\, v_1(t) - d_2\, v_2(t) + BSu(t - \tau) + Sv_1(t) + S\, v_2(t)$$

$$= Bu(t - \tau)\left[R\left[1 - \frac{u(t-\tau)}{K}\right] + S\right] - (S - d_1)v_1(t) - (S - d_2)v_2(t)$$

$$\leq Bu(t - \tau)\left[R\left[1 - \frac{u(t-\tau)}{K}\right] + S\right]$$

$$\dot{\mathbb{V}}(t) + S\mathbb{V}(t) \leq M \text{ where } M = \frac{BK(R+S)^2}{4}$$

$$\therefore \mathbb{V}(t) \leq \frac{M}{S} + \left[v(0) - \frac{M}{S}\right]e^{-st} \tag{6}$$

$$\lim_{t \to \infty} \mathbb{V}(t) \leq \frac{M}{S} \,\forall\, t > 0.$$

Hence all the solutions of the considered system are uniformly bounded. Hence the theorem.

2.3. Equilibria

Solving the system of equations (1-3) we obtain the trivial equilibrium point $E_0(0,0,0)$, predator free equilibrium $E_1(K, 0,0)$ and $E_c(u^c, v_1^c, v_2^c)$. The coexistence equilibrium $E_c(u^c, v_1^c, v_2^c)$ exists if the following conditions hold

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$$BA - r_1 > 0, K_1RB > (1 - \lambda)(BA - r_1)$$
(C₁)

where

$$u^c = K - \frac{K(1 - \lambda)(BA - r_1)}{K_1RB}; v_1^c = \frac{d_2}{D} v_2^c; v_2^c = \frac{u^c(1 - \lambda)(BA - r_1)}{K_1r_1}$$

where

$$r_1 = d_2 \left(1 + \frac{d_1}{D}\right) \tag{7}$$

If $\hat{E} = (\hat{u}, \hat{v}_1, \hat{v}_2)$ be any arbitrary equilibrium point, then the variational matrix of the system (1-3) at this point is given by

$$J = \begin{bmatrix} R - \frac{2R\hat{u}}{K} - \frac{A(1-\lambda)K_1\hat{v}_2^2}{[\hat{u}(1-\lambda)+K_1\hat{v}_2]^2} & 0 & -\frac{A\hat{u}^2(1-\lambda)^2}{[\hat{u}(1-\lambda)+K_1\hat{v}_2]^2} \\ \frac{K_1BA(1-\lambda)\hat{v}_2^2e^{-\lambda\tau}}{[\hat{u}(1-\lambda)+K_1\hat{v}_2]^2} & -D - d_1 & \frac{BA(1-\lambda)^2\hat{u}^2e^{-\lambda\tau}}{[\hat{u}(1-\lambda)+K_1\hat{v}_2]^2} \\ 0 & D & -d_2 \end{bmatrix} \tag{8}$$

2.4. Local Stability

Theorem 2.2: The zero equilibrium $E_0(0,0,0)$ is unstable for the system (1-3).

Proof: Working out the characteristic equation from the variational matrix and substituting the equilibrium point $E_0(0,0,0)$ we get,

$$(\gamma - R)(\gamma + D + d_1)(\gamma + d_2) = 0 \tag{9}$$

We observe that one of the roots is positive in (9) and hence the equilibrium E_0 is always unstable. Hence the theorem.

Theorem 2.3: The predator free equilibrium $E_1(K, 0,0)$ of the system (1-3) is locally asymptotically stable if the following holds

$$(BA - r_1) < 0 \text{ (C}_2\text{)}$$

Proof: At the equilibrium point $E_1(K, 0,0)$, the characteristic equation of (8) is

$$(\gamma + R)[(\gamma + D + d_1)(\gamma + d_2) - ABDe^{-\gamma\tau}] = 0 \tag{10}$$

Clearly from (10) one root is negative. Therefore, the remaining roots are determined by

$$\gamma^2 + (D + d_1 + d_2)\gamma + (D + d_1)d_2 - ABDe^{-\gamma\tau} = 0 \tag{11}$$

When $\tau = 0$, (11) becomes

$$\gamma^2 + (D + d_1 + d_2)\gamma + (D + d_1)d_2 - ABD = 0 \tag{12}$$

Using Routh Hurwitz criterion, the boundary equilibrium $E_1(K, 0,0)$ is locally asymptotically stable for (C₂).

The existence of purely imaginary root of (11) is analyzed for $\tau > 0$. Let $i\Omega_1$ where $\Omega_1 > 0$ be the root of (11). Then,

$$-\Omega_1^2 + (D + d_1 + d_2)i\Omega_1 + (D + d_1)d_2 - ABD\cos\Omega_1\tau + iABD\sin\Omega_1\tau = 0$$

Separating the real and imaginary part,

$$-\Omega_1^2 + (D + d_1)d_2 = ABD\cos\Omega_1\tau$$
(13)

$$(D + d_1 + d_2)\Omega_1 = ABD\sin\Omega_1\tau$$

(14)

which gives

$$\Omega_1^4 + \left[(D + d_1)^2 + d_2{}^2\right]\Omega_1^2 + (D + d_1)^2 d_2{}^2 - A^2 B^2 D^2 = 0$$ (15)

If the condition (C_2) is satisfied, then all the eigen values of (11) have negative real parts for all $\tau \geq 0$, which in turn shows that the boundary equilibrium is locally asymptotically stable.(by theorem 3.4.1 [22]). Hence the theorem.

