Novel Approach of Existence of Solutions to the Exponential Equation $(3m^2 + 3)^x + (7m^2 + 1)^y = z^2$

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Article History: Received: 11 January 2021; Accepted: 27 February 2021; Published online: 5 April 2021

Abstract— In this manuscript the exponential equation $(3m^2 + 3)^x + (7m^2 + 1)^y = z^2$ where m ∊ Z in three variables for the occurrence of solutions belonging to the set Z of all integers or the concerned equation has no solution for various alternatives of m is investigated.

Keywords— Exponential Diophantine equation, Pell equation, integer solutions

I. Introduction

In Mathematics, a Diophantine equation is a polynomial equation conventionally in two or more unknowns, such that only the integer solutions are sought or studied. Diophantine Analysis deals with numerous techniques of solving Diophantine equations in multivariable and multi-degrees. Suppose that a, b, c are pairwise co-prime positive integers. Then we call the equation $a^x + b^y = c^z$, $x, y, z \in N$ as an Exponential Diophantine equation. Nobuhiro Terai [1] proved that if *a*, *b*, *c* gratify $a = 4m^2 + 1, b = 5m^2 - 1, c = 3m, m \in N,$ then this equation has only the solution $(x, y, z) = (1, 1, 2)$, provided that $m \equiv 3 \pmod 6$ or $m \le 20$. JuanliSu and Xiaoxue Li [2], proved that if $m > 90$ and $3|m$, then the equation $(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z$ has only the positive integer solution $(x, y, z) = (1, 1, 2)$ by utilizing some results on the subsistence of primitive divisors of Lucas numbers. In this context one may refer [3-8]. In this paper, the exponential equation $(3m^2 + 3)^x + (7m^2 + 1)^y = z^2$ is discussed for the existence of solutions in integers or this equation has no solution for different choices of m.

II. PROCESS OF TESTING THE HYPOTHESIS

The exponential equation for searching out solutions exists or not in integer is taken as

$$(3m^2 + 3)^x + (7m^2 + 1)^y = z^2 \qquad (1)$$

where $m \in Z$

Investigate the hypothesis of (1) for the ensuing three cases.

(i) $x+y=1$
(ii) $x+y=2$
(iii) $x+y=3$

The possibilities of the above three cases are exemplified below.

(i) $x = 0, y = 1 \quad x = 1, y = 0$
(ii) $x=0, y=2$, $x = 1, y = 1$ and $x = 0, y = 2$
(iii) $x=0, y=3$, $x = 1, y = 2$, $x = 2, y = 1$ and $x = 3, y = 0$

The detailed explanation of analyzing and all the above nine cases are given below.

Case (i): Suppose $x = 0,\ y = 1$

These two values of x and y direct (1) to the second-degree equation in two variables as follows

$$z^2 = 7m^2 + 2 \tag{2}$$

The very least roots of (2) are monitored by

$$z_0 = 3\ ,\ m_0 = 1$$

The other possible roots of (2) are located through the equivalent Pellian equation

$$z^2 = 7m^2 + 1 \tag{3}$$

The lowest positive values of the couple (z, m) fulfilling (3) is estimated by

$$\widetilde{z_0} = 8,\ \widetilde{m_0} = 3.$$

The n^{th} solution to (3) is generalized by the equations

$$\widetilde{z_n} = \frac{1}{2}\left[\left(8+3\sqrt{7}\right)^{n+1} + \left(8-3\sqrt{7}\right)^{n+1}\right]$$
$$\widetilde{m_n} = \frac{1}{2\sqrt{7}}\left[\left(8+3\sqrt{7}\right)^{n+1} - \left(8-3\sqrt{7}\right)^{n+1}\right],$$

$n = 0,1,2,3 \ldots \ldots$

The sequence of solutions for (3) belonging to the set Z of all integers is specified as earlier by the formulae

$$z_{n+1} = z_0\widetilde{z_n} + 7m_0\widetilde{m_n}$$
$$m_{n+1} = m_0\widetilde{z_n} + z_0\widetilde{m_n}$$

which means that

$$z_{n+1} = 3\frac{A}{2} + \frac{B}{2\sqrt{7}} \tag{4}$$

$$m_{n+1} = \frac{A}{2} + \frac{3B}{2\sqrt{7}} \tag{5}$$

where

$$A = \left(8+3\sqrt{7}\right)^{n+1} + \left(8-3\sqrt{7}\right)^{n+1}$$
$$B = \left(8+3\sqrt{7}\right)^{n+1} - \left(8-3\sqrt{7}\right)^{n+1},$$

$n = 0,1,2,\ldots.$

Hence the numerous solutions to (1) for the preferred choices of m as given in (5) are scrutinized by the subsequent equation

$$(x, y, z) = (0, 1,\ z_{n+1})$$

The deduction of different types of equations for a variety of values of m and their corresponding solutions by utilizing (4) and (5) are tabularised in Table (I).

Case (ii): Let $x = 1,\ y = 0$

These two preferences of x and y deviate (1) to the equation occupying z and m as follows

$$z^2 = 3m^2 + 4 \tag{6}$$

The extremely least roots of (6) are checked manually and it is indicated by

$$z_0 = 4\ ,\ m_0 = 2$$

The alternative feasible solutions of (6) are sited through the indistinguishable equation

$$z^2 = 3m^2 + 1 \tag{7}$$

The pair (z, m) flattering (3) is computed by

$$\widetilde{z_0} = 2,\ \widetilde{m_0} = 1.$$

The common solutions to (7) is communicated through the following equations for the convenience that

$$\widetilde{z_n} = \frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right]$$
$$\widetilde{m_n} = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right]$$

Exploiting the formulae as there in case (i), the array of solutions for (6) existing in the set Z of all integers is concluded by

$$z_{n+1} = 2C + 3D \tag{8}$$

$$m_{n+1} = C + \frac{2D}{\sqrt{3}} \tag{9}$$

where

$$C = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}$$
$$D = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1},$$

$n = 0,1,2,....$

Hence, the enormous solutions to (1) for the favourable choices of m as furnished in (9) are inspected by the succeeding equation

$$(x,y,z) = (1,0, z_{n+1})$$

The implication of nature of equations for a selection of values of m and their resultant solutions by operating (8) and (9) are tabularised in Table (II)

Case (iii): Consider $x = 1,\ y = 1$

Under these assumptions, the parallel equation of (1) is derived by

$$z^2 = 10m^2 + 4 \qquad (10)$$

The extremely first solutions to (10) are supervised by the character

$$z_0 = 38\ ,\ m_0 = 12.$$

All other possible solutions of (10) are perceived through the equation

$$z^2 = 10m^2 + 1 \qquad (11)$$

Following the analogous procedure as mentioned in case (i) and case (ii) by employing the primary solution $(\widetilde{z_0}, \widetilde{m_0}) = (19,6)$ and also the general solutions to (11), the cycle of solutions for (10) to be the members of the set Z of all integers is demonstrated by the succeeding equations

$$z_{n+1} = 19E + \frac{6}{\sqrt{10}}F \qquad (12)$$
$$m_{n+1} = 6E + \frac{19F}{\sqrt{10}} \qquad (13)$$

where

$$E = \left(19 + 6\sqrt{10}\right)^{n+1} + \left(19 - 6\sqrt{10}\right)^{n+1}$$
$$F = \left(19+6\sqrt{10}\right)^{n+1} - \left(19-6\sqrt{10}\right)^{n+1},$$

$n = 0,1,2,\ldots.$

Hence, the plentiful solutions to (1) for the elected choices of m as given in (13) are displayed by the next equation

$$(x,y,z) = (1,1, z_{n+1})$$

The inference of altered equations for the range of values of m and their equivalent solutions by consuming (12) and (13) are tabularised in Table(III)

Case (iv): $x = 0,\ y = 2$

These supposition simplifies (1) to the fourth degree equation with two variables as declared below

$$\left(7m^2+1\right)^2 = z^2 - 1 \qquad (14)$$

Since, the square of an integer minus one can never be a square, the above postulation is always not possible.
Consequently, the equation (14) and hence equation (1) does not possess a solution.

Case (v): $x = 2,\ y = 0$

These hypotheses make things easier to (1) as the succeeding equation involving two variables with degree four

$$(3m^2 + 3)^2 = z^2 - 1 \qquad (15)$$

According to explanation given in case (iv), the statement produced above does not hold.
As a result, the equation (15) and hence equation (1) does not have a solution.

Case (vi): $x = 3,\ y = 0$

Repercussion of these selections reduces (1) to the equation consisting two unknowns with degree six as

$$(3m^2 + 3)^3 + 1 = z^2 \qquad (16)$$

which can be modified by

$$s^3 = z^2 - 1 \qquad (17)$$

where

$$s = 3m^2 + 3 \qquad (18)$$

The most promising solution to (17) is pointed out by $(s, z) = (2,\ 3)$. Match up to the value of s with (18) established that

$$3m^2+3=2$$
$$\Rightarrow 3m^2=-1$$

which is absurd for integer values of m.

The conclusion of the problem is equation (16) and hence equation (1) does not have a solution when $m \in Z$.

Note:

If $m \in C$, the set of all complex numbers, then the one and only one solution in integer to (1) is $(x, y, z) = (3,0,3)$

Case (vii): $x = 1, y = 2$

Influence of these options reduces (1) to the equation consisting two unknowns with degree four as

$$49m^4+17m^2+4=z^2 \qquad (19)$$

which can be reshuffled by

$(14t + 17)(14t - 17) = (14z + 28)(14z - 28)$ where

$$t = 7m^2 + \frac{17}{14} \qquad (20)$$

Resolving the equivalent structure of the ensuing proportion of the equation (20)

$$\frac{(14t+17)}{(14z+28)}=\frac{(14t-17)}{(14z-28)}=\frac{a}{b},\ b\neq 0 \qquad (21)$$

make available with the solutions as

$$t=\frac{17a^2+17b^2-56ab}{14(a^2-b^2)}\Rightarrow m^2=\frac{17b^2-28ab}{14(a^2-b^2)} \qquad (22)$$

$$z=\frac{17ab-14a^2-14b^2}{7(a^2-b^2)} \qquad (23)$$

The only prospect of integer for m as mentioned in (22) is $m = 0$ for the next two alternatives of a and b

(i) $a = 17k$ and $b = 28k$, (ii) $a = k$ and $b = 0$,

For all other values of a and b, it is noted that $m \notin Z$.The above two selections of a and b for all values of k certify (23) that $z = 2$ and $z = -2$ respectively. Hence, the specified equation (1) is reduced into the equation with two unknowns as

$3^x + 1 = z^2$ which has only two solutions $(x, z) = (1,\ 2)$ and $(x, z) = (1, -2)$

Remark:

Instead of taking the fraction (21) as any other possible ratios also endow with the same solutions $(x, z) = (1,\ 2)$ and $(x, z) = (1, -2)$ to the equation $3^x + 1 = z^2$

Case (viii): $x = 2, y = 1$

This speculation simplifies (1) to the fourth-degree equation with two variables as affirmed below

$$9m^4+25m^2+10=z^2 \qquad (24)$$

Modify (24) into factors as given below

$$(6u + 29)(6u - 29) = (6z + 24)(6z - 24)$$

where

$$u = 3m^2 + \frac{25}{6} \qquad (25)$$

The systematic procedure as in case (vii) for the proportion $\frac{6u+29}{6z+24} = \frac{6u-29}{6z-24} = \frac{c}{d}, d \neq 0$ offers the subsequent values of u and z

$$u = \frac{29c^2+29d^2-48cd}{6(c^2-d^2)} \Rightarrow m^2 = \frac{4c^2+54d^2-48cd}{18(c^2-d^2)} \qquad (26)$$

$$z=\frac{58cd-24c^2-24d^2}{6(c^2-d^2)} \qquad (27)$$

None of the values of a and b in (26) and (27) generate m and z in integers.

Hence, there does not exit a solution to (1) in integers.

Remark:

As a replacement of all other fraction for (25) also provide no solutions in integers to (1)

III. TABLES

Table (I)

n	m_{n+1}	Reduced form of (1)	(x, y, z_{n+1})
1	271	$(220326)^x + (514088)^y = z^2$	(0,1,717)
2	4319	$(55961286)^x + (130576328)^y = z^2$	(0,1,11427)
3	68833	$(1.421394567 \times 10^{10})^x + (3.316587322 \times 10^{10})^y = z^2$	(0,1,182115)
4	1097009	$(3.610286238 \times 10^{12})^x + (8.424001223 \times 10^{12})^y = z^2$	(0,1,2902413)
5	17483311	$(9.169984906 \times 10^{14})^x + (2.139663145 \times 10^{14})^y = z^2$	(0,1,46256493)

Table (II)

n	m_{n+1}	Reduced form of (1)	(x, y, z_{n+1})
1	30	$(2703)^x + (6301)^y = z^2$	(1,0,52)
2	112	$(37635)^x + (87809)^y = z^2$	(1,0,194)
3	418	$(524175)^x + (1223069)^y = z^2$	(1,0,724)
4	1560	$(7300803)^x + (17035201)^y = z^2$	(1,0,2702)
5	5822	$(101687055)^x + (237269789)^y = z^2$	(1,0,10084)

Table (III)

n	m_{n+1}	Reduced form of (1)	(x, y, z_{n+1})
1	17316	$(899531571)^x + (2098906993)^y = z^2$	(1,1,54758)
2	657552	$(1.297123898 \times 10^{12})^x + (3.026622429 \times 10^{12})^y = z^2$	(1,1,2079362)
3	24969660	$(1.870451762 \times 10^{15})^x + (4.364387444 \times 10^{15})^y = z^2$	(1,1,78960998)
4	948189528	$(2.697190143 \times 10^{18})^x + (6.293443667 \times 10^{18})^y = z^2$	(1,1,2998438562)
5	360062324×10^{10}	$(3.889346315 \times 10^{37})^x + (9.075141401 \times 10^{37})^y = z^2$	$(1,1,1.138617043 \times 10^{19})$

IV. CONCLUSION

In this text, the Exponential Diophantine equation $(3m^2+3)^x + (7m^2+1)^y = z^2$, $m \in Z$ in three variables has numerous integer solutions for particular choices m and the equation has no solution for some other alternatives of m is scrutinized. The conclusion is one can search the solutions to similar type of exponential Diophantine equation having higher powers of m and z than two by using various concept of theory of numbers.

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