Connected and Total Vertex covering in Graphs

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Abstract: A Subset S of vertices of a Graph G is called a vertex cover if S includes at least one end point of every edge of the Graph. A Vertex cover S of G is a connected vertex cover if the induced subgraph of S is connected. The minimum cardinality of such a set is called the connected vertex covering number and it is denoted by . A Vertex cover S of G is a total vertex cover if the induced subgraph of S has no isolates. The minimum cardinality of such a set is called the total vertex covering number and it is denoted by . In this paper a few properties of connected vertex cover and total vertex covers are studied and specific values of and of some well-known graphs are evaluated.

Keywords: Vertex cover, connected vertex cover, Total vertex

1. Introduction

By a graph \( G = (V, E) \) we mean a finite, undirected and connected graph with neither self loops nor multiple edges. The order and size of \( G \) are denoted by \( n \) and \( m \) respectively. For graph theoretic terminology, we refer to Chartrand and Lesniak [1].

We start with following definitions and theorems.
1. A subset S of vertices of a graph G is called a vertex cover if S includes at least one end point of every edge of the graph and the minimum cardinality of such a vertex cover is called vertex covering number and it is denoted by \( \alpha(G) \).
2. A subset S of vertices of a graph G is called a dominating set of a graph if each vertex not in \( S \) is adjacent with some vertex in S. The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of G.
3. A dominating set S of a connected graph is called connected dominating set if the induced subgraph of S is connected and the minimum cardinality of such a set is called connected domination number and it is denoted by \( \gamma_c(G) \).
4. A Total dominating set S of a graph G is a dominating set in which the induced subgraph of \( S \) has no isolates and the minimum cardinality of such a set is called the Total domination number and it is denoted by \( \gamma_t(G) \).
5. A set S of vertices in a graph is said to be an independent set if no two vertices in S are adjacent. A maximal independent set is an independent set to which no other vertex can be added to it without destroying its independence property. The number of vertices in the largest independent set is called the independence number and it is denoted by \( \beta(G) \).
6. A property P of a set of vertices is said to be hereditary if whenever a set S has property P, so does every proper subset \( S' \subset S \). A property P is super hereditary if whenever a set S has property P, so does every proper superset \( S' \supset S \).
7. A matching is any independent set of edges. A maximal matching is a matching in X so that V-\( \{u\} \) is an independent set of vertices.

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11. A perfect matching in a graph \( G \) is a matching \( X \) so that \( V(X) = V(G) \). Let \( \beta_1(G) \) denote the size of a maximum in \( G \). The number of edges in a smallest maximal independent set of edges in \( G \) is denoted by \( \beta_1(G) \).

12. A split graph is a graph in which the vertices can be partitioned into a clique and an independent set.

Cockayne et al [2] obtained the following fundamental results for hereditary and super hereditary properties.

**Proposition 1.1** [2] Let \( G = (V,E) \) be a graph and let \( P \) be a hereditary property. Then a set \( S \) is maximal \( P \)-set if and only if \( S \) is a 1-maximal \( P \)-set.

**Proposition 1.2** [2] Let \( G = (V,E) \) be a graph and let \( P \) be a super hereditary property. Then a set \( S \) is minimal \( P \)-set if and only if \( S \) is a 1-minimal \( P \)-set.

**Theorem 1.3** [3] For any graph \( G \) of order \( n \), then \( \alpha(G) + \beta(G) = n \).

A vertex cover \( S \) of \( G \) is a connected vertex cover if the induced subgraph of \( S \) is connected. The minimum cardinality of such a set is called the connected vertex covering number and it is denoted by \( \alpha_c(G) \). A vertex cover \( S \) of \( G \) is a total vertex cover if the induced subgraph of \( S \) has no isolates. The minimum cardinality of such a set is called the total vertex covering number and it is denoted by \( \alpha_t(G) \).

In this paper a few properties of connected vertex cover and total vertex covers are studied and specific values of \( \alpha_c(G) \) and \( \alpha_t(G) \) of some well-known graphs are evaluated.

Many variants of connected domination and total domination number have been already studied.

2. **Connected and Total Vertex Covering In Graphs**

The property of being a connected vertex cover is a super hereditary property. Hence a connected vertex cover is minimal if and only if it is 1-minimal.

**Theorem 2.1**

A connected vertex covering \( S \) is a minimal connected vertex covering if and only if for each \( v \in S \), one of the following holds.

(a) \( v \) is a cut vertex in \( \langle S \rangle \).
(b) There exists a vertex \( u \in V - S \) such that \( uv \in E(G) \).

