\textbf{gη - Separation Axioms In Topological Spaces}

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\textbf{ABSTRACT:} In this paper a new class of separation axioms in topological spaces is used in \(g\eta\)-closed sets. The concept of \(g\eta-T_k\) spaces for \(k = 0, 1, 2\), \(g\eta-D_k\) spaces for \(k = 0, 1, 2\) and \(g\eta-R_k\) spaces for \(k = 0, 1, 2\) and some of their properties are investigated.

\textbf{Keywords} \(g\eta\)-closed sets, \(g\eta\)-open sets, \(g\eta-T_k\) spaces for \(k = 0, 1, 2\), \(g\eta-D_k\) spaces for \(k = 0, 1, 2\) and \(g\eta-R_k\) spaces for \(k = 0, 1\).

\section{1. INTRODUCTION}
In recent years a number of generalizations of open sets have been developed by many mathematicians. In 1963, Levine [5] introduced the notion of semi-open sets in topological spaces. In 1984, Andrijevic [1] introduced some properties of the topology of \(a\)-sets. In 2016, Sayed and Mansour introduced [6] new near open set in Topological Spaces. The aim of this paper is to introduce new types of separation axioms [2, 3, 4, 7] via \(g\eta\)-open sets, and investigate the relations among these concepts.

\section{2. PRELIMINARIES}

\textbf{Definition : 2.1}
A subset \(A\) of topological space \((X, \tau)\) is called
(i) \(\eta\)-open set if \(A \subseteq \text{int} (\text{cl} (\text{int} (A))) \cup \text{cl} (\text{int} (A)), \eta\)-closed set if \(\text{cl} (\text{int} (\text{cl} (A))) \cap \text{int} (\text{cl} (A)) \subseteq A\).
(ii) \(g\eta\)-closed set if \(\eta\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

\textbf{Definition : 2.2}
A topological space \((X, \tau)\) is said to be
(i) \(g\eta-T_0\) if for each pair of distinct points \(x, y \in X\), there exists a \(g\eta\)-open set \(U\) such that either \(x \in U\) and \(y \notin U\) or \(x \notin U\) and \(y \in U\).
(ii) \(g\eta-T_1\) if for each pair of distinct points \(x, y \in X\), there exist an two \(g\eta\)-open sets \(U\) and \(V\) such that \(x \in U\) but \(y \notin U\) and \(y \in V\) but \(x \notin V\).
(iii) \(g\eta-T_2\) if for each pair of distinct points \(x, y \in X\), there exist an two disjoint \(g\eta\)-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively.

\textbf{Example : 2.3}
(i) Let \(X = \{a, b, c\}\) with the topology \(\tau = \{X, \varnothing, \{b, c\}\}\). Here \(g\eta\)-open sets are \(\{X, \varnothing, \{b\}, \{c\}, \{b, c\}\}\). Since for the distinct points \{b\} and \{c\}, there exist a \(g\eta\)-open set \(U = \{b\}\) such that \(b \in U\) and \(c \notin U\) or \(U = \{c\}\) such that \(b \notin U\) and \(c \in U\). Therefore \(X\) is \(g\eta-T_0\) space.
(ii) Let \(X = \{a, b, c\}\) with the topology \(\tau = \{X, \varnothing, \{a\}\}\). Here \(g\eta\)-open sets are \(\{X, \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}\). For the distinct points \{a\} and \{c\} there exist two \(g\eta\)-open sets \(U = \{a\}\) and \(V = \{c\}\) such that \(a \in U\) but \(c \notin U\) and \(a \notin V\) but \(c \in V\). In a similar manner for any two distinct points \(g\eta\)-open sets may be found out. Therefore \(X\) is \(g\eta-T_1\) space.
(iii) Let \(X = \{a, b, c\}\) with the topology \(\tau = \{X, \varnothing, \{a\}\}\). Here \(g\eta\)-open sets are \(\{X, \varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}\). Since for the distinct points \{a\} and \{c\} there exist two disjoint \(g\eta\)-open sets \(U = \{a\}\) and \(V = \{c\}\) containing \{a\} and \{c\} satisfying \(g\eta-T_2\) conditions. And this is true for other pairs of distinct points. Therefore \(X\) is \(g\eta-T_2\) space.
Remark: 2.4 Let \((X, \tau)\) be a topological space, then the following statements are true:
(i) Every \(g\eta\)-T2 space is \(g\eta\)-T1.
(ii) Every \(g\eta\)-T1 space is \(g\eta\)-T0.

