Structures, Operations and their Applications to Topology

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Abstract: A structure on a non empty set X is a collection of subsets of X. Any kind of topology on a non empty set X is a special structure on X. A filter and a filter base on X are examples of structures. Also any ideal of subsets of X is a structure. In this paper several structures are classified and the binary relations and operations on structures are discussed. Furthermore structures on a topological space are also discussed.

Keywords: Structure, hyper relation, hyper union, hyper intersection, micro relation.

1. Introduction

A structure on a non empty set X is a collection of subsets of X**.** Any kind of topology on a non empty set X is a special structure on X. A filter and a filterbase in X are examples of a structure. Also any ideal (Jankovic & Hamlet, 1990) of subsets of X is a structure. In this paper several structures are classified and the binary relations and operations on structures are discussed. In particular several structures on a topological space and their common properties are discussed. The second section deals with the preliminaries that are needed for the paper. The notions of hyper intersction and hyper union of structures have been introduced and investigated in Section-3. The hyper difference operator on structures has been introduced and studied in the fourth section and the fifth section deals with the application of the above operators to the structures induced by a topology.

2. Preliminaries

In this paper certain basic concepts and results in topology are given. Let A and B be the subsets of a topological space (X,τ). The Interior and Closure operators on A are respectively denoted by *Int* A and *Cl* A. The following expressions will be useful in sequel.

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2.1. Expression

$Int A \subseteq Int\ Cl\ Int\ A \subseteq Cl\ Int A \subseteq Cl\ Int\ Cl A \subseteq Cl A$.

2.2. Expression

$Int A \subseteq Int\ Cl\ Int\ A \subseteq IntCl\ A \subseteq Cl\ Int\ Cl A \subseteq Cl A$.

2.3. Definition

A is called

(i).b-open(Andrijevic,1996) in (X,τ) if $A \subseteq Cl\ Int A \cup Int\ Cl A$ and b-closed if $Cl\ Int A \cap Int\ Cl A \subseteq A$,

(ii).*b-open(Indira et.al.,2012) if $A \subseteq Cl\ Int A \cap Int\ Cl A$ and *b-closed if $Cl\ Int A \cup Int\ Cl A \subseteq A$,

(iii).$b^{\#}$-open(Usha Parameswari et.al.,2014) if $A = Cl\ Int A \cup Int\ Cl A$ and $b^{\#}$-closed if $Cl\ Int A \cap Int\ Cl A = A$.

Let X be a set. By a structure on X we wean a collection of subsets of X. For example if X = {a,b,c} then the subsets {a},{b} and {a,c} of X constitute a structure of X, denoted by [{a}, {b}, {a,c}]. Throughout this paper P, Q, R, S, Ω are structures on X.

2.4. Definitions

(i) 2^X denotes the whole structure on X.

(ii) If A is a subset of X then the structure [A] is known as a singleton structure of X.

(iii) [X] denotes the absolute structure of .

(iv) ϕ̰ = the empty structure or the null structure on X.

(v) If A and B are two distinct subsets of X then [A, B]=[B, A] is a doublton structure of X.

Generally structures can be compared by the set inclusion relations namely $\subseteq$, $\supseteq$, $\subset$ and $\supset$. The hyper relations namely ⋐, ⋑ and micro relations ⋐, ⋑ on structures have been already discussed in (Jeyalakshmi et.al., 2021). It has been established that the relations ⋐ and ⋑ are both transitive and reflexive.

2.5. Definition

(i) If $P \subseteq Q$ then P is a substructure of Q and Q is a superstructure of P.

(ii) P is a hyper substructure of Q denoted by P⋐Q if for all $A \in P$ there exists $B \in Q$ with $A \subseteq B$.

(iii) P is a hyper superstructure of Q denoted by P⋑Q if for all $A \in P$ there is a $B \in Q$ with $A \supseteq B$.

(iv) P is a micro substructure of Q denoted by P⋐Q if $A \in P \Rightarrow A \subseteq B$ for every $B \in Q$.

(v) P is a micro superstructure of Q denoted by P⋑Q if $A \in P \Rightarrow A \supseteq B$ for every $B \in Q$.

A topology of X induces several structures on X. The following are structures induced by a topology.

2.6. Examples

(i) $bO(X,\tau)$ - structure of b-open sets and $bC(X,\tau)$ - structure of b-closed sets.

(ii) $*bO(X,\tau)$ - structure of *b-open sets and $*bC(X,\tau)$-structure of *b-closed sets.

(iii) $b^{\#}O(X,\tau)$-structure of $b^{\#}$-open sets and $b^{\#}C(X,\tau)$-structure of $b^{\#}$- closed sets.

The following hyper inclusion diagrams always hold for any topological space (X,τ).

2.7. Diagram

(i) $*bO(X,\tau)$⋐$bO(X,\tau)$ and $b^{\#}O(X,\tau)$⋐$bO(X,\tau)$.

(ii) $*bC(X,\tau)$⋐$bC(X,\tau)$ and $b^{\#}C(X,\tau)$⋐$bC(X,\tau)$.