Now, let us investigate the stability of the equilibriumpoint E_c. The characteristic equation at the coexistence equilibrium point is

$$\gamma^3 + h_2\gamma^2 + h_1\gamma + h_0 + (g_1\gamma + g_0)e^{-\gamma\tau} = 0$$ (16)

where

$$h_2 = D + d_1 + d_2 + \mathcal{A}_1 + \frac{2Ru^c}{K} - R$$

$$h_1 = \mathcal{Q}_1 + \left[\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right][D + d_1 + d_2]$$

$$h_0 = \mathcal{Q}_1\left[\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right]$$

$$g_1 = -\mathcal{B}_1$$

$$g_0 = -\mathcal{B}_1\left[\frac{2Ru^c}{K} - R\right]$$

and

$$\mathcal{A}_1 = \frac{A(1-\lambda)K_1 v_2^{c^2}}{[u^c(1-\lambda)+K_1 v_2^c]^2} \; ; \mathcal{B}_1 = \frac{ABD(1-\lambda)^2 u^{c^2}}{[u^c(1-\lambda)+K_1 v_2^c]^2} = \frac{\mathcal{Q}_1 d_2\left(1+\frac{d_1}{D}\right)}{AB} \; ; \mathcal{Q}_1 = (D + d_1)d_2$$

when $\tau = 0$, (16) becomes

$$\gamma^3 + h_2\gamma^2 + (h_1 + g_1)\gamma + h_0 + g_0 = 0$$

From (C_1), we obtain $\mathcal{B}_1 < \mathcal{Q}_1$
where

$$h_2 = D + d_1 + d_2 + \mathcal{A}_1 + \frac{2Ru^c}{K} - R$$

$$h_1 + g_1 = [\mathcal{Q}_1 - \mathcal{B}_1] + \left[\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right][D + d_1 + d_2]$$

$$h_0 + g_0 = \mathcal{Q}_1\mathcal{A}_1 + [\mathcal{Q}_1 - \mathcal{B}_1]\frac{2Ru^c}{K} - [\mathcal{Q}_1 - \mathcal{B}_1]R$$

$$h_2(h_1 + g_1) - (h_0 + g_0) = \left[D + d_1 + d_2 + \mathcal{A}_1 + \frac{2Ru^c}{K} - R\right][\mathcal{Q}_1 - \mathcal{B}_1] + \left[\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right]$$

$$+ \left[\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right]\left[(D + d_1)\left(D + d_1 + \mathcal{A}_1 + \frac{2Ru^c}{K} - R\right)\right] + d_2\left[\mathcal{A}_1 + \frac{2Ru^c}{K} - R + d_2\right]$$

$$+Q_1\mathcal{A}_1 + [Q_1 + \mathcal{B}_1]\frac{2Ru^c}{K} - [Q_1 + \mathcal{B}_1]R$$

Denote

$$P_1 = \frac{Q_1\mathcal{A}_1}{[Q_1 + \mathcal{B}_1]} + \frac{2Ru^c}{K} \; ; \; P_2 = \mathcal{A}_1 + \frac{2Ru^c}{K} \; ; P_3 = \frac{Q_1\mathcal{A}_1}{[Q_1 - \mathcal{B}_1]} + \frac{2Ru^c}{K} \; ;$$

$$P_4 = \frac{[D + d_1 + d_2 + P_2 - R][Q_1 - \mathcal{B}_1]}{[Q_1 + \mathcal{B}_1]} + \frac{[P_2 - R][(D + d_1)(D + d_1 + P_2 - R) + d_2(P_2 - R + d_2)]}{[Q_1 + \mathcal{B}_1]} + P_1$$

If $R < min\{P_2, P_4\}$ and (C$_1$) hold then $h_2 > 0, (h_1 + g_1) > 0, (h_0 + g_0) > 0$ and $h_2(h_1 + g_1) - (h_0 + g_0) > 0$ .Utilizing the criterion given by Hurwitz, the system (1-3) is locally asymptotically stable at the positive equilibrium $E_c(u^c, v_1^c, v_2^c)$.

For $\tau > 0, i\Omega$ where $\Omega > 0$ is a solution of (16) if and only if it satisfies

$$-i\Omega^3 - h_2\Omega^2 + ih_1\Omega + h_0 + ig_1\Omega \cos\Omega\,\tau + g_1\Omega \sin\Omega\,\tau + g_0 \cos\Omega\,\tau - ig_0 \sin\Omega\,\tau = 0 \qquad (17)$$

Separating the real and imaginary part,

$$\Omega^3 - h_1\Omega = g_1\Omega\cos\Omega\tau - g_0\sin\Omega\tau \qquad (18)$$

$$h_2\Omega^2 - h_0 = g_1\Omega\sin\Omega\tau + g_0\cos\Omega\tau \qquad (19)$$

which gives,

$$\Omega^6 + (h_2^2 - 2h_1)\Omega^4 + (h_1^2 - 2h_2h_0 - g_1^2)\Omega^2 + h_0^2 - g_0^2 = 0 \qquad (20)$$

where

$$(h_2^2 - 2h_1) = (D + d_1)^2 + d_2^2 + \left(\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right)^2 > 0$$

$$h_1^2 - 2h_2h_0 - g_1^2 = \left[Q_1 + \left(\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right)(D + d_1 + d_2)\right]^2$$

$$-2\left[D + d_1 + d_2 + \mathcal{A}_1 + \frac{2Ru^c}{K} - R\right]\left[Q_1\left(\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right)\right] - \mathcal{B}_1^2$$

$$= \left[\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right]^2 [(D + d_1)^2 + d_2^2] > 0$$

$$h_0^2 - g_0^2 = Q_1^2\left[\mathcal{A}_1 + \frac{2Ru^c}{K} - R\right]^2 - \mathcal{B}_1^2\left[\frac{2Ru^c}{K} - R\right]^2$$

$$= \left[Q_1\mathcal{A}_1 + (Q_1 + \mathcal{B}_1)\frac{2Ru^c}{K} - (Q_1 + \mathcal{B}_1)R\right]\left[Q_1\mathcal{A}_1 + (Q_1 - \mathcal{B}_1)\frac{2Ru^c}{K} - (Q_1 - \mathcal{B}_1)R\right]$$

If $R < P_1$ and (C$_1$) holds then, $h_0^2 - g_0^2 > 0$ which shows that (20) has no positive roots. Therefore by a theorem [22], for all $\tau \geq 0$, all the eigen values of (20) have negative real parts. Hence at the positive equilibrium $E_c(u^c, v_1^c, v_2^c)$ the system is locally asymptotically stable for all $\tau \geq 0$.