Theorem: 2.5 A topological space \((X, \tau)\) is \(g\eta\)-T0 if and only if for each pair of distinct points \(x, y\) of \(X\), \(g\eta\)-cl\({x}\) \(\neq g\eta\)-cl\({y}\)).

Proof:
Necessity: Let \((X, \tau)\) be a \(g\eta\)-T0 space and \(x, y\) be any two distinct points of \(X\). There exists a \(g\eta\)-open set \(U\) containing \(x\) or \(y\), say \(x\) but not \(y\). Then \(X - U\) is a \(g\eta\)-closed set which does not contain \(x\) but contains \(y\). Since \(g\eta\)-cl\({y}\) is the smallest \(g\eta\)-closed set containing \(y\), \(g\eta\)-cl\({y}\) \(\subseteq X - U\) and therefore \(x \notin g\eta\)-cl\({y}\)). Consequently \(g\eta\)-cl\({x}\) \(\neq g\eta\)-cl\({y}\)).

Sufficiency: Suppose that \(x, y \in X\) and \(x \neq y\) and \(g\eta\)-cl\({x}\) \(\neq g\eta\)-cl\({y}\)). Let \(z\) be a point of \(X\) such that \(z \in g\eta\)-cl\({x}\) but \(z \notin g\eta\)-cl\({y}\)). We claim that \(x \notin g\eta\)-cl\({y}\)). For if \(x \in g\eta\)-cl\({y}\) then \(g\eta\)-cl\({x}\) \(\subseteq g\eta\)-cl\({y}\)). This contradicts the fact that \(z \notin g\eta\)-cl\({y}\)). Consequently \(x\) belongs to the \(g\eta\)-open set \(X - g\eta\)-cl\({y}\)) to which \(y\) does not belong to. Hence \((X, \tau)\) is a \(g\eta\)-T0 space.

Theorem: 2.6 In a topological space \((X, \tau)\), if the singletons are \(g\eta\)-closed then \(X\) is \(g\eta\)-T1 space and the converse is true if \(g\eta\)-O\(X, \tau\) is closed under arbitrary union.

Proof: Let \(\{z\}\) is \(g\eta\)-closed for every \(z \in X\). Let \(x, y \in X\) with \(x \neq y\). Now \(x \neq y\) implies \(y \in X - \{x\}\). Hence \(X - \{x\}\) is a \(g\eta\)-open set that contains \(y\) but not \(x\). Similarly \(X - \{y\}\) is a \(g\eta\)-open set containing \(x\) but not \(y\). Therefore \(X\) is a \(g\eta\)-T1 space.

Conversely, let \((X, \tau)\) be \(g\eta\)-T1 and \(x, y\) be any point of \(X\). Choose \(y \in X - \{x\}\) then \(x \neq y\) and so there exists a \(g\eta\)-open set \(U\) such that \(y \in U\) but \(x \notin U\). Consequently \(y \in U \subseteq X - \{x\}\), that is \(X - \{x\} = \{U : y \in X - \{x\}\}\) which is \(g\eta\)-open. Hence \(x\) is \(g\eta\)-closed.

Theorem: 2.7 Let \((X, \tau)\) be a topological space, then the following statements are true:
(i) \(X\) is \(g\eta\)-T2.
(ii) Let \(x \in X\). For each \(y \neq x\), there exists a \(g\eta\)-open set \(U\) containing \(x\) such that \(y \notin g\eta\)-cl\({U}\).
(iii) For each \(x \in X\), \(\cap \{g\eta\)-cl\({U}\) : \(U \in g\eta\)-O\(X, \tau\) and \(x \in U\}\} = \{x\}.

Proof: (i) \(\Rightarrow\) (ii) Let \(x \in X\), and for any \(y \in X\) such that \(x \neq y\), there exist disjoint \(g\eta\)-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively, since \(X\) is \(g\eta\)-T2. So \(U \subseteq X - V\). Therefore, \(g\eta\)-cl\({U}\) \(\subseteq X - V\). So \(y \notin g\eta\)-cl\({U}\).

