Scope for application of Topological spaces in Data Granulation through a new class of nearly open set Semi* Regular*- Open sets

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Article History: Received: 11 January 2021; Accepted: 27 February 2021; Published online: 5 April 2021

Abstract: The purpose of this paper is to define and study a new class of weaker form of regular*-open sets called semi*-regular*- open sets in topological spaces. Finally we conclude with the scope of this new class of open sets in applications of Data granulation.

Keywords: regular*-open, generalized closure, semi*-regular*-open

1. Introduction

Recent day's data analysis is in booming. Basic ideology behind data analysis is to dividing a whole into separate components for individual examination in such a way that is unresponsive to the specific metric chosen. As it inherits the concept of topology the tools for data analytic like granular computing are studied through the idea of topology such as quotient spaces and topological rough spaces. The inheriting properties continuity, compactness and connectedness are based on open sets in topology. So it is needed to create and study new class of open sets to support the advancements in emerging fields.
In 1937 Stone [8] investigated on regular open sets and impose the concept of regular open sets in Boolean algebra which has application in electronics automation tools and data analytic. Levine initiated to define a new class of nearly open sets. In 1963 he defined and studied semi-open sets which are weaker form of open sets. Following him many researchers worked on semi open sets and studied various topological concepts based on semi open sets. These types of generalized open sets play a very important role in fuzzy topology which is an extension of topology.
Levine [5] defined and studied generalized closed sets in 1970. In [2] Dunham introduced generalized closure. Recently authors defined new class of open sets using the concept of generalized closure.
In this paper a new class of set is defined using the concept of generalized closure and regular*-open set. Throughout this paper X, Y and Z will always denote topological spaces on which no separation axioms are assumed, unless explicitly stated. If A is a subset of a space (X, τ), $cl(A)$ and $int(A)$ denote the closure and the interior of A respectively.

Definition 1.1 A subset A of a space X is generalized closed (briefly g-closed) [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 1.2 If A is a subset of a space X, the generalized closure [2] of A is defined as the intersection of all g-closed sets in X containing A and is denoted by $Cl^*(A)$ and the g-interior of A[2]is the union of all g-open sets contained in A and is denoted by $Int^*(A)$.

Definition 1.3 A subset A of a topological space (X, τ) is semi-open [4] (respectively semi*-open [7]) if there is an open set U in X such that $U \subseteq A \subseteq Cl(U)$ (respectively $U \subseteq A \subseteq Cl^*(U)$) or equivalently if $A \subseteq Cl(Int(A))$ (respectively $A \subseteq Cl^*(Int(A))$).

Definition 1.4 A subset A of a topological space (X, τ) is said to be regular-open [8](respectively regular*open [9]) if $A = Int(Cl(A))$ (respectively $A = Int(Cl^*(A))$

Definition 1.5 A subset A of a topological space (X, τ) is called a semi-regular*-open set[10] if there is a regular*-open set U in X such that $U \subseteq A \subseteq cl(U)$

Definition 1.6 The semi-interior[4] (respectively semi*-interior [7] , regular-interior [8], semi-regular*-interior[10] and regular*-interior[9]) of a subset A is defined to be the union of all semi-open (respectively semi*-open , regular-open, semi-regular*open and regular* open) subsets of A. It is denoted by $sInt(A)$ (respectively $s^*Int(A)$, $rInt(A)$, $sr^*Int(A)$ and $r^*Int(A)$).

Lemma 1.7 Let $A \subseteq X$, then
(i) $X \setminus Cl^*A = Int^*(X \setminus A)$
(ii) $X \setminus Int^*A = cl^*(X \setminus A)$

Theorem 1.8 [9] Intersection of any two regular*-open sets is regular*-open.

2. Semi*-regular*-open

Definition 2.1 A subset A of a topological space (X, τ) is called a semi*-regular*- open set if there is a regular*- open set U in X such that $U \subseteq A \subseteq cl^*(U)$

The class of all semi*- regular* open sets in (X, τ) is denoted by $S^*R^*O(X, \tau)$ simply $S^*R^*O(X)$.

Example 2.2 Let $X = \{a, b, c\}$and $\tau = \{\phi, \{a\}, \{c\}, \{a, c\} X\}$. Then $\{b, c\}$ is semi*-regular*-open.