If $P_1 < R < min\{P_2, P_4\}$, then $h_0^2 - g_0^2 < 0$. Then there exists a unique positive root Ω_* which satisfies (20). From (18) and (19) we have,

$$g_1\Omega_*^4 + (h_2g_0 - h_1g_1)\Omega_*^2 - h_0g_0 = (g_1{}^2\Omega_*^2 + g_0^2) \cos\Omega_*\,\tau$$

$$\cos\Omega_*\,\tau = \frac{g_1\Omega_*^4 + (h_2g_0 - h_1g_1)\Omega_*^2 - h_0g_0}{(g_1{}^2\Omega_*^2 + g_0^2)}$$

Let

$$\tau_{*n} = \frac{1}{\Omega_*} arc\ cos\ \frac{g_1 \Omega_*^4 + (h_2 g_0 - h_1 g_1)\Omega_*^2 - h_0 g_0}{(g_1^2 \Omega_*^2 + g_0^2)} + \frac{2n\pi}{\Omega_*}, n = 0,1,2,\dots$$

If $h_0^2 - g_0^2 < 0$, then E_c remains stable for $\tau < \tau_0 := \tau_{*0}$.

Now we derive that $\left\{\frac{d(Re\ \gamma)}{d\tau}\right\}_{\tau = \tau_0} > 0$. Differentiating (16) with respect to τ, it follows that

$$\left[\frac{d\gamma}{d\tau}\right]^{-1} = -\frac{3\gamma^2 + 2h_2\gamma + h_1}{\gamma(\gamma^3 + h_2\gamma^2 + h_1\gamma + h_0)} + \frac{g_1}{\gamma(g_1\gamma + g_0)} - \frac{\tau}{\gamma}$$

Simplifying,

$$sgn\left\{\frac{d(Re\ \gamma)}{d\tau}\right\}_{\tau = \Omega_*} = sgn\left\{Re\left[\frac{d\gamma}{d\tau}\right]^{-1}\right\}$$

$$= sin\left\{-\frac{[h_1 - 3\Omega_*^2][\Omega_*^2 - h_1] + 2h_2[h_0 - h_2\Omega_*^2]}{[\Omega_*^3 - h_1\Omega_*]^2 + [h_0 - h_2\Omega_*^2]^2} - \frac{[g_1^2]}{[g_1^2\Omega_*^2 + g_0^2]}\right\}$$

From (18) and (19) we get,

$$[\Omega_*^3 - h_1\Omega_*]^2 + [h_0 - h_0\Omega_*^2]^2 = [g_1^2\Omega_*^2 + g_0^2]$$

Therefore

$$sin\left\{\frac{d(Re\ \gamma)}{d\tau}\right\} = sin\left\{\frac{3\Omega_*^4 + 2(h_2^2 - 2h_1)\Omega_*^2 + h_1^2 - 2h_0h_2 - g_1^2}{[g_1^2\Omega_*^2 + g_0^2]}\right\} > 0$$

Thus, the traversal condition holds and Hopf bifurcation occurs at $\Omega = \Omega_*, \tau = \tau_0$. Summarizing the above we get

Theorem: 2.4(1) If $R < min\{P_1, P_2, P_4\}$ and (C$_1$) hold, then the positive equilibrium E_c of the system (1-3) is locally asymptotically stable for all ≥ 0.

(2) If $P_1 < R < min\{P_2, P_4\}$ and (C$_1$) holds, then there exists a $\tau_0 > 0$ such that when $\tau\epsilon[0, \tau_0)$, the positive equilibrium E_c is locally asymptotically stable. Also, the system (1-3) undergoes a Hopf bifurcation at $E_c = (u^c, v_1^c, v_2^c)$ when $= \tau_0$.

3. Permanence

Lemma 3.1: Let $(u(t), v_1(t), v_2(t))$ be any positive solution of the system (1-3) with initial conditions (4). Assume (C$_1$) holds then,

$$\lim_{t\to\infty} sup\ u\ (t) \leq L_1, \lim_{t\to\infty} sup\ v_1\ (t) \leq L_2, \lim_{t\to\infty} sup\ v_2\ (t) \leq L_3 \tag{21}$$

where

$$L_1 = K, L_2 = \frac{(1-\lambda)BADL_1 - (1-\lambda)d_2(D + d_1)L_1}{D(D + d_1)K_1}, L_3 = \frac{D}{d_2}L_2$$