(ii) \(\Rightarrow\) (iii) If possible for some \(y \neq x\), \(y \in \cap \{g\eta\)-cl\({U}\) : \(U \in g\eta\)-O\(X, \tau\) and \(x \in U\}\}. This implies \(y \in g\eta\)-cl\({U}\) for every \(g\eta\)-open set \(U\) containing \(x\), which contradicts (ii). Hence \(\cap \{g\eta\)-cl\({U}\) : \(U \in g\eta\)-O\(X, \tau\) and \(x \in U\}\} = \{x\}.

(iii) \(\Rightarrow\) (i) Let \(x, y \in X\) and \(x \neq y\). Then there exists a \(g\eta\)-open set \(U\) containing \(x\) such that \(y \notin g\eta\)-cl\({U}\)). Let \(V = X - g\eta\)-cl\({U}\)), then \(y \in V\) and \(x \in U\) and also \(U \cap V = \varnothing\). Therefore \(X\) is \(g\eta\)-T2.

Definition: 2.8 A subset \(A\) of a topological space \(X\) is called a \(g\eta\)-difference set (briefly \(g\eta\)-D set) if there exists \(U, V \in g\eta\)-O\((X, \tau)\) such that \(U \neq X\) and \(A = U - V\).

Theorem: 2.9 Every proper \(g\eta\)-open set is a \(g\eta\)-D set.
Proof: Let \(U\) be a \(g\eta\)-open set different from \(X\). Take \(V = \varnothing\). Then \(U = U - V\) is a \(g\eta\)-D set. But, the converse is not true as shown in the following example.

Example: 2.10 Let \(X = \{a, b, c\}\) with \(\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\}\). Here \(g\eta\)-O\((X, \tau)\) = \(\{X, \varnothing, \{a\}, \{c\}, \{a, b, \{a, c\}\}\}\). Then \(U = \{a, b\} \neq X\) and \(V = \{a, c\}\) are \(g\eta\)-open sets in \(X\). Let \(A = U - V = \{a, b\} - \{a, c\} = \{b\}\). Then \(A = \{b\}\) is a \(g\eta\)-D set but it is not \(g\eta\)-open.

Definition: 2.11 A topological space \((X, \tau)\) is said to be
(i) \(g\eta\)-D\(0\) if for any pair of distinct points \(x\) and \(y\) of \(X\) there exists a \(g\eta\)-D set of \(X\) containing \(x\) but not \(y\) or a \(g\eta\)-D set of \(X\) containing \(y\) but not \(x\).
(ii) \(g\eta\)-D if for any pair of distinct points \(x\) and \(y\) of \(X\) there exists a \(g\eta\)-D set of \(X\) containing \(x\) but not \(y\) and a \(g\eta\)-D set of \(X\) containing \(y\) but not \(x\).

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(iii) $g\eta$-D$_2$ if for any pair of distinct points $x$ and $y$ of $X$ there exists two disjoint $g\eta$-D sets of $X$ containing $x$ and $y$, respectively.

**Remark 2.12** For a topological space $(X, \tau)$, the following properties hold:

(i) If $(X, \tau)$ is $g\eta$-T$_0$, then it is $g\eta$-D$_k$, for $k = 0, 1, 2$.

(ii) If $(X, \tau)$ is $g\eta$-D$_0$, then it is $g\eta$-D$_{k-1}$ for $k = 1, 2$.

(iii)

**Theorem 2.13** A topological space $(X, \tau)$ is $g\eta$-D$_0$ if and only if it is $g\eta$-T$_0$.

**Proof:** Suppose that $X$ is $g\eta$-D$_0$. Then for each distinct pair $x, y \in X$, at least one of $x, y$ say $x$, belongs to a $g\eta$-D set $P$ but $y \not\in P$. As $P$ is $g\eta$-D set, $P$ can be written as $P = U_1 - U_2$ where $U_1 \neq X$ and $U_1, U_2 \in g\eta$-O$(X, \tau)$. Then $x \in U_1$, and for $y \not\in P$ we have two cases: (i) $y \not\in U_1$, (ii) $y \in U_1$ and $y \in U_2$. In case (i), $x \in U_1$ but $y \not\in U_1$. In case (ii), $y \in U_2$ but $x \not\in U_2$. Thus in both the cases, we obtain that $X$ is $g\eta$-T$_0$.