Proof:

Let \( S \) be a minimal connected vertex covering of a graph \( G \). Then for every vertex \( v \in S \), \( S - \{v\} \) is not a connected vertex covering. This means that either \( \langle S - \{v\} \rangle \) is not connected or an edge in \( V - S \) is not covered by \( S - \{v\} \). This implies that either \( v \) is a cut vertex in \( \langle S \rangle \) or there exists a vertex \( u \in V - S \) such that \( uv \in E(G) \).

Conversely suppose that \( S \) is a connected vertex covering of a graph \( G \) and for each \( v \in S \), one of the stated conditions hold. We show that \( S \) is a minimal connected vertex covering of a graph. Suppose \( S \) is not a minimal connected vertex covering, then there exists a vertex \( v \in S \) such that \( S - \{v\} \) is a connected vertex covering. Hence \( \langle S - \{v\} \rangle \) is connected and \( v \) is not a cut vertex of \( \langle S \rangle \). Also if \( S - \{v\} \) is a connected vertex covering, then there is no edge in \( V - \{S - \{v\} \} \) which is not covered by \( S - \{v\} \). This means that there is no vertex \( u \in V - S \) such that \( uv \in E(G) \). Hence \( S \) is a minimal connected vertex covering of a graph \( G \).

We observe that the property of being a total vertex covering is a super hereditary property. Hence the total vertex covering \( S \) is minimal if and only if it is 1-minimal.

**Theorem 2.2**

A total vertex covering \( S \) is a minimal total vertex covering if and only if for each \( u \in S \), one of the following conditions hold:

(a) there exists a vertex \( w \in S \) such that \( N(w) \cap S = \{u\} \).
(b) there exists a vertex \( v \in V - S \) such that \( uv \in E(G) \).

Proof:

Let \( S \) be a minimal total vertex covering of a graph \( G \). Then for every \( u \in S \), \( S - \{u\} \) is not a total vertex covering. This means that \( \langle S - \{u\} \rangle \) has an isolate or an edge in \( (V - S) \cup \{u\} \) is not covered by \( S - \{u\} \). This
means that there exists a vertex \( w \in S \) such that \( N(w) \cap S = \{u\} \) or there exists a vertex \( v \in V - S \) such that \( uv \in E(G) \).

Conversely suppose that \( S \) is a total vertex covering and for each \( u \in S \), one of the stated conditions hold. We show that \( S \) is a minimal total vertex covering of a graph \( G \). Suppose \( S \) is not a minimal total vertex covering, then there exists a vertex \( u \in S \) such that \( \{u\} \subseteq N(u) \) and hence \( S \) has no isolates. Then there is no vertex \( w \in S \) such that \( \{w\} \subseteq N(w) \) or there exists a vertex \( v \in V - S \) such that \( uv \in E(G) \).

Observations 2.2
(i) \( \alpha_r(G) = 1 \) if and only if \( G \) is star.
(ii) Since every connected vertex cover is a connected dominating set, we have \( \gamma_c(G) \leq \alpha_r(G) \).
(iii) Since every total vertex covering is a total dominating set, \( \gamma_t(G) \leq \alpha_c(G) \).
(iv) Since every connected vertex cover is a total vertex cover of \( G \), \( \alpha_c(G) \leq \alpha_t(G) \).
(v) There is no relation between \( \alpha_t(G) \) and \( \gamma_c(G) \).

For example, if \( G \) is a star \( K_{1,n} \), then \( \alpha_r(G) = 2 \) and \( \gamma_c(G) = 1 \).

If \( G \) is a path of length 10, then \( \alpha_r(G) = 6 \) and \( \gamma_c(G) = 8 \).

Observations 2.3
(i) \( \alpha_r(G) = 1 \) if and only if \( G \) is star.
(ii) Since every connected vertex cover is a connected dominating set, we have \( \gamma_c(G) \leq \alpha_r(G) \).
(iii) Since every total vertex covering is a total dominating set, \( \gamma_t(G) \leq \alpha_c(G) \).
(iv) Since every connected vertex cover is a total vertex cover of \( G \), \( \alpha_c(G) \leq \alpha_t(G) \).
(v) There is no relation between \( \alpha_t(G) \) and \( \gamma_c(G) \).

Theorem 2.4
For any graph \( G \), \( \alpha_t(G) + \beta_0(G) \geq n \).

Proof
Let \( S \) be a connected vertex covering of \( G \). Then \( S \) is a vertex cover of \( G \) implies that \( V - S \) is independent. Thus \( |V - S| \leq \beta_0(G) \)
\[ \Rightarrow n - \alpha_r(G) \leq \beta_0(G) \]
\[ \Rightarrow \alpha_r(G) + \beta_0(G) \geq n \]

We illustrate this with an example. Consider the complete bipartite graph \( K_{4,5} \).