Theorem 2.3 If a subset A of X is semi*- regular*-open, then $A \subseteq cl^*(int(cl^*(A)))$

Proof Assume A is semi*-regular*- open set. Then there exists a regular*- open set U in X such that $U \subseteq A \subseteq cl^*(U)$. Now $U \subseteq A \Rightarrow U = int(cl^*(U)) \subseteq int(cl^*(A)) \Rightarrow cl^*(U) \subseteq cl^*(int(cl^*(A))) \Rightarrow A \subseteq cl^*(U) \subseteq cl^*(int(cl^*(A)))$

Remark 2.4 The above condition is not sufficient

Example 2.5 Let $X = \{a, b, c\}$and $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}\}$, then $\{a, c\} \subseteq cl^*(int(cl^*(\{a, c\})))$ but $\{a, c\}$ is not semi*-regular*-open.

Remark 2.6 In any topological space (X, τ), ϕ and X are semi*-regular*-open

Theorem 2.7 The union of infinitely many semi*-regular*-open sets in (X, τ), is semi*-regular*-open in X.

Proof Assume $\{A_\alpha\}$ is semi*-regular*-open. Using Theorem 2.3, we have $A_\alpha \subseteq cl^*(int(cl^*(A_\alpha)))$. This implies $\cup A_\alpha \subseteq \cup (cl^*(int(cl^*(A_\alpha)))) \subseteq cl^* \cup (int(cl^*(A_\alpha))) \subseteq cl^*(int \cup (cl^*(A_\alpha))) \subseteq cl^*(int(cl^*(\cup A_\alpha)))$. Hence $\cup A_\alpha$ is semi*-regular*-open in X.

Remark 2.8 The intersection of two semi*-regular*-open sets need not to be semi*-regular*-open as seen from the following example.

Example 2.9 Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. If $A = \{a, b\}$ and $B = \{b, c\}$ then A and B are semi*-regular*-open. But $A \cap B = \{b\}$ is not semi*-regular*-open set.

Theorem 2.10A subset A of X is semi*-regular*-open if and only if A contains a semi*-regular*-open set about each of its points.
Proof Necessity: Obvious.
Sufficiency: Let $x \in A$. Then by assumption, there is a semi*-regular*-open set U_x containing x such that $U_x \subseteq A$. Then we have$\cup \{U_x: x \in A\} = A$. By using Theorem 2.7,$A$ is semi*-regular*-open

Theorem 2.11 If A is semi*-regular*-open and B is regular*-open in a discrete space X, then $A \cap B$ is semi*-regular*-open.
Proof Since A is semi*-regular*-open in X, there is a regular*-open set U such that $U \subseteq A \subseteq cl^*(U)$. This implies $U \cap B \subseteq A \cap B \subseteq cl^*(U) \cap B$. By Theorem 1.8, $U \cap B$ is regular*-open. Therefore there exists a regular*-open set $U \cap B$ in X such that $U \cap B \subseteq A \cap B \subseteq cl^*(U) \cap B$. Hence $A \cap B$ is semi*-regular*-open in X.

Theorem 2.12 (i)Every regular*-open set is semi*-regular*-open.
(ii)Every semi*-regular*-open set is semi-regular*-open.
Proof
(i) Obvious.
(ii) Suppose A is semi*-regular*-open in X. Then there exists a regular* open set U such that $U \subseteq A \subseteq cl^*(U) \subseteq cl(U)$. Hence A is semi-regular*-open.

Remark 2.13 The converse of the above statements need not be true.

Example 2.14 Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. Then $\{a, b\}$ is semi*-regular*-open but not regular*-open.

Example 2.15 Let $X = \{a, b, c, d\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then $\{b, c, d\}$ is semi-regular*-open but not semi*-regular*-open.

Remark 2.16 If X is a $T_{1/2}$ space, regular open set coincides with regular* open set. Therefore the class of semi*-regular*-open set and semi-regular*-open set are coincide.

Theorem 2.17 If A is semi*-regular*-open, then $cl^*(A) = cl^*(int(cl^*(A)))$.
Proof Since A is semi*-regular*-open, $A \subseteq cl^*(int(cl^*A))$. Hence $cl^*A \subseteq cl^*(cl^*(int(cl^*A))) = (cl^*(int(cl^*A))$.

Remark 2.18 Let A be a subset of a space X. The semi*-regular*-interior is denoted by $s^*r^*Int(A)$ is the union of all semi*-regular*-open sets in X contained in A .That is $s^*r^*Int(A) = \cup \{U: U \subseteq A, U \in s^*r^*O(X)\}$.

Theorem 2.19 In any topological space (X, τ), the following statements hold:
(i) $s^*r^*Int(\phi) = \phi$
(ii)$s^*r^*Int(X) = X$
Proof Obvious.

Theorem 2.20 Let A be a subset of X.Then A is semi*-regular*-open if and only if $s^*r^*Int(A) = A$.
Proof Follows from Definition 2.1 and Theorem 2.7.