Conversely, if $X$ is $g\eta$-T$_0$, by Remark 2.12 (ii) $X$ is $g\eta$-D$_0$.

**Theorem 2.14** Suppose $g\eta$-O$(X, \tau)$ is closed under arbitrary union, then $X$ is $g\eta$-D$_1$ if and only if it is $g\eta$-D$_2$.

**Proof:**

**Necessity:** Let $x, y \in X$ and $x \neq y$. Then there exist a $g\eta$-D sets $P_1, P_2$ in $X$ such that $x \in P_1, y \not\in P_1$ and $y \in P_2, x \not\in P_2$.

Let $P_1 = U_1 - U_2$ and $P_2 = U_1 - U_4$. Where $U_1, U_2$, $U_3$ and $U_4$ are $g\eta$-open sets in $X$. From $x \not\in P_2$, the following two cases arise: Case (i): $x \not\in U_3$. Case (ii): $x \in U_3$ and $x \in U_4$.

Case (i): $x \in U_3$. By $\eta \in P_1$, we have two sub cases:

(a) $y \not\in U_1$. Since $x \in U_2 - U_4$, it follows that $x \in U_2 - (U_3 - U_4)$, and since $y \in U_3 - U_4$ we have $y \not\in U_3 - (U_1 - U_4)$, and $(U_1 - (U_2 - U_3)) \cap (U_3 - (U_1 - U_4)) = \emptyset$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 - U_2$ and $y \in U_2$, and $(U_1 - U_2) \cap U_2 = \emptyset$.

Case (ii): $x \in U_3$ and $x \in U_4$. We have $y \in U_3 - U_4$, and $x \in U_4$. Hence $(U_3 - U_4) \cap U_4 = \emptyset$. Thus both case (i) and in case (ii), $X$ is $g\eta$-D$_2$.

**Sufficiency:** Follows from Remark 2.12 (ii).

**Corollary 2.15** If a topological space $(X, \tau)$ is $g\eta$-D$_1$, then it is $g\eta$-T$_0$.

**Proof:** Follows from 2.12 (ii) and Theorem 2.13.

**Definition 2.16** A point $x \in X$ which has only $X$ as the $g\eta$-neighbourhood is called a $g\eta$-neat point.

**Proposition 2.17** For a $g\eta$-T$_0$ topological space $(X, \tau)$ which has at least two elements, the following are equivalent:

(i) $(X, \tau)$ is $g\eta$-D$_1$ space.

(ii) $(X, \tau)$ has no $g\eta$-neat point.

**Proof:** (i) $\Rightarrow$ (ii): Since $(X, \tau)$ is a $g\eta$-D$_1$ space, each point $x$ of $X$ is contained in a $g\eta$-D set $A = U - V$ and thus in $U$. By definition $U \neq X$. This implies that $x$ is not a $g\eta$-neat point. Therefore $(X, \tau)$ has no $g\eta$-neat point.

(ii) $\Rightarrow$ (i): Let $X$ be a $g\eta$-T$_0$, then for each distinct pair of points $x, y \in X$, at least one of them, $x$ (say) has a $g\eta$-neighbourhood $U$ containing $x$ and not $y$. Thus $U$ which is different from $X$ is a $g\eta$-D set. If $X$ has no $g\eta$-neat point, then $y$ is not $g\eta$-neat point. This means that there exists a $g\eta$-neighbourhood $V$ of $y$ such that $V \neq X$. Thus $y \in V - U$ but not $x$ and $V - U$ is a $g\eta$-D set. Hence $X$ is $g\eta$-D$_1$.

**Definition 2.18** A topological space $(X, \tau)$ is said to be $g\eta$-symmetric if for any pair of distinct points $x$ and $y$ in $X$, $x \in g\eta$-cl$(\{y\})$ implies $y \in g\eta$-cl$(\{x\})$.

**Theorem 2.19** If $(X, \tau)$ is a topological space, then the following are equivalent:

(i) $(X, \tau)$ is a $g\eta$-symmetric space.

(ii) $\{x\}$ is $g\eta$-closed, for each $x \in X$.