Proof: Let $(u(t), v_1(t), v_2(t))$ be any positive solution of the system (1-3) with initial conditions (4). From (1) we get,

$$\dot{u}(t) = Ru(t)\left(1 - \frac{u(t)}{k}\right) - \frac{Au(t)(1-\lambda)v_2(t)}{u(t)(1-\lambda) + K_1v_2(t)}$$

$$\leq Ru(t)\left(1 - \frac{u(t)}{k}\right) \tag{22}$$

Using lemma (2.3) in [23] to the above, it immediately follows that

$$\lim_{t\to\infty} sup\ u\ (t) \leq K := L_1 \tag{23}$$

Then for > 0 , sufficiently small, there exists a $T_1 > 0$, such that if $t > T_1, u(t) \leq L_1 + \varepsilon$. From equations (2) and (3), we get for $t > T_1 + \tau$,

$$\dot{v}_1(t) = -(D + d_1)v_1(t) + \frac{BA(1 - \lambda)(L_1 + \varepsilon)v_2(t - \tau)}{(L_1 + \varepsilon)(1 - \lambda) + K_1 v_2(t - \tau)}$$

$$\dot{v}_2(t) = Dv_1(t) - d_2 v_2(t) \quad (24)$$

Consider the following auxiliary equations

$$\dot{u}_1(t) = -(D + d_1)u_1(t) + \frac{BA(1 - \lambda)(L_1 + \varepsilon)u_2(t - \tau)}{(L_1 + \varepsilon)(1 - \lambda) + K_1 u_2(t - \tau)}$$

$$\dot{u}_2(t) = Du_1(t) - d_2 u_2(t) \quad (25)$$

Following the proof of Lemma 2.4 in [23] we get,

$$\lim_{t \to \infty} u_1(t) = \frac{DBA(1 - \lambda)(L_1 + \varepsilon) - (D + d_1)d_2(L_1 + \varepsilon)(1 - \lambda)}{(D + d_1)K_1 D} := L_{2\varepsilon}$$

$$\lim_{t \to \infty} u_2(t) = \frac{D}{d_2}L_{2\varepsilon} := L_{3\varepsilon} \quad (26)$$

By comparison we obtain,

$$\lim_{t \to \infty} \sup u_1(t) \leq L_{2\varepsilon}, \lim_{t \to \infty} \sup u_2(t) \leq L_{3\varepsilon} (27)$$

Let $\to 0$, it follows that

$$\lim_{t \to \infty} \sup v_1(t) \leq L_2, \lim_{t \to \infty} \sup v_2(t) \leq L_3 \quad (28)$$

Hence the proof.

Note: It follows from (C_1) that $L_2 = \frac{DBA(1-\lambda)L_1 - (D+d_1)d_2 L_1(1-\lambda)}{(D+d_1)K_1 D} > 0$. Hence there exists a positive constant ε such that $L_{2\varepsilon} > 0, L_{3\varepsilon} > 0$.

Lemma 3.2: Let $(u(t), v_1(t), v_2(t))$ be any positive solution of the system (1-3) with initial conditions (4). If (C_1) and

$$R > \frac{A(1-\lambda)}{K_1} \text{holds, then} \qquad\qquad\qquad (29)$$

$$\lim_{t \to \infty} \inf u(t) \geq l_1, \lim_{t \to \infty} \inf v_1(t) \geq l_2, \lim_{t \to \infty} \inf v_2(t) \geq l_3 (30)$$

where$l_1 = \frac{K[K_1 R - A(1-\lambda)]}{RK_1}$, $l_2 = \frac{(1-\lambda)BADl_1 - (1-\lambda)d_2(D+d_1)l_1}{D(D+d_1)K_1}$, $\qquad l_3 = \frac{D}{d_2}l_2$

Proof: It follows from (C_1) and condition (29) that

$$l_1 = \frac{K[K_1 R - A(1 - \lambda)]}{RK_1} > 0,$$

$$l_2 = \frac{(1 - \lambda)BADl_1 - (1 - \lambda)d_2(D + d_1)l_1}{D(D + d_1)K_1} > 0$$

Hence, there exists enough positive constants ε sufficiently small such that

$$l_{1\varepsilon} = \frac{K[K_1 R - A(1 - \lambda)]}{RK_1} > 0$$

$$l_{2\varepsilon} = \frac{(1-\lambda)BAD(l_1-\varepsilon) - (1-\lambda)d_2(D+d_1)(l_1-\varepsilon)}{D(D+d_1)K_1} > 0$$

$$l_{3\varepsilon} = \frac{D}{d_2}l_{2\varepsilon} > 0$$

Let $\big(u(t), v_1(t), v_2(t)\big)$ be any positive solution of the system (1-3) with initial conditions (4). For the above $\varepsilon > 0$, sufficiently small, it follows from the previous lemma that there exists a $T_2 > 0$ such that if $t > T_2, v_1(t) \le L_2 + \varepsilon, v_2(t) \le L_3 + \varepsilon$. Hence for $t>T_2$, we have from equation (1)

$$\dot{u}(t) \ge u(t)\left[R - \frac{R}{K}u(t) - \frac{A(1-\lambda)}{K_1}\right]$$

Using condition (29) we have,

$$\lim_{x\to\infty} \inf u(t) \ge \frac{[K_1 R - A(1-\lambda)]K}{RK_1} := l_1 > 0 \tag{31}$$

For the above $\varepsilon > 0$ sufficiently small, there exists a $T_3 \ge T_2, u(t) \ge l_1 - \varepsilon$. Therefore from equations(2) and (3), for $t > T_3 + \tau$

$$\dot{v}_1(t) = -(D+d_1)v_1(t) + \frac{BA(1-\lambda)(l_1-\varepsilon)v_2(t-\tau)}{(l_1-\varepsilon)(1-\lambda) + K_1 v_2(t-\tau)}$$

$$\dot{v}_2(t) = Dv_1(t) - d_2 v_2(t) \tag{32}$$

Consider the following auxiliary equations

$$\dot{u}_1(t) = -(D+d_1)u_1(t) + \frac{BA(1-\lambda)(l_1-\varepsilon)u_2(t-\tau)}{(l_1-\varepsilon)(1-\lambda) + K_1 u_2(t-\tau)}$$

$$\dot{u}_2(t) = Du_1(t) - d_2 u_2(t) \tag{33}$$

Following the proof of the lemma 2.4 in [23], we obtain that

$$\lim_{t\to\infty} u_1(t) \ge l_{2\varepsilon}; \quad \lim_{t\to\infty} u_2(t) \ge l_{3\varepsilon} \tag{34}$$

By comparison we get,

$$\lim_{t\to\infty} \inf v_1(t) \ge l_{2\varepsilon}, \lim_{t\to\infty} \inf v_2(t) \ge l_{3\varepsilon}$$

Let $\varepsilon \to 0$, then

$$\lim_{t\to\infty} \inf v_1(t) \ge l_2, \lim_{t\to\infty} \inf v_2(t) \ge l_3$$

The proof is complete.