Here \( \alpha_r(G) = 5, \beta_0(G) = 5 \).

Thus \( \alpha_r(G) + \beta_0(G) = 5 + 5 = 10 \geq 9 \).

Hence \( \alpha_r(G) + \beta_0(G) \geq n \).

Observations 2.5
We have \( \beta_0(G) = n - \delta(G) \) where \( \delta(G) \) is the minimum degree of graph \( G \).

Thus \( \alpha_r(G) + \beta_0(G) \geq n \)
\[ \Rightarrow \alpha_r(G) \geq n - \beta_0(G) = n - (n - \delta(G)) = \delta(G) \]

Hence for any connected graph we have
\[ \delta(G) \leq \alpha_c(G) \leq n-1 \]

Both the bounds are sharp.

We observe that \( \alpha_c(G) = n-1 \) if \( G \) is a complete graph or cycle on ‘n’ vertices and \( \alpha_r(G) = \delta \) if \( G \) is a star \( K_{1,n} \).

Theorem 2.6
Let \( a \) and \( b \) be two positive integers such that \( 2 \leq a \leq b \). Then there exists a graph \( G \) with \( \gamma_r(G) = a, \alpha_c(G) = b \).

Proof
Case (i): \( a = b \)
Let us take a path on ‘\( a \)’ vertices denoted by \( P_a \). Consider corona of path \( P_a \). That is \( P_a \circ K_1 \). For this graph \( \gamma_r(G) = a, \alpha_c(G) = b = a \)

Case (ii): \( a < b \)
Consider the path on ‘\( a \)’ vertices say \( P_a = (v_1, v_2, \ldots, v_a) \). Attach \( 2(b-a) \) pendent vertices to \( v_1 \) and let it be \( u_1, u_2, \ldots, u_{2(b-a)} \) and attach two pendants to all other vertices and then join the vertices \( u_1, u_2, u_3, u_4 \) and \( u_5, u_6, \ldots, u_{2(b-a)-1}, u_{2(b-a)} \).

Clearly \( \gamma_r(G) = a, \alpha_c(G) = a + (b-a) = b \)

Let us illustrate this construction with examples.

Example (1): Take \( a = b = 4 \). Then \( P_4 \circ K_1 \) is

Here \( \gamma_r(G) = 4, \alpha_c(G) = 4 \)

Example (2): Take \( a = 5, b = 8 \).
Then \( P_5 \) is

Attach \( 2(b-a) \) pendent vertices to \( v_1 \) and two pendent vertices to all other vertices. Then the graph \( G \) will be

Clearly \( \gamma_r(G) = 5, \alpha_c(G) = 5 + 3 = 8 \)

\( \alpha_c(G) \) AND \( \alpha_r(G) \) FOR SOME GRAPHS

(1) For complete graph \( K_n \),
\( \alpha_c(G) = n-1, \alpha_r(G) = n-1 \).

(2) For star \( K_{1,n} \), \( \alpha_c(G) = 1, \alpha_r(G) = 2 \).

(3) For Bistar \( K_{n,n} \), \( \alpha_c(G) = 2, \alpha_r(G) = 2 \).

Next, we characterise graphs for which \( \alpha_c(G) = 2 \) and \( \alpha_r(G) = 2 \).

Theorem 3.1
For any graph $G$ of order $n$, $\alpha_2(G) = 2$ if and only if $G$ is a split graph with the split partition $S$ and $V-S$ such that $|S| = 2, |V-S| = n-2$ with $\langle S \rangle = K_2$ and $\langle V-S \rangle$ is independent.

**Proof**

Suppose $\alpha_2(G) = 2$.

Let $S$ be a connected vertex covering of $G$ with 2 vertices. Since $S$ is connected, $S$ is isomorphic to $K_2$. Since $S$ is a vertex cover, $V-S$ is independent. Hence $G$ is isomorphic to a split graph with split partition $S$ and $V-S$ such that $\langle S \rangle = K_2$ and $\langle V-S \rangle$ is independent and converse is obvious.

Similarly, we have the following Theorem.

**Theorem 3.2**

For any graph $G$ of order $n$, $\alpha_2(G) = 2$ if and only if $G$ is a split graph with the split partition $S$ and $V-S$ such that $|S| = 2, |V-S| = n-2$ with $\langle S \rangle = K_2$ and $\langle V-S \rangle$ is independent.

**Open Problems**

1. Characterise graphs for which $\alpha_1(G) = \alpha_2(G)$.
2. Obtain upper bounds for $\alpha_1(G)$ and $\alpha_2(G)$ for special types of graphs like Trees, Petersen graph, etc.
3. Find bounds for $\gamma_1(G)$ and $\alpha_2(G)$.

**References**