3. Hyper Intersection and Hyper Union

The concepts of hyper intersection and hyper union operators have been introduced and discussed in this section.

3.1. Definition

P⋒Q={$A \cap B$:$A \in P$ and $B \in Q$}=The hyper intersection of P with Q.

P⋓Q ={$A \cup B$:$A \in P$ and $B \in Q$}=The hyper union of P with Q.

3.2. Example

Let X={a,b,c}, P=[{a,b},{a,c},{b,c}] and Q=[{a},{b},{c},{a,b},{a,c}],$P \cap Q$ = [{a,b}, {a,c}], P⋒Q= [$\emptyset$, {a}, {b}, {c}, {a,b}, {a,c}], $P \cup Q$ = [{a}, {b}, {c}, {a,b}, {a,c}, {b,c}],
P⋓Q = [{a,b},{a,c}, {b,c}, X].

3.3. Proposition

(i) P⋓ϙ = P and P ⋒ϙ= ϙ .

(ii) P⋓[$\emptyset$] =P and P ⋒ [$\emptyset$] = [$\emptyset$].

(iii) P ⋓ [X] = [X] and P ⋒ [X] = P.

(iv) P ⋓2^X =2^X and $P \subseteq$ P⋒$2^X \subseteq 2^X$.

3.4. Proposition

(i) P $\subseteq$P⋓P and P $\subseteq$ P⋒P.

(ii) P⋒P⋐P ⋐ P ⋓ P.

(iii) P⋓Q = Q ⋓ P and P⋒Q = Q ⋒ P.

(iv) P⋓(Q⋓R)=(P⋓Q)⋓R and P⋒(Q⋒R)=(P⋒Q)⋒R.

(v) P⋒(Q⋓R)$\subseteq$ (P⋒Q)⋓(P⋒R) and P⋓(Q⋒R)$\subseteq$(P⋓Q)⋒(P⋓R).

3.5. Proposition

If P = [A] then P⋒P = P = P ⋓ P and if P = [A,B] where $A \neq B$ and $A \subseteq B$ then P⋒P = P = P⋓P.

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3.6. Definition

P is a nested structure if for any two members A, B of P either $A \subseteq B$ or $A \supseteq B$ holds.

3.7. Proposition

If P is a nested structure then P⋒P = P = P ⋓ P.

3.8. Proposition

(i) If $P \Subset Q$ and $R \Subset S$ and $P \Cap R \Subset Q \Cap S$ and $P \Cup R \Subset Q \Cup S$.

(ii) If $P \sqsubset Q$ and $R \sqsubset S$ and $P \Cap R \sqsubset Q \Cap S$ and $P \Cup R \sqsubset Q \Cup S$.

3.9. Proposition

(i) If $P \Supset Q$ and $R \Supset S$ and $P \Cap R \Supset Q \Cap S$ and $P \Cup R \Supset Q \Cup S$.

(ii) If $P \sqsupset Q$ and $R \sqsupset S$ and $P \Cap R \sqsupset Q \Cap S$ and $P \Cup R \sqsupset Q \Cup S$.

4. Hyper Difference Operator

4.1. Definition

$P \ominus Q = [A \backslash B: A \in P \text{ and } B \in Q]$ = The hyper difference of Q from P.

4.2. Proposition

(i) $P \ominus [\varnothing] = P$ and $[\varnothing] \ominus P = [\varnothing]$.

(ii) $P \ominus [X] = [\varnothing]$ and $[X] \ominus P = [X \backslash A: A \in P]$.

(iii) $P \subseteq P \ominus 2^X$ and $[X] \ominus P \subseteq 2^X \ominus P$.

(iv) $[A] \ominus [A] = [\varnothing]$ and $P \ominus P \neq [\varnothing]$ if P contains more than one mmeber.

4.3. Proposition

(i) $P \ominus Q \subseteq P \ominus (P \Cap Q)$.

(ii) If $P \Cap Q = [\varnothing]$ then $P \ominus Q = P$.

(iii) If $P \ominus Q = [\varnothing]$ then $P \Subset Q$ and $P \sqsubset Q$.

4.4. Proposition

(i) $(P \ominus Q) \ominus R = P \ominus (Q \Cup R) \subseteq (P \ominus R) \ominus (Q \ominus R)$.

(ii) $P \ominus (Q \ominus R) \subseteq (P \ominus Q) \Cup (P \Cap R)$.

(iii) $(P \Cup Q) \ominus R \subseteq (P \ominus R) \Cup (Q \ominus R)$.

(iv) $(P \Cap Q) \ominus R \subseteq (P \ominus R) \Cap (Q \ominus R)$

(v) $P \Cap (Q \ominus R) \subseteq (P \Cap Q) \ominus (P \Cap R)$.

(vi) $P \Cup (Q \ominus R) \subseteq (P \ominus Q) \Cup (P \ominus Q) \Cup (P \Cap Q \Cap R)$.

(vii) $P \ominus (Q \Cup R) \subseteq (P \ominus Q) \Cap (P \ominus R)$

(viii) $P \ominus (Q \Cap R) \subseteq (P \ominus Q) \Cup (P \ominus R)$.