**Proof:** (i) $\Rightarrow$ (ii): Let $(X, \tau)$ be a $g\eta$-symmetric space. Assume that $\{x\} \subseteq U \in g\eta$-O$(X, \tau)$, but $g\eta$-cl$(\{x\}) \not\subseteq U$. Then $g\eta$-cl$(\{x\}) \cap (X - U) = \emptyset$. Now, we take $y \in g\eta$-cl$(\{x\}) \cap (X - U)$, then by hypothesis $x \in g\eta$-cl$(\{x\}) \subseteq X - U$ that is, $x \not\in U$, which is a contradiction. Therefore $\{x\}$ is $g\eta$-closed, for each $x \in X$.

(iii) $\Rightarrow$ (i): Assume that $x \in g\eta$-cl$(\{y\})$, but $y \not\in g\eta$-cl$(\{x\})$. Then $\{y\} \subseteq X - g\eta$-cl$(\{x\})$ and hence $g\eta$-cl$(\{y\}) \subseteq X - g\eta$-cl$(\{x\})$. Therefore $x \in X - g\eta$-cl$(\{x\})$, which is a contradiction and hence $y \in g\eta$-cl$(\{x\})$.

(iv)

**Corollary 2.20** Let $g\eta$-O$(X, \tau)$ be closed under arbitrary union. If the topological space $(X, \tau)$ is a $g\eta$-T$_1$ space, then it is $g\eta$-symmetric.

**Proof:** In a $g\eta$-T$_1$ space, every singleton set is $g\eta$-closed by theorem 2.6 therefore, by theorem 2.19, $(X, \tau)$ is $g\eta$-symmetric.
Corollary : 2.21 If a topological space \((X, \tau)\) is \(g\eta\)-symmetric and \(g\eta\)-T\(_0\) then \((X, \tau)\) is \(g\eta\)-T\(_1\) space.

Proof: Let \(x \neq y\) and as \((X, \tau)\) is \(g\eta\)-T\(_0\), we may assume that \(x \in U \subseteq X - \{y\}\) for some \(U \in g\eta\-O(X, \tau)\). Then \(x \notin g\eta\-cl(\{y\})\) and hence \(y \notin g\eta\-cl(\{x\})\). There exists a \(g\eta\)-open set \(V\) such that \(y \in V \subseteq X - \{x\}\) and thus \((X, \tau)\) is \(g\eta\)-T\(_1\) space.

Corollary : 2.22 For a \(g\eta\)-symmetric space \((X, \tau)\), the following are equivalent:

(i) \((X, \tau)\) is \(g\eta\)-T\(_0\) space.
(ii) \((X, \tau)\) is \(g\eta\)-D\(_1\) space.
(iii) \((X, \tau)\) is \(g\eta\)-T\(_1\) space.

Proof: (i) \(\Rightarrow\) (iii): follows from Corollary 2.21.

(iv) \(\Rightarrow\) (ii): follows from Remark 2.2.7 and Corollary 2.15.

Definition : 2.23 A topological space \((X, \tau)\) is said to be \(g\eta\)-R\(_0\) if \(U\) is a \(g\eta\)-open set and \(x \in U\) then \(g\eta\-cl(\{x\}) \subseteq U\).

Theorem : 2.24 For a topological space \((X, \tau)\) the following properties are equivalent:

(i) \((X, \tau)\) is \(g\eta\)-R\(_0\) space.
(ii) For any \(P \in g\eta\-C(X, \tau)\), \(x \notin P\) implies \(P \subseteq U\) and \(x \notin U\) for some \(U \subseteq g\eta\-O(X, \tau)\).
(iii) For any \(P \in g\eta\-C(X, \tau)\), \(x \notin P\) implies \(P \cap g\eta\-cl(\{x\}) = \emptyset\).
(iv) For any two distinct points \(x, y\) of \(X\), either \(g\eta\-cl(\{x\}) = g\eta\-cl(\{y\})\) or \(g\eta\-cl(\{x\}) \cap g\eta\-cl(\{y\}) = \emptyset\).

Proof: (i) \(\Rightarrow\) (ii): Let \(P \in g\eta\-C(X, \tau)\) and \(x \notin P\). Then by (1), \(g\eta\-cl(\{x\}) \subseteq X - P\). Set \(U = X - g\eta\-cl(\{x\})\), then \(U\) is a \(g\eta\)-open set such that \(P \subseteq U\) and \(x \notin U\).