Theorem 3.1: Assume that (C_1) and (29) hold. Then the system (1-3) is permanent.
Proof: It follows from lemma (3.1) and (3.2).

4. Global Stability

Here we use Lyapunov functional and Laselle invariance principal to study the global attractivity of the coexistence equilibrium E_c of the system.

Theorem 4.1: Assume that (C_1) and (29) are satisfied. If

$$\frac{R}{K}(l_1 + u^c) - R \ge \bar{\rho}_1, u^{c^2}(1-\lambda) \ge \bar{\rho}_2 \tag{35}$$

where

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$$\bar{\rho}_1 = \frac{AK_1\left[L_1^2 u^c v_2^c (1-\lambda) + u^{c^2} L_3 K_1 v_2^c + 2L_1 v_2^{c^2} K_1 L_3\right]}{2l_1 u^c [l_1(1-\lambda) + K_1 l_3][u^c(1-\lambda) + K_1 v_2^c]},$$

$$\bar{\rho}_2 = \frac{[L_1^2(1-\lambda) + u^c L_3 K_1]}{2l_1}$$

then at the positive equilibrium $E_c(u^c, v_1^c, v_1^c)$ the system (1-3) is globally stable.

Proof: Let $(u(t), v_1(t), v_2(t))$ be any positive solution of the system (1-3) with initial conditions (4). Define

$$V(t) = u - u^c - u^c \log \frac{u}{u^c} + C_1\left[v_1 - v_1^c - v_1^c \log \frac{v_1}{v_1^c}\right] + C_2\left[v_2 - v_2^c - v_2^c \log \frac{v_2}{v_2^c}\right] \tag{36}$$

Taking the time derivative of $V(t)$ along the positive solution of the system (1-3), we get

$$\dot{V}(t) = \left(1 - \frac{u^c}{u}\right)\left[Ru\left(1 - \frac{u}{K}\right) - \frac{Au(1-\lambda)v_2}{u(1-\lambda) + K_1 v_2}\right] + C_1\left(1 - \frac{v_1^c}{v_1}\right)\left[-(D + d_1)v_1\right.$$
$$\left. + \frac{BA(1-\lambda)u(t-\tau)v_2(t-\tau)}{u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)}\right]$$

$$+ C_2\left(1 - \frac{v_2^c}{v_2}\right)[Dv_1 - d_2 v_2] \tag{37}$$

$$= \frac{(u - u^c)^2}{u}\left[R - \frac{R}{K}(u + u^c)\right] + \left[1 - \frac{u^c}{u}\right]\frac{Au^c(1-\lambda)v_2^c}{u^c(1-\lambda) + K_1 v_2^c} - Au(1$$
$$- \lambda)v_2\left[\frac{u^c(1-\lambda) + K_1 v_2^c}{[u(1-\lambda) + K_1 v_2]u^c(1-\lambda)} - \frac{u^c}{u^2(1-\lambda)}\right]$$

$$+ \frac{K_1(1-\lambda)[v_2^c u^2 - v_2 u^{c^2}]}{u^c(1-\lambda)[u(1-\lambda) + K_1 v_2]}\frac{Av_2}{u} + C_1(D + d_1)v_1^c + \frac{C_1 BA(1-\lambda)u(t-\tau)v_2(t-\tau)}{u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)}$$
$$- \frac{C_1 v_1^c BA(1-\lambda)u(t-\tau)v_2(t-\tau)}{v_1[u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)]}$$

$$- C_2 Dv_1 \frac{v_2^c}{v_2} - C_2 d_2 v_2 + C_2 d_2 v_2^c$$

$$= \frac{(u - u^c)^2}{u}\left[R - \frac{R}{K}(u + u^c)\right] + \frac{Au^c v_2^c}{u}\left[\frac{u}{u^c} - \frac{u^c[u(1-\lambda) + K_1 v_2]}{u[u^c(1-\lambda) + K_1 v_2^c]}\right] - \frac{A K_1 v_2^c [v_2^c u^2 - v_2 u^{c^2}]}{u^2[u^c(1-\lambda) + K_1 v_2^c]}$$

$$- Auv_2\left[\frac{[u^c(1-\lambda) + K_1 v_2^c]}{u^c[u(1-\lambda) + K_1 v_2]} - \frac{u^c}{u^2}\right] + \frac{Av_2}{u}\frac{K_1[v_2^c u^2 - v_2 u^{c^2}]}{u^c[u(1-\lambda) + K_1 v_2]} + C_1(D + d_1)v_1^c$$
$$+ \frac{C_1 BA(1-\lambda)u(t-\tau)v_2(t-\tau)}{u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)}$$

$$- \frac{C_1 v_1^c BA(1-\lambda)u(t-\tau)v_2(t-\tau)}{v_1[u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)]} - C_2 Dv_1 \frac{v_2^c}{v_2} - C_2 d_2 v_2 + C_2 d_2 v_2^c \tag{38}$$

Define V_1 as

$$V_1 = V + C_1 BA(1-\lambda)\int_{t-\tau}^{t}\left[\frac{u(s)v_2(s)}{[u(s)(1-\lambda) + K_1 v_2(s)]} - \frac{u^c v_2^c}{[u^c(1-\lambda) + K_1 v_2^c]}\right.$$

$$\left. - \frac{u^c v_2^c}{[u^c(1-\lambda) + K_1 v_2^c]} \times \log \frac{[u^c(1-\lambda) + K_1 v_2^c]u(s)v_2(s)}{u^c v_2^c[u(s)(1-\lambda) + K_1 v_2(s)]}\right] ds \tag{39}$$