4.5. Proposition

(i) $P \Cap (2^X \ominus P) \neq [\varnothing]$ if $P \neq [\varnothing]$ and $P \Cup (2^X \ominus P) \neq 2^X$.

(ii) $2^X \ominus (2^X \ominus P) \neq P$ and $X \ominus (X \ominus P) = P$

(iii) $P \Cap ([X] \ominus P) = P \ominus P$ and $P \Cup ([X] \ominus P) = [X] \ominus (P \ominus P)$

4.6. Proposition

(i) $P \Cap ([X] \ominus Q) = P \ominus Q$ and $P \Cup ([X] \ominus Q) = [X] \ominus (Q \ominus P)$.

(ii) $P \Cap Q = [\varnothing] \Rightarrow P \Subset [X] \ominus Q$.

(iii) $[X] \ominus (Q \Cup R) = ([X] \ominus Q) \Cap ([X] \ominus R)$.

(iv) $[X] \ominus (Q \Cap R) = ([X] \ominus Q) \Cup ([X] \ominus R)$

4.7. Remark

The results (iii) and (iv) of the above proposition are also valid for arbitrary hyper union and arbitrary hyper intersection of structures as given below.

4.8. Proposition

(i) $[X] \ominus (\Cup \{P_\alpha: \alpha \in J\}) = \Cap \{[X] \ominus P_\alpha: \alpha \in J\}$

(ii) $[X] \ominus (\Cap \{P_\alpha: \alpha \in J\}) = \Cup \{[X] \ominus P_\alpha: \alpha \in J\}$

4.9. Proposition

(i) If $P \Subset Q$ then $[X] \ominus Q \Subset [X] \ominus P$

(ii) If P⋑Q then [X]⊖Q⋑[X]⊖P

5. Applicatuons to Topology

Let T denote the collection of all topologies on a non empty set X.

5.1. Lemma

(i) P⋒Q = ∪[P⋒[B]:B∈Q]

(ii) P⋓Q= ∪[P⋓[B]:B∈Q]

5.2. Proposition

Let P∈T, Q∈T.

(i) P⋒[∅, X] = P

(ii) P is a hyper substructure of [∅, X]

(iii) P⋓[∅, X] = P .

(iv) P⋒2^X = 2^X

(iii) P⋒Q=Q⋒P will be a basis for some topology on X

(iv) P⋓Q is a generalized topology(Csaszar,20002) on X

5.3. Proposition

Let P∈T .

(i) P⋒P = P = P⋓P

(ii) P⊖P = P⋒P′ where P′ is the collection of closed sets with respect to P.

Let (X,τ) be a topological space and Ω be a collection of subsets of X. Let *Cl* Ω=[*Cl* A:A∈Ω] and *Int* Ω = [*Int*A: A∈Ω].

5.4. Proposition

(i) bO(X,τ) ⋐ *Cl Int* bO(X,τ) ⋓ *Int Cl* bO(X,τ).

(ii) bC(X,τ) ⋑ *Cl Int* bC(X,τ) ⋒ *Int Cl* bC(X,τ).

(iii) *Cl* bO(X,τ) = *Cl Int Cl* bO(X,τ).

(iv) *Int* bC(X,τ) = *Int Cl Int* bC(X,τ).

5.5. Proposition

(i) *bO(X,τ) ⋐ *Cl Int* *bO(X,τ) ⋒ *Int Cl* *bO(X,τ).

(ii) *bC(X,τ) ⋑ *Cl Int* *bC(X,τ) ⋓ *Int Cl* *bC(X,τ).

(iii) *Cl* *bO(X,τ) =*Cl Int* *bO(X,τ) =.*Cl Int Cl* *bO(X,τ).

(iv) *Int* *bC(X,τ)= *IntCl* *bC(X,τ) = *IntCl Int* *bC(X,τ).

5.6. Proposition

(i) $b^{\#}$O(X,τ) ⋐ *Cl Int* $b^{\#}$O(X,τ) ⋓ *Int Cl* $b^{\#}$O(X,τ)

(ii) $b^{\#}$C(X,τ) ⋐ *Cl Int* $b^{\#}$C(X,τ) ⋒ *Int Cl* $b^{\#}$C(X,τ).

5.7. Proposition

Let (X,τ) be a topological space .

(i) *Int Cl* (2^X⋒2^X) ⋐ *Int Cl* (2^X) ⋒ *Int Cl* (2^X)

(ii) *Cl Int* (2^X⋒2^X) ⋐ *Cl Int* (2^X) ⋒ *Cl Int* (2^X)

(iii) *Int Cl* (2^X) ⋓ *Int Cl* (2^X) ⋐ *Int Cl* (2^X⋓2^X)

(iv) *Cl Int* (2^X) ⋓ *Cl Int* (2^X) ⋐ *Cl Int* (2^X⋓2^X)

(v) *Cl Int Cl* (2^X) ⋓ *Cl Int Cl* (2^X) = *Cl Int Cl* (2^X⋓2^X)

(vi) $Int\ Cl\ Int\ (2^X) \cap Int\ Cl\ Int\ (2^X) = Int\ Cl\ Int\ (2^X \cap 2^X)$

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