(ii) \(\Rightarrow\) (iii): Let \(P \in g\eta\-C(X, \tau)\) and \(x \notin P\). There exists \(U \subseteq g\eta\-O(X, \tau)\) such that \(P \subseteq U\) and \(x \notin U\). Since \(U \subseteq g\eta\-O(X, \tau)\), \(P \cap g\eta\-cl(\{x\}) = \emptyset\) implies \(P \setminus g\eta\-cl(\{x\}) = \emptyset\).

(iii) \(\Rightarrow\) (iv): Suppose that \(g\eta\-cl(\{x\}) \neq g\eta\-cl(\{y\})\) for two distinct points \(x, y \in X\). Then \(z \notin \) \(g\eta\-cl(\{x\})\) or \(z \notin \) \(g\eta\-cl(\{y\})\). There exists \(V \subseteq g\eta\-O(X, \tau)\) such that \(y \notin V\) and \(z \notin V\) in \(X, \tau\). Hence \(x \in V\). Therefore, there exists \(x \notin g\eta\-cl(\{y\})\). By (iii), we obtain \(g\eta\-cl(\{x\}) \neq g\eta\-cl(\{y\})\).

(iv) \(\Rightarrow\) (i): Let \(V \subseteq g\eta\-O(X, \tau)\) and \(x \in X\). For each \(y \notin X\), \(x \notin y\) and \(x \notin g\eta\-cl(\{y\})\). This shows that \(g\eta\-cl(\{x\}) \neq g\eta\-cl(\{y\})\).

Theorem : 2.25 If a topological space \((X, \tau)\) is \(g\eta\)-R\(_0\) and \(g\eta\)-R\(_0\) space then it is \(g\eta\)-T\(_1\) space.

Proof: Let \(x\) and \(y\) be any two distinct points of \(X\). Since \((X, \tau)\) is \(g\eta\)-R\(_0\), there exists a \(g\eta\)-open set \(U\) such that \(x \in U\) and \(y \notin U\). As \(x \in U\), \(g\eta\-cl(\{x\}) \subseteq U\) as \(X\) is \(g\eta\)-R\(_0\) space. Hence \(y \notin U\), so \(y \notin g\eta\-cl(\{x\})\). Hence \(y \in V = X - g\eta\-cl(\{x\})\) and it is clear that \(x \notin V\). Hence it follows that there exist \(g\eta\)-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively, such that \(y \notin U\) and \(x \notin V\). This implies that \((X, \tau)\) is \(g\eta\)-T\(_1\) space.

Theorem : 2.26 For a topological space \((X, \tau)\) the following properties are equivalent:

(i) \((X, \tau)\) is \(g\eta\)-R\(_0\) space.
(ii) \(x \in g\eta\-cl(\{y\})\) if and only if \(y \in g\eta\-cl(\{x\})\), for any two points \(x\) and \(y\) in \(X\).

Proof: (i) \(\Rightarrow\) (ii): Assume that \((X, \tau)\) is \(g\eta\)-R\(_0\). Let \(x \in g\eta\-cl(\{y\})\) and \(V\) be any \(g\eta\)-open set such that \(y \in V\). Now by hypothesis, \(x \in V\). Therefore, every \(g\eta\)-open set which contain \(y\) contains \(x\). Hence \(y \notin g\eta\-cl(\{x\})\).

(ii) \(\Rightarrow\) (i): Let \(U\) be a \(g\eta\)-open set and \(x \in U\). If \(y \notin U\), then \(x \notin g\eta\-cl(\{y\})\) and hence \(y \notin g\eta\-cl(\{x\})\). This implies that \(g\eta\-cl(\{x\}) \subseteq U\). Hence \((X, \tau)\) is \(g\eta\)-R\(_0\) space.

Remark : 2.27 From Definition 2.18 and Theorem 2.26 the notion of \(g\eta\)-symmetric and \(g\eta\)-R\(_0\) are equivalent.

Theorem : 2.28 A topological space \((X, \tau)\) is \(g\eta\)-R\(_0\) space if and only if for any two points \(x\) and \(y\) in \(X\), \(g\eta\-cl(\{x\}) \neq g\eta\-cl(\{y\})\) implies \(g\eta\-cl(\{x\}) \cap g\eta\-cl(\{y\}) = \emptyset\).