From the above we get,

$$\dot{V}_1 = \frac{(u - u^c)^2}{u}\left[R - \frac{R}{K}(u + u^c)\right] + v_2^c A\left[1 - \frac{u^{c^2}[u(1-\lambda) + K_1 v_2]}{u^2[u^c(1-\lambda) + K_1 v_2^c]}\right] + \frac{K_1 A[v_2^c u^2 - v_2 u^{c^2}]}{u}$$

$$\left[\frac{v_2 u[u^c(1-\lambda) + K_1 v_2^c] - v_2^c u^c[u(1-\lambda) + K_1 v_2]}{uu^c[u(1-\lambda) + K_1 v_2][u^c(1-\lambda) + K_1 v_2^c]}\right] + \frac{Av_2 u^c}{u} + v_2^c A$$

$$- \frac{v_2^c A v_1^c}{u^c v_2^c v_1}\left[\frac{[u^c(1-\lambda) + K_1 v_2^c]u(t-\tau)v_2(t-\tau)}{[u(t-\tau)(1-\lambda) + K_1 v_2(t-\tau)]}\right] - v_2^c A \frac{v_1 v_2^c}{v_1^c v_2}$$

$$-Av_2 + v_2^c A + v_2^c \log \frac{[u^c(1-\lambda)+K_1 v_2^c]u(t-\tau)v_2(t-\tau)}{u^c v_2^c[u(t-\tau)(1-\lambda)+K_1 v_2(t-\tau)]} + v_2^c A \log \frac{u^c v_2^c[u(1-\lambda)+K_1 v_2]}{[u^c(1-\lambda)+K_1 v_2^c]uv_2} \tag{40}$$

Note that

$$C_1(D + d_1)v_1^c = C_2 d_2 v_2^c = v_2^c A = \frac{C_1 BA(1-\lambda)u^c v_2^c}{u^c(1-\lambda) + K_1 v_2^c}$$

and

$$\left[1 - \frac{u^c}{u}\right]\frac{Au^c(1-\lambda)v_2^c}{[u^c(1-\lambda)+K_1 v_2^c]} = \frac{Au^c v_2^c}{u}\left[\frac{u}{u^c} - \frac{u^c[u(1-\lambda)+K_1 v]}{u[u^c(1-\lambda)+K_1 v_2^c]}\right] - \frac{Av_2 K_1}{u[u^c(1-\lambda)+K_1 v_2^c]}\left[\frac{v_2^c u^2 - v_2 u^{c^2}}{u}\right]$$

$$\dot{V}_1 = \frac{(u-u^c)^2}{u}\left[R - \frac{R}{K}(u+u^c)\right] + \frac{K_1 A[v_2^c u^2 - v_2 u^{c^2}]}{u}\left[\frac{v_2 u[u^c(1-\lambda)+K_1 v_2^c] - v_2^c u^c[u(1-\lambda)+K_1 v_2]}{uu^c[u(1-\lambda)+K_1 v_2][u^c(1-\lambda)+K_1 v_2^c]}\right]$$

$$-v_2^c A\left[\frac{u^{c^2}[u(1-\lambda)+K_1 v_2]}{u^2[u^c(1-\lambda)+K_1 v_2^c]} - 1 - \log \frac{u^{c^2}[u(1-\lambda)+K_1 v_2]}{u^2[u^c(1-\lambda)+K_1 v_2^c]}\right]$$

$$-v_2^c A\left[\frac{v_1^c}{u^c v_2^c v_1}\frac{[u^c(1-\lambda)+K_1 v_2^c]u(t-\tau)v_2(t-\tau)}{[u(t-\tau)(1-\lambda)+K_1 v_2(t-\tau)]} - 1\right.$$
$$\left. - \log \frac{v_1^c}{u^c v_2^c v_1}\frac{[u^c(1-\lambda)+K_1 v_2^c]u(t-\tau)v_2(t-\tau)}{[u(t-\tau)(1-\lambda)+K_1 v_2(t-\tau)]}\right]$$

$$-v_2^c A\left[\frac{v_1 v_2^c}{v_1^c v_2} - 1 - \log \frac{v_1 v_2^c}{v_1^c v_2}\right] - v_2^c A \log \frac{u^c}{u} - Av_2(u - \frac{u^c}{u})$$

$$\dot{V}_1 \leq -\frac{(u-u^c)^2}{u}\left[\frac{R}{K}(u+u^c) - R - \rho_1\right] - \frac{K_1 A(v_2 - v_2^c)^2[u^{c^2}(1-\lambda) - \rho_2]}{u[u(1-\lambda)+K_1 v_2][u^c(1-\lambda)+K_1 v_2^c]}$$

$$-v_2^c A\left[\frac{u^{c^2}[u(1-\lambda)+K_1 v_2]}{u^2[u^c(1-\lambda)+K_1 v_2^c]} - 1 - \log \frac{u^{c^2}[u(1-\lambda)+K_1 v_2]}{u^2[u^c(1-\lambda)+K_1 v_2^c]}\right]$$

$$-v_2^c A\left[\frac{v_1^c}{u^c v_2^c v_1}\frac{[u^c(1-\lambda)+K_1 v_2^c]u(t-\tau)v_2(t-\tau)}{[u(t-\tau)(1-\lambda)+K_1 v_2(t-\tau)]} - 1\right.$$
$$\left. - \log \frac{v_1^c}{u^c v_2^c v_1}\frac{[u^c(1-\lambda)+K_1 v_2^c]u(t-\tau)v_2(t-\tau)}{[u(t-\tau)(1-\lambda)+K_1 v_2(t-\tau)]}\right]$$

$$-v_2^c A\left[\frac{v_1 v_2^c}{v_1^c v_2} - 1 - \log \frac{v_1 v_2^c}{v_1^c v_2}\right] - v_2^c A \log \frac{u^c}{u} - Av_2\left(u - \frac{u^c}{u}\right) \tag{41}$$

where

$$\rho_1 = \frac{AK_1 u^2 u^c v_2^c(1-\lambda) + u^{c^2}v_2 K_1 v_2^c + 2uv_2^{c^2}K_1 v_2}{2uu^c[u(1-\lambda)+K_1 v_2][u^c(1-\lambda)+K_1 v_2^c]}$$