Theorem 2.21 If A and B are subsets of X, then
(i) $s^*r^*Int(A) \subseteq A$
(ii) $A \subseteq B \Rightarrow s^*r^*Int(A) \subseteq s^*r^*Int(B)$
(iii) $s^*r^*Int(s^*r^*Int(A)) = s^*r^*Int(A)$

(iv) $r^*(Int(A)) \subseteq s^*r^*(Int(A)) \subseteq sr^*(Int(A))$
(v) $s^*r^*Int(A) \cup s^*r^*Int(B) \subseteq s^*r^*Int(A \cup B)$
(vi) $s^*r^*Int(A \cap B) \subseteq s^*r^*Int(A) \cap s^*r^*Int(B)$
Proof Obvious.

Remark 2.22 The equality in the statement (vi) of the above theorem need not be true as shown from the following example.

Example 2.23 Let $X = \{a,b,c\}, \tau = \{\phi, \{a\}, \{c\}, \{a,c\}, X\}$. If $A = \{a,b\}, B = \{b,c\}$, then $s^*r^*Int(A \cap B) = \phi$ and $s^*r^*Int(A) \cap s^*r^*Int(B) = \{b\}$.

1. Semi*-Regular*-Closed Sets

In this section we introduced semi*-regular*-closed sets and investigated some basic properties.
Definition 3.1 A subset A of space (X,τ) is called semi*-regular*-closed if $X \setminus A$ is semi*-regular*-open in (X,τ).

The collection of all semi*-regular*-closed sets in X is denoted by $S^*R^*C(X)$
Remark 3.2 If a subset A of a space X is semi*-regular*-closed then $int^*(cl(int^*(A))) \subseteq A$

Remark 3.3 Let A be a subset of a space X. The semi*-regular*-closure of A denoted by $s^*r^*cl(A)$ is the intersection of all semi*-regular*-closed sets in X containing A . That is $s^*r^*cl(A) = \cap \{F: A \subseteq F, F \in S^*R^*C(X)\}$.

Remark 3.4(i) Arbitrary intersection of semi*-regular*-closed set is semi*-regular*-closed.
(ii)The union of two semi*-regular*-closed sets need not be semi*-regular*-closed.
Theorem 3.5 Let A be a subset of X. Then A is a Semi*-regular*-closed set in X if and only if $s^*r^*cl(A) = A$
Proof Suppose A is semi*-regular*-closed set in X .Then $s^*r^*cl(A) = A$ by definition 2.1. Conversely, suppose $s^*r^*cl(A) = A$.Then A is a semi*-regular*-closed set in X by Remark 3.4 (i)

Theorem 3.6 If A is a subset of X, then
(i) $X \setminus s^*r^*cl(A) = s^*r^*Int(X \setminus A)$.
(ii) $X \setminus s^*r^*Int(A) = s^*r^*cl(X \setminus A)$.
Proof Obvious.

Theorem 3.7 Let $x \in X$. Then $x \in s^*r^*Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every semi*-regular*-open set Ucontaining x.
Proof Let $x \in s^*r^*Cl(A)$ and there exists semi*-regular*-open set U containing x such that $U \cap A = \phi$. Then $A \subseteq X \setminus U$ and $X \setminus U$ is semi*-regular*-closed. Therefore $s^*r^*Cl(A) \subseteq s^*r^*cl(X \setminus U)$. $s^*r^*Cl(A) \subseteq s^*r^*Cl(X \setminus U) = X \setminus U$.This implies $x \notin s^*r^*Cl(A)$, which is a contradiction. Conversely, assume that $U \cap A \neq \phi$ for every semi*-regular*-open set U containing x and $x \notin s^*r^*cl(A)$.Then there exists semi*-regular*-closed subset F containing A such that $x \notin F$. Hence $x \in X \setminus F$ and $X \setminus F$ is semi*-regular*-open. Therefore $A \subseteq F$, $(X \setminus F) \cap A = \phi$. This is a contradiction to our assumption.

Future Scope

The regular open sets constitute a complete Boolean Algebra of sets with respect to the distinguished Boolean elements and operations defined by

$$0 = \phi$$
$$1 = X$$
$$A \wedge B = A \cap B$$
$$A \vee B = (A \cup B)^{\perp\perp}$$
$$P' = P^{\perp}$$
$$Int(\overline{A}) = A^{\perp\perp}$$

Therefore it is possible to relate the idea of Boolean algebra with the concept of newly constructed set and it will lead to many applications in electronic automation tools and.

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