Proof: Necessity: Suppose that \((X, \tau)\) is \(g\eta\)-R\(_0\) and \(x\) and \(y\) in \(X\) such that \(g\eta\-cl(\{x\}) \neq g\eta\-cl(\{y\})\). Then, there exists \(z \in g\eta\-cl(\{x\})\) such that \(z \notin g\eta\-cl(\{y\})\) or \(z \in g\eta\-cl(\{y\})\) such that \(z \notin g\eta\-cl(\{x\})\). There exists \(V \subseteq g\eta\-O(X, \tau)\) such that \(y \notin V\) and \(z \notin V\), hence \(x \notin V\). Therefore, we have \(x \notin g\eta\-cl(\{y\})\). Thus \(x \notin [X - g\eta\-cl(\{y\})] \subseteq g\eta\-O(X, \tau)\), which implies \(g\eta\-cl(\{x\}) \subseteq [X - g\eta\-cl(\{y\})]\) and \(g\eta\-cl(\{x\}) \cap g\eta\-cl(\{y\}) = \emptyset\).
Sufficiency: Let $V \in \eta\cap \Omega(\mathcal{X}, \tau)$ and let $x \in V$. To show that $\eta\cap \text{cl}(\{x\}) \subseteq V$. Let $y \notin V$, that is $y \in \mathcal{X} - V$. Then $x \neq y$ and $x \notin \eta\cap \text{cl}(\{y\})$. This shows that $\eta\cap \text{cl}(\{x\}) \neq \eta\cap \text{cl}(\{y\})$. By assumption, $\eta\cap \text{cl}(\{x\}) \cap \eta\cap \text{cl}(\{y\}) = \varnothing$. Hence $y \notin \eta\cap \text{cl}(\{x\})$ and therefore $\eta\cap \text{cl}(\{x\}) \subseteq V$. Hence $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_0$ space.

Definition: 2.29 A topological space $(\mathcal{X}, \tau)$ is said to be $\eta\cap \text{R}_1$ if for $x, y$ in $\mathcal{X}$ with $\eta\cap \text{cl}(\{x\}) \neq \eta\cap \text{cl}(\{y\})$, there exist disjoint $\eta\cap$-open sets $U$ and $V$ such that $\eta\cap \text{cl}(\{x\}) \subseteq U$ and $\eta\cap \text{cl}(\{y\}) \subseteq V$.

Theorem: 2.30 A topological space $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space if it is $\eta\cap \text{T}_2$ space.

Proof: Let $x$ and $y$ be any two points $X$ such that $\eta\cap \text{cl}(\{x\}) \neq \eta\cap \text{cl}(\{y\})$. By remark 2.4 (i), every $\eta\cap \text{T}_2$ space is $\eta\cap \text{T}_1$ space. Therefore, by theorem 2.6, $\eta\cap \text{cl}(\{x\}) = \{x\}$, $\eta\cap \text{cl}(\{y\}) = \{y\}$ and hence $\{x\} \neq \{y\}$. Since $(\mathcal{X}, \tau)$ is $\eta\cap \text{T}_2$, there exist a disjoint $\eta\cap$-open sets $U$ and $V$ such that $\eta\cap \text{cl}(\{x\}) = \{x\} \subseteq U$ and $\eta\cap \text{cl}(\{y\}) = \{y\} \subseteq V$. Therefore $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space.

Theorem: 2.31 If a topological space $(\mathcal{X}, \tau)$ is $\eta\cap$-symmetric, then the following are equivalent:

(i) $(\mathcal{X}, \tau)$ is $\eta\cap \text{T}_2$ space.
(ii) $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space and $\eta\cap \text{T}_1$ space.
(iii) $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_0$ space and $\eta\cap \text{T}_0$ space.

Proof: (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) obvious.