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$$\rho_2 = \frac{v_2^c[u^2(1-\lambda) + u^c K_1 v_2^c]}{2u}$$

Hence if (35) holds, then it follows from (41) that $\dot{V}(t) \le 0$, with equality if and only if $u = u^c$, $\frac{v_1^c[u^c(1-\lambda)+K_1v_2^c]u(t-\tau)v_2(t-\tau)}{u^cv_2^cv_1[u(t-\tau)(1-\lambda)+K_1v_2(t-\tau)]} = \frac{v_1v_2^c}{v_1^cv_2} = 1$. Looking for the invariant $\mathcal{H}$ within the set $H = \left\{(u, v_1, v_2); u = u^c, \frac{v_1^c[u^c(1-\lambda)+K_1v_2^c]u(t-\tau)v_2(t-\tau)}{u^cv_2^cv_1[u(t-\tau)(1-\lambda)+K_1v_2(t-\tau)]} = \frac{v_1v_2^c}{v_1^cv_2} = 1\right\}$. Here $u = u^c$ on $\mathcal{H}$ and also $0 = \dot{u}(t) = Ru^c(1-\frac{u^c}{K}) - \frac{Au^c(1-\lambda)v_2}{u^c(1-\lambda)+K_1v_2}$, hence we get $v_2(t) = v_2^c$. From (3) $0 = \dot{v}_2(t) = Dv_1(t) - d_2v_2^c$ which gives $v_1 = v_1^c$. Therefore, the only invariant set in H is $\mathcal{H} = \{(u^c, v_1^c, v_2^c)\}$. Using Lasalle invariance principle E_c is globally attractive for the system.

5. Numerical Simulations

Example 5.1 Consider the system (1-3) with $\lambda = .5$, $\tau = 6$, R = .65, D = .9, d_1= .9, d_2= .6, B = .6, A = .9, K_1= .9, K = 7. Calculations shows $BA - r_1 = -0.66 < 0$ and predator free equilibrium $E_1 = (7,0,0)$. By theorem (2.3) E_1 is locally asymptotically stable. Figure 1 and Figure 2.
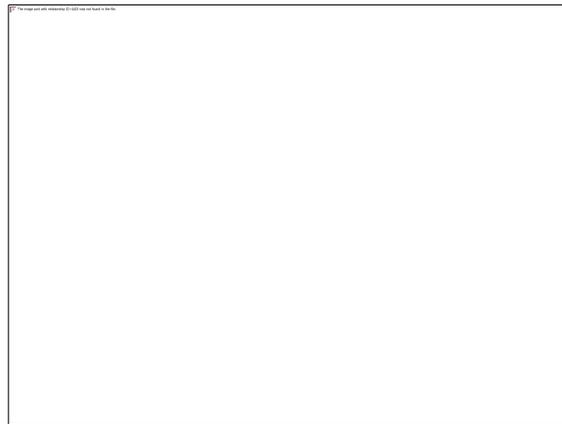
Figure1. The boundary equilibrium E_1 = (7, 0, 0) is locally asymptotically stable
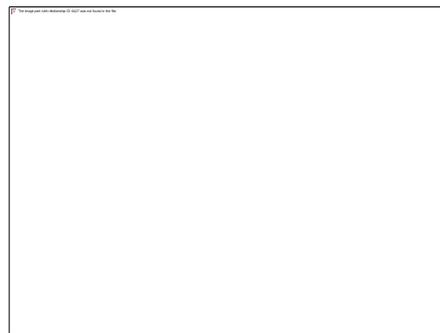
Figure 2. The boundary equilibrium E_1 = (7, 0, 0) at different initial values is locally asymptotically stable

Example 5.2 For the system (1-3) when $\lambda = .3$, R = 4, D = 3, d_1= 1, d_2= 1, B = 3, A = 2, K_1= 3, K = 10 and τ = 5, we find $BA - r_1 = 4.6667 > 0$, $36 = K_1RB > (1-\lambda)(BA - r_1) = 3.2667$. Also $P_1 = 7.5051, P_2 = 7.5564, P_4 = 38.2518$ and $E_c = (9.0926, 2.4752, 7.4256)$. The conditions in theorem (2.4)(1) are satisfied and hence E_c is locally asymptotically stable. Figure 3 and Figure 4.
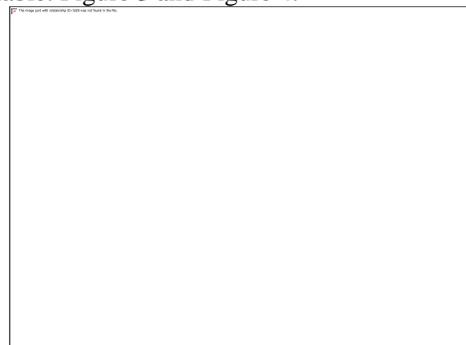
Figure 3. The coexistence equilibrium E_c = (9.0926, 2.4752, 7.4256) is locally asymptotically stable
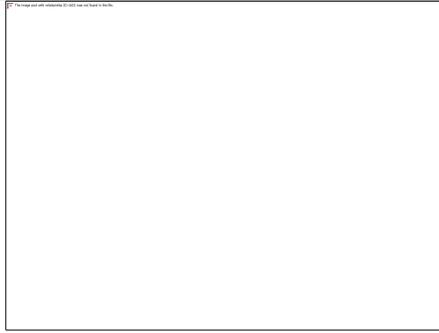
Figure 4. The coexistence equilibrium E_c = (9.0926, 2.4752, 7.4256) at different initial values is locally asymptotically stable

Example 5.3 Let for the system (1-3) λ = .4, R = .8, D = .9, d_1= .3, d_2= .1, B = .6, A = .7, K_1= .5, K = 20 and τ = 5. Calculation yields $BA - r_1 = 0.2867 > 0$, $0.24 = K_1 RB > (1 - \lambda)(BA - r_1) = 0.172$. $P_1 = 0.7504, P_2 = 0.8447, P_4 = 1.8730$ and $E_c = (5.6667, 1.6244, 14.62)$. The conditions in theorem (2.4)(2) are met. Also $\tau_0 = 14.3451$. When $\tau = 12 < 14.3451 = \tau_0$, the equilibrium E_c is locally asymptotically stable. Figure 5 and Figure 6. When $\tau = 16 > 14.3451 = \tau_0$, the equilibrium E_c yields a periodic solution. Figure 7 and Figure 8.
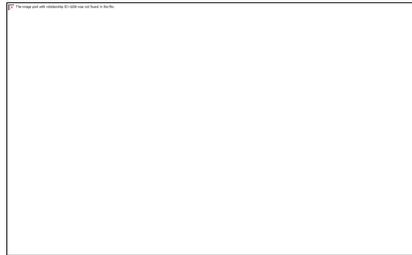
Figure 5. The coexistence equilibrium E_c = (5.6667, 1.6244, 14.62) is locally asymptotically stable for τ = 12
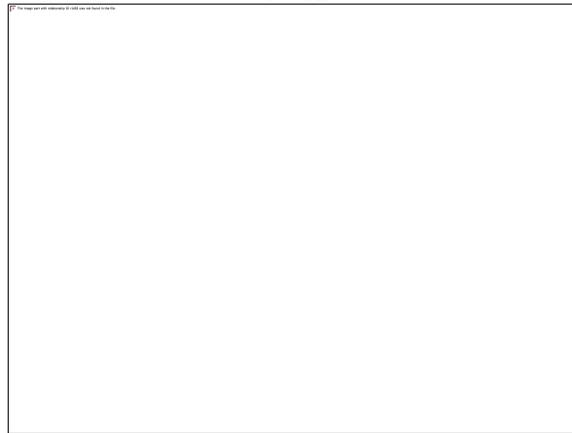
Figure 6. The coexistence equilibrium E_c = (5.6667, 1.6244, 14.62) is locally asymptotically stable for τ = 12<14.3451 = τ_0 as time increases.
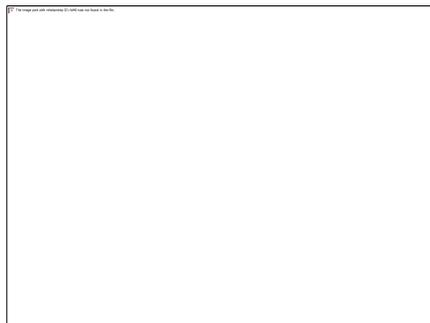
Figure 7. The coexistence equilibrium E_c = (5.6667, 1.6244, 14.62) undergoes oscillation

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for $\tau = 16 > 14.3451 = \tau_0$

Figure 8. The coexistence equilibrium $E_c = (5.6667, 1.6244, 14.62)$ undergoes oscillation for
$\tau = 16 > 14.3451 = \tau_0$

Example 5.4 In (1) let $\lambda = 0.7$, $R = 4$, $D = 3$, $d_1 = 1$, $d_2 = 1$, $B = 3$, $A = 2$, $K_1 = 3$, $K = 20$ and $\tau = 5$. Calculation shows that $BA - r_1 = 4.66 > 0$, $4 = R > \frac{A(1-\lambda)}{K_1} = 0.2$, $3.64 = \frac{R}{K}(l_1 + u^c) - R > \overline{\rho_1} = 0.474$ and $110 = u^{c^2}(1 - \lambda) \geq \overline{\rho_2} = 89.909$.hence by theorem (4.1), the positive equilibrium $E_c = (19.2222, 2.2425, 6.723)$ is stable globally. Figure 9.
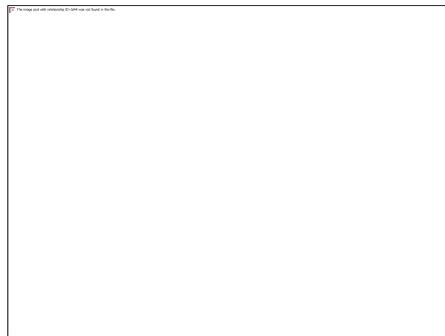
Figure 9. The coexistence equilibrium $E_c = (19.222, 2.2425, 6.723)$ with different initial values is globally stable

6. Conclusion

The mathematical model of the prey predator system considered for analysis is uniformly bounded which implies that the model is well behaved biologically. The boundary equilibrium is asymptotically stable under appropriate conditions. The conditions for the coexistence equilibrium to be locally stable is obtained. Also, it is found that the delay in time can make the stable equilibrium to be unstable, as the time lag crosses a critical value and making way for Hopf bifurcation. By suitable construction of the Lyapunov functional global stability at the positive equilibrium point is established.

Conflicts of Interest

No conflict of interest was declared by the authors.

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