(iii) $\Rightarrow$ (i) Let $x, y \in \mathcal{X}$ such that $x \neq y$. Since $(\mathcal{X}, \tau)$ is $\eta\cap \text{T}_0$ space. By theorem 2.5, $\eta\cap \text{cl}(\{x\}) \neq \eta\cap \text{cl}(\{y\})$, since $X$ is $\eta\cap \text{R}_1$, there exist disjoint $\eta\cap$-open sets $U$ and $V$ such that $\eta\cap \text{cl}(\{x\}) \subseteq U$ and $\eta\cap \text{cl}(\{y\}) \subseteq V$. Therefore, there exist disjoint $\eta\cap$-open set $U$ and $V$ such that $x \in U$ and $y \in V$. Hence $(\mathcal{X}, \tau)$ is $\eta\cap \text{T}_2$ space.

Remark: 2.32 For a topological space $(\mathcal{X}, \tau)$ the following statements are equivalent.

(i) $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space.
(ii) If $x, y \in \mathcal{X}$ such that $\eta\cap \text{cl}(\{x\}) \neq \eta\cap \text{cl}(\{y\})$, then there exist $\eta\cap$-closed sets $P_1$ and $P_2$ such that $x \in P_1, y \notin P_1, y \in P_2, x \notin P_2$, and $X = P_1 \cup P_2$.

Theorem: 2.33 A topological space $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space, then $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_0$ space.

Proof: Let $U$ be a $\eta\cap$-open such that $x \in U$. If $y \notin U$, then $x \notin \eta\cap \text{cl}(\{y\})$, therefore $\eta\cap \text{cl}(\{x\}) \neq \eta\cap \text{cl}(\{y\})$. So, there exists a $\eta\cap$-open set $V$ such that $\eta\cap \text{cl}(\{y\}) \subseteq V$ and $x \notin V$, which implies $y \notin \eta\cap \text{cl}(\{x\})$. Hence $\eta\cap \text{cl}(\{x\}) \subseteq U$. Therefore, $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_0$ space.

Theorem: 2.34 A topological space $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space if and only if $x \in \mathcal{X} - \eta\cap \text{cl}(\{y\})$ implies that $x$ and $y$ have disjoint $\eta\cap$-open neighbourhoods.

Proof: Necessity: Let $(\mathcal{X}, \tau)$ be a $\eta\cap \text{R}_1$ space. Let $x \in \mathcal{X} - \eta\cap \text{cl}(\{y\})$. Then $\eta\cap \text{cl}(\{x\}) \neq \eta\cap \text{cl}(\{y\})$, so $x$ and $y$ have disjoint $\eta\cap$-open neighbourhoods.

Sufficiency: First to show that $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_0$ space. Let $U$ be a $\eta\cap$-open set and $x \in U$. Suppose that $y \notin U$. Then, $\eta\cap \text{cl}(\{y\}) \cap U = \varnothing$ and $x \notin \eta\cap \text{cl}(\{y\})$. There exist a $\eta\cap$-open sets $U_x$ and $U_y$ such that $x \in U_x, y \in U_y$, and $U_x \cap U_y = \varnothing$. Hence, $\eta\cap \text{cl}(\{x\}) \subseteq \eta\cap \text{cl}(\{U_x\})$ and $\eta\cap \text{cl}(\{x\}) \cap U_y \subseteq \eta\cap \text{cl}(\{U_y\}) \cap U_y = \varnothing$. Thus $\eta\cap \text{cl}(\{U_x\}) \subseteq \eta\cap \text{cl}(\{U_y\})$. Consequently, $\eta\cap \text{cl}(\{x\})$ is $\eta\cap \text{R}_0$ space and $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space. Next, $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space. Suppose that $\eta\cap \text{cl}(\{x\}) \neq \eta\cap \text{cl}(\{y\})$. Then, assume that there exists $z \in \eta\cap \text{cl}(\{x\})$ such that $z \notin \eta\cap \text{cl}(\{y\})$. Then, there exist a $\eta\cap$-open sets $V_z$ and $V_y$ such that $z \in V_z, y \notin V_z$ and $V_z \cap V_y = \varnothing$. Since $z \notin \eta\cap \text{cl}(\{x\}), x \in V_z$. Since $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space, we obtain $\eta\cap \text{cl}(\{x\}) \subseteq V_z, \eta\cap \text{cl}(\{y\}) \subseteq V_y$ and $V_z \cap V_y = \varnothing$. Therefore $(\mathcal{X}, \tau)$ is $\eta\cap \text{R}_1$ space.

REFERENCES: