Research Article

Radio Mean Labeling Of Paths And Its Total Graph

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***Article History**: Received: 11 January 2021; Accepted: 27 February 2021; Published online: 5 April 2021*

Abstract— A graph labeling problem is an assignment of labels to the vertices or edges (or both) of a graph *G* satisfying some mathematical condition. Radio Mean Labeling, a vertex-labeling of graphs with non-negative integers has a significant application in the study of problems related to radio channel assignment. The maximum label used in a radio mean labeling is called its span, and the lowest possible span of a radio mean labeling is called the radio mean number of a graph. In this paper, we obtain the radio mean number of paths and total graph of paths.

Keywords— Channel assignment problem; Graph theory; Path graph; Radio Mean Labeling; Total graph of a graph

I. Introduction

For basic graph theory terminology, we refer [16]. The basic principle of a Radio communication network is transmission and reception of radio signals. Each radio station is assigned a channel number or frequency; transmitter sends signals; a receiver then picks it up and translates it to the sounds heard through the radio. However, the reception will be degraded by the unnecessary interference by transmitters of closely related channel number, if any. Hence the channel assignment problem is to assign radio channels to transmitters with minimum span in such a way that it minimizes interference between radio stations that are in the same neighborhood. This problem of Radio channel assignment can be converted into a Graph theoretic problem as follows: The radio network can be considered as a graph in which vertices corresponds to transmitter locations and two vertices are adjacent if the locations of the radio stations corresponding to these vertices are close. The main objective is to label vertices of this graph with minimum span where the labels given to the vertices determine the channel on which it transmits [15]. Chartrand et al. converted this problem to a vertex labeling problem as follows: For a connected graph *G*, radio labeling was defined as a one-to-one function φ from $V(G)$ to $\mathbb{Z}^+$, the set of all positive integers where $d(u,v)+|\varphi(u)-\varphi(v)| \geq 1 + diam(G), \forall\, u,v \in V(G)$. Authors in [17] studied the Radio labeling of Strong product of K_3 and P_n. Graphs for which the largest label used is same as the order of the graph are called radio graceful. In [10], [11] the authors study this concept of radio gracefulness of a graph.

The idea of radio mean labeling of graphs was conceived the paper [5], published in the year 2015. The radio mean labeling of a connected graph *G* was defined as an injective function

$f: V(G) \rightarrow \mathbb{Z}^+$ where

$$\left\lceil \frac{f(u)+f(v)}{2} \right\rceil + d(u,v) \geq diam(G)+1, \forall u,v \in V(G).$$

The radio mean number of *f* or *rmn(f)* is the maximum integer assigned to any $v \in V(G)$ under this mapping *f*. Further the radio mean number of *G*, denoted by *rmn(G)* is the smallest value of *rmn(f)* taken over all radio mean labelings *f* of *G*. It is obvious by the definition that $rmn(G) \geq |V(G)|$. If $rmn(G) = |V(G)|$, then *G* is called a radio mean graceful graph [3].

In [5, 6, 7, 8, 9], Ponraj, R., S. Sathish Narayanan and R. Kala have investigated the radio mean labeling of many classes of graphs including graphs with maximum distance between distinct pairs of vertices either two or three. The radio mean number of Triangular Ladder graph, corona P_n with $\square K_2$, corona K_n with $\square K_2$ and corona W_n with $\square K_2$ are obtained in [13] by Sunitha, K., C. David Raj and A. Subramanian and that of subdivision graph of complete graphs, Mongolian tent graph, subdivision of Friendship graphs, and diamond graphs in [10] by Lavanya Y. and K. N. Meera. Smitha, KM Baby and K. Thirusangu studied the radio mean labeling of corona K_m with K_n, corona W_m with $\overline{K_n}$, corona star S_m with $\overline{K_n}$ and corona Helm H_m with $\overline{K_n}$ in [12]. In [11], Raj, Deva and Brindha studied the radio mean labeling of Degree Splitting graph of P_n, $K_{1,n}$ and corona P_n with K_1. In [4] the mean in the definition is replaced by geometric mean and radio geometric graceful graphs are studied.

The total graph of a graph G is a graph whose vertex set is $V(G) \cup E(G)$, and two vertices are adjacent in the total graph if and only if they are adjacent or incident in G. We denote total graph of G by $T(G)$.

II. RADIO MEAN LABELING OF PATHS

Consider the Path P_n on n vertices $v_1, v_2, \cdots, v_n$. Note that for path graph P_n, diam(P_n) = n − 1. A labeling $f: V(P_n) \to \mathbb{Z}^+$ is a Radio mean labeling of P_n, if f is injective and satisfies the condition:

$$\lceil \frac{f(v_i) + f(v_j)}{2} \rceil + d(v_i, v_j) \geq n, \text{ for all } v_i, v_j \in V(P_n). (1)$$

A. Case I: n = 2, 3

Define $f: V(P_n) \to \mathbb{Z}^+$ defined by $f(v_i) = i$, where $v_i \in V(P_n)$ for each n ∈ {2, 3}. It can easily be seen that paths P_2, P_3 admit Radio mean Graceful labeling under the injective mapping f.

B. Case II: $n \geq 4$

Theorem: II-B.1

The Path P_n , n = 4, 5, 6 admits Radio mean labeling with $rmn(P_n) = 2n - 4$.

Proof. For n = 4, 5, 6, define a function $f: V(P_n) \to \mathbb{Z}^+$ by $f(v_1) = n - 3$, $f(v_2) = 2n - 4$, $f(v_3) = n - 2$, $f(v_i) = 2n - i - 1 : 4 \leq i \leq n$. Clearly, f is an injective function. We shall now show that the function f satisfies *(1)*. Consider any pair $(v_i , v_j) \in V(P_n)$.

$$\begin{aligned}
\lceil \frac{f(v_i) + f(v_j)}{2} \rceil + d(v_i, v_j) &\geq \lceil \frac{f(v_1) + f(v_3)}{2} \rceil + d(v_1, v_3) \\
&\geq \lceil \frac{n-3+n-2}{2} \rceil + 2 \\
&\geq n - 2 + 2 \\
&\geq n
\end{aligned}$$

Under this labeling f, every pair of vertices in P_n, n = 4, 5, 6 satisfies radio mean condition and hence f is a radio mean labeling of P_n. The maximum integer used as a label under this function f is 2n − 4 and so $rmn(f)$ = 2n − 4. When n = 4, f is a graceful labeling and so $rmn(P_n)$ = 2n − 4 = n and for n = 5, 6, $rmn(P_n) \leq$ 2n – 4.

It is clear from the definition of f that any radio mean labeling of P_n, n = 5, 6 whose range consists of only integers greater than n − 3 has a span greater than that of f. Let us now consider any radio mean labeling h of P_n , n = 5, 6, $h : V(P_n) \to \{n-4, n-3, n-2, \cdots\}$. Then it follows from the Radio mean condition that any vertices receiving labels n − 4 and n − 3 are at least n − 2 distance apart, any vertices receiving labels n − 3 and n − 2 are at least n – 3 distance apart and any vertices receiving labels n−4 and n−2 are at least n − 2 distance apart. A labeling of P_n using integers {n − 4, n − 3, n − 2} satisfying the above constraints on distance is not feasible. This indicates that not all of the integers {n−4, n−3, n−2} are in the Range of h and so span of h is greater than 2n−5. In other words, $rmn(h) \geq$ 2n − 4. Thus, we can show that any radio mean labeling of P_n, n = 5, 6 whose range set consists of integers less than n − 3 has span greater than or equal to 2n − 4. Hence, $rmn(P_n)$ = 2n − 4, n = 5, 6. Hence, for path P_n, n = 4, 5, 6 we have $rmn(P_n)$ = 2n − 4.

Lemma: II-B.1

Suppose n is any integer which belongs to an interval of the form : $[4+S_k, 6+S_k+k]$ where $S_k = 3+4+5+\cdots+(3+k-1)$ and $k = 1, 2, 3, \cdots$. Then there exists a Radio mean labeling of P_n, $n \geq 7$ with radio mean number, $rmn(P_n) = 2n-k-4$.

Proof. Suppose n ∈ $[4+S_k, 6+S_k+k]$ where $S_k = 3+4+5+ \cdots + (3 + k - 1)$ and k = 1, 2, 3, ⋯. Let us define a function f,

$f : V(P_n) \longrightarrow \mathbb{Z}^+$ using indices $d_1, d_2, \cdots, d_{k+2}$ where $d_1 = 1$, and $d_i = d_{i-1} + n - f(v_{d_{i-1}}) - 1; i = 2, 3, \cdots, k+2$.

The function f is given by

$$\begin{aligned}
&(i) \quad f(v_1) = f(v_{d_1}) = n - k - 3 \\
&(ii) \quad f(v_{d_i}) = f(v_{d_1}) + i - 1 = n - k + i - 4, \text{ if } i = 2, 3, \cdots, k+2 \\
&(iii) \quad f(v_i) = f(v_{d_{k+2}}) + n - i + 1 = 2n - i - 1, \text{ if } (d_{k+2}) + 1 \leq i \leq n \\
&(iv) \quad f(v_i) = f(v_{(d_l+1)}) + d_l - i, \text{ if } i \in (d_{l-1}, d_l) \text{ where } l \in \{k+2, k+1, k, \cdots, 3, 2\}
\end{aligned}$$

Clearly, f is an injective function. We shall now show that the f satisfies *(1)*. Following are the different cases we consider:

(1) Consider the pair (v_i, v_j) where $v_i, v_j \in \{v_{d_1}, v_{d_2}, \cdots, v_{d_{k+2}}\}$

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j)}{2}\rceil + d(v_i,v_j) &\geq \lceil\frac{f(v_{d_1})+f(v_{d_2})}{2}\rceil \\ &\quad + d(v_{d_1}, v_{d_2}) \\ &\geq f(v_{d_1}) + 1 + d_2 - 1 \\ &\geq n\end{aligned}$$

(2) Consider the pair (v_i, v_j) where $(d_{k+2}) + 1 \leq i$ and $j \leq n, i \neq j$.

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j)}{2}\rceil + d(v_i,v_j) &\geq 2n - 1 + \frac{i+j}{2} + \\ &\quad d(v_i, v_j) \\ &\geq n\end{aligned}$$

(3) Consider the pair (v_i, v_j) where $i \in \{d_1, d_2, \cdots, d_{k+2}\}$ and $(d_{k+2}) + 1 \leq j \leq n$.

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j)}{2}\rceil + d(v_i,v_j) &\geq \frac{3}{2}n + \frac{j}{2} - \frac{k}{2} - 3 \\ &\geq n\end{aligned}$$

(4) Consider the pair (v_i, v_j) where $v_i, v_j \in V(P_n) \setminus (\{v_{d_1}, v_{d_2}, \cdots, v_{d_{k+2}}\} \bigcup \{v_{(d_{k+2})+1}, v_{(d_{k+2})+2}, \cdots, v_n\})$

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j)}{2}\rceil + d(v_i,v_j) &\geq f(v_{d_{k+2}-1}) + d(v_i, v_j) \\ &\geq 2n - d_{k+2} - 2 \\ &\quad + d(v_i, v_j) \\ &\geq n\end{aligned}$$

(5) Consider the pair (v_i, v_j) where $v_i \in \{v_{d_1}, \cdots, v_{d_{k+2}}\}$ and $v_j \in V(P_n) \setminus (\{v_{d_1}, \cdots, v_{d_{k+2}}\} \bigcup \{v_{(d_{k+2})+1}, \cdots, v_n\})$

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j)}{2}\rceil + d(v_i,v_j) &\geq \lceil\frac{n - 3 + 2n - d_{k+2}}{2}\rceil \\ &\quad + 1 \\ &\geq \lceil\frac{3n - d_{k+2} - 3)}{2}\rceil + 1 \\ &\geq n\end{aligned}$$

(6) Consider the pair (v_i, v_j) where $v_i \in \{v_{(d_{k+2})+1}, v_{(d_{k+2})+2}, \cdots, v_n\}$ and $v_j \in V(P_n) \setminus (\{v_{d_1}, \cdots, v_{d_{k+2}}\} \bigcup \{v_{(d_{k+2})+1}, \cdots, v_n\})$

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j)}{2}\rceil + d(v_i,v_j) &\geq \lceil\frac{2f(v_{(d_{k+2})+1}) + 1}{2}\rceil \\ &\quad + 2 \\ &\geq f(v_{(d_{k+2})+1}) + 3 \\ &\geq 2n - (d_{k+2} + 1) + 2 \\ &\geq n\end{aligned}$$

From all the above cases it follows that f satisfies Radio mean condition for all pairs of vertices of P_n, $n \geq 7$. The maximum number assigned to any vertex of P_n under this mapping is $2n - k - 4$. Hence the radio mean number of f, $rmn(f) = 2n - k - 4$.

Theorem: II-B.2

For any Path P_n, $n \in [4 + S_k, 6 + S_k + k]$ where $S_k = 3 + 4 + 5 + \cdots + (3+k-1)$ and $k = 1, 2, 3, \cdots$ as in previous lemma the radio mean number of P_n is $2n - k - 4$.

Proof. It is clear from the function f defined in previous lemma that any radio mean labeling of P_n whose range contains only integers greater than n – k − 3 has a span greater than that of f. Now we shall investigate about the span of radio mean labelings of P_n whose range consists of integers less than n − k − 3. It is observed that under the labeling f,

$$d(v_{d_1}, v_{d_2})+d(v_{d_2}, v_{d_3})+\cdots+d(v_{d_{k+1}}, v_{d_{k+2}})+1 \leq diam(P_n)$$

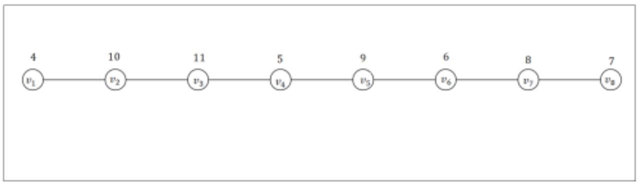
i.e $(k+2)+(k+1)+k+\cdots+2+1 = 4+S_k-1 \leq n-1$

Consider any radio mean labeling h of P_n, $h : V(P_n) \rightarrow \{n-k-4, n-k-3, n-k-2, \cdots\}$. Then it follows from Radio mean condition that any vertices receiving labels n − k − 4 and n − k − 3 must be at least k + 3 distance apart, any vertices receiving labels n − k − 3 and n − k − 2 must be at least k + 2 distance apart, any vertices receiving labels n − k − 2 and n − k − 1 must be at least k + 1 distance apart, · · ·, any vertices receiving labels (n − k − 3) + k and (n − k − 3) + (k + 1) must be at least 2 distance apart. If {(n − k − 4),(n − k − 3),(n − k − 2), · · ·,(n − k − 3) + k,(n− k − 3) + (k + 1)} are in the Range of h, then we must have

$$(k+3) + (k+2) + (k+1) \cdots + 3 + 2 + 1 \leq n - 1$$

That implies $(k + 3) + 4 + S_k - 1 \leq n - 1$, a contradiction since $4 + S_k \leq n \leq 6 + S_k + k$. This means not all of the integers

Fig:1 Path on 8 vertices



{(n − k − 4),(n − k − 3),(n − k − 2), · · ·,(n − k −3) +k,(n−k −3) + (k + 1)} are in the Range of h which implies that the maximum integer assigned to any vertex of P_n under the mapping h, $rmn(h) > 2n - k - 5$. In other words, $rmn(h) \geq 2n - k - 4$.

Similarly, we can show that any radio mean labeling of P_n whose range set includes integers less than n − k − 3 has span greater than or equal to 2n − k − 4. Therefore, for path P_n, $4 + S_k \leq n \leq 6 + S_k + k$, $rmn(P_n) = 2n - k - 4$.

III. RADIO MEAN LABELING OF TOTAL GRAPH OF PATHS

The total graph of a path P_n is a graph whose vertex set consists of the vertices and edges of P_n and two vertices are adjacent in $T(P_n)$ if and only if their corresponding elements are either adjacent or incident in P_n. Let P_n be a path of n vertices namely $v_1, v_2, \cdots, v_n$ and edges $e_1, e_2, \cdots, e_n$. Then the total graph of P_n denoted by $T(P_n)$ is the graph with vertex set

$$\{v_1, v_2, \cdots, v_n\} \cup \{v_1', v_2', \cdots, v_{n-1}'\}$$

Where v_i' is the vertex corresponding to edge e_i of P_n and two vertices of $T(P_n)$ are adjacent if their corresponding elements are adjacent in P_n. It is obvious that the diameter of $T(P_n)$ is equal to n − 1. A labeling $f: V(T(P_n)) \rightarrow \mathbb{Z}^+$ is a radio mean labeling of $T(P_n)$, if $T(P_n)$ is injective and satisfies the condition

$$\lceil \frac{f(u)+f(v)}{2} \rceil + d(u,v) \geq n, \text{for all } u,v \in V(\mathcal{T}(P_n)). \quad (2)$$

A. Case I: n = 2, 3, 4

Theorem: III-A.1

The total graph $T(P_n)$ of Path P_n, n = 2, 3, 4 admits Radio mean graceful labeling.

Proof. Define $f: V(T(P_n)) \rightarrow \mathbb{Z}^+$ as follows.

$$f(v_1) = 2, \; f(v_n) = 1,$$

$$f(v_i) = 2+2(i-1) : 2 \leq i < n,$$
$$f(v_i') = 3+2(i-1) : 1 \leq i \leq n-1$$

Clearly, f is an injective function. We shall now show that the f satisfies *(2)*. Following are the different cases we consider:

(1) Consider the pair $(v_i, v_j^{'})$ where $v_i, v_j^{'} \in V(\mathcal{T}(P_2))$

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j^{'})}{2}\rceil + d(v_i, v_j^{'}) &\geq \lceil\frac{f(v_1)+f(v_2)}{2}\rceil + \\ &\quad d(v_1, v_2) \\ &\geq \lceil\frac{1+2}{2}\rceil + 1 = 3 \\ &> 2\end{aligned}$$

(2) If $v_i, v_j \in V(\mathcal{T}(P_n)), n = 3, 4$

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j)}{2}\rceil + d(v_i, v_j) &\geq \lceil\frac{f(v_1)+f(v_n)}{2}\rceil + \\ &\quad d(v_1, v_n) \\ &\geq \lceil\frac{2+1}{2}\rceil + n - 1 \\ &\geq 2 + n - 1 \\ &> n\end{aligned}$$

(3) If $v_i^{'}, v_j^{'} \in V(\mathcal{T}(P_n)), n = 3, 4$

$$\begin{aligned}\lceil\frac{f(v_i^{'})+f(v_j^{'})}{2}\rceil + d(v_i^{'}, v_j^{'}) &\geq \lceil\frac{f(v_1^{'})+f(v_2^{'})}{2}\rceil + \\ &\quad d(v_1^{'}, v_2^{'}) \\ &\geq \lceil\frac{3+5}{2}\rceil + 1 = 5 \\ &> n\end{aligned}$$

(4) If $v_i, v_j^{'} \in V(\mathcal{T}(P_n)), n = 3, 4$

$$\begin{aligned}\lceil\frac{f(v_i)+f(v_j^{'})}{2}\rceil + d(v_i, v_j^{'}) &\geq \lceil\frac{f(v_1)+f(v_1^{'})}{2}\rceil + \\ &\quad d(v_1, v_1^{'}) \\ &\geq \lceil\frac{2+3}{2}\rceil + 1 = 4 \\ &\geq n\end{aligned}$$

From all the above cases it follows that *f* satisfies Radio mean condition for all pairs of vertices of $T(P_n)$, n = 2, 3, 4 and the largest integer utilized in this labeling is n and so *rmn(f)* = n and *f* is a graceful radio mean labeling.

B. Case II: n = 5, 6, 7

Theorem: *III*-B.1

The total graph $T(P_n)$ of path P_n, n = 5, 6, 7 admits Radio mean labeling with $rmn(T(P_n)) = 3n - 6$.

Proof. Define $f: V(T(P_n)) \to \mathbb{Z}^+$ using indices $d_0 = n$, $d_1 = 1$, $d_2 = 3$ as follows:

$$\begin{aligned}f(v_1) &= n-3, \\ f(v_2) &= 3n-7, \\ f(v_i) &= n+2i-7 : 3 \leq i \leq n-1, \\ f(v_n) &= n-4, \\ f(v_1^{'}) &= 3n-6, \\ f(v_i^{'}) &= n+2i-6 : 2 \leq i \leq n-1.\end{aligned}$$

Clearly, *f* is an injective function. We can also verify as in earlier case that *f* satisfies *(2)* and so it follows that *f* is a radio mean labeling. Since maximum integer assigned to any vertex under this labeling is 3n − 6, *rmn(f)* = 3n − 6. When n = 5, *f* is a graceful labeling and so $rmn(T(P_n))$ = 3n − 6, n = 5. And $rmn(T(P_n)) \leq 3n - 6$, n = 6, 7. It is clear from the function f that any radio mean labeling of $T(P_n)$, n = 6, 7 whose range consists of only integers greater than n − 4 has a span greater than that of *f*.

Let us now consider any radio mean labeling h of $T(P_n)$, n = 6, 7, $h : V(T(P_n)) \rightarrow \{n-5, n-4, n-3, \cdots\}$. Then it follows from Radio mean condition that any vertices receiving labels
n − 5 and n − 4 must be at least 4 distance apart, any vertices receiving labels n − 4 and n − 3 must be at least 3 distance apart, any vertices receiving labels n − 3 and n − 2 must be at least 2 distance apart and any vertices receiving labels n − 2 and n − 1 must be 1 distance apart. It can easily be seen that it is not feasible to label the vertices of $T(P_n)$ using integers from n − 5 to n − 1 satisfying the above constraints on distance. This indicates that not all of these integers are in the Range of h which in turn says that the maximum integer assigned to any vertex under this labeling, rmn(h) > 3n − 7. In other words, rmn(h) ≥ 3n − 6. Thus, we can show that any radio mean labeling of $T(P_n)$, n = 6, 7 whose range set consists of integers less than n − 4 has span greater than or equal to 3n − 6. Hence, $rmn(T(P_n))$ = 3n − 6, n = 6, 7.

Therefore, for the total graph $T(P_n)$ of path P_n, n ∈ {5, 6, 7}, $rmn(T(P_n))$ = 3n − 6.

C. Case III: $n \geq 8$

Theorem: III-C.1

The total graph $T(P_n)$, $n \in [8, 12]$ admits Radio mean labeling with $rmn(T(P_n)) = 3n - 7$.

Proof. Let n ∊ [8, 12]. Define $f: V(T(P_n)) \rightarrow \mathbb{Z}^+$ using indices $d_0 = n$, $d_1 = 1$, $d_2 = 3$, $d_3 = 5$, $d_4 = 5$ as follows:

$$\begin{aligned}
f(v_1) &= n-4,\\
f(v_i) &= 3n+2i-16 : i = 2,3,4,\\
f(v_5) &= n-2,\\
f(v_i) &= n+2i-12 : 6 \leq i \leq n-1,\\
f(v_n) &= n-5,\\
f(v_i') &= 3n+2i-13 : i = 1,2,\\
f(v_3') &= n-3,\\
f(v_4') &= 3n-7,\\
f(v_5') &= n-1,\\
f(v_i') &= n+2i-11 : 6 \leq i \leq n-1.
\end{aligned}$$

Clearly, f is an injective function. We can also verify that f satisfies *(2)* and so it follows that f is a radio mean labeling. Since maximum integer assigned to any vertex under this labeling is 3n−7, $rmn(f)$ = 3n−7. And $rmn(T(P_n)) \leq$ 3n − 7, n ∊ [8, 12]. It is clear from the function f that any radio mean labeling of $T(P_n)$, n ∊ [8, 12] whose range consists of only integers greater than n − 5 has a span greater than that of f. Let us now consider any radio mean labeling h of $T(P_n)$, n ∊ [8, 12], $h : V(T(P_n)) \rightarrow \{n-6, n-5, n-4, \cdots\}$. Then it follows from Radio mean condition that any vertices receiving labels n − 6 and n − 5 must be at least 5 distance apart, any vertices receiving labels n − 5 and n − 4 must be at least 4 distance apart, any vertices receiving labels n − 4 and n − 3 must be at least 3 distance apart and any vertices receiving labels n − 3 and n − 2 must be 2 distance apart. It can easily be seen that it is not feasible to label the vertices of $T(P_n)$ using integers n − 6 to n − 2 satisfying the above constraints on distance. This indicates that not all of these integers in the range of h and hence the maximum integer assigned to any vertex under this labeling, $rmn(h)$ > 3n − 8. In other words, $rmn(h) \geq$ 3n − 7. Thus, we can show that any radio mean labeling of $T(P_n)$, n ∊ [8, 12] whose range set consists of integers less than n − 5 has span greater than or equal to 3n − 7. Hence, $rmn(T(P_n))$ = 3n − 7, n ∊ [8, 12].

Lemma: III-C.1

Suppose n is any integer which belongs to an interval of the form : $[8+S_k, 12+S_k+k]$ where $S_k = 5+6+7+\cdots+(5+k-1)$ and $k = 1, 2, 3, \cdots$. Then there exists a Radio mean labeling of $T(P_n)$, $n \geq 13$ with radio mean number $3n-k-7$.

Proof. Let us define a function $f: V(T(P_n)) \rightarrow \mathbb{Z}^+$ using indices $d_0, d_1, d_2, \cdots$ where

$(i) d_0 = n,$

$(ii) d_1 = 1,$

$(iii) d_2 = n - [f(v_{d_1}) + 1],$

$(iv)\ d_i = d_{i-1} + n - [f(v'_{d_{i-1}}) + 1]$, for i is odd, $i \leq k+4$,

$(v)\ d_i = d_{i-1} + n - [f(v_{d_{i-1}}) + 1] - 1$, for i is even, $i \leq k+4$.

The function f is given by

$$
\begin{array}{lll}
(i) & f(v_n) = & f(v_{d_0}) = n-k-5 \\
(ii) & f(v_1) = & f(v_n)+1 = n-k-4 \\
(iii) & f(v_{d_i}) = & f(v_{d_1})+i-1 = n-k+i-5 \\
 & & \text{: if } i \text{ is odd }, i \leq k+4 \\
(iv) & f(v'_{d_i}) = & f(v_{d_1})+i-1 = n-k+i-5 \\
 & & \text{: if } i \text{ is even }, i \leq k+4 \\
(v) & \text{if k+4 is odd:} & \\
 & f(v'_i) = & f(v'_{d_{k+3}})+2(i-d_{k+3})\text{where} \\
 & & d_{k+3}+1 \leq i \leq n-1 \\
 & f(v_i) = & f(v_{d_{k+4}})+2(i-d_{k+4})\text{where} \\
 & & d_{k+4}+1 \leq i \leq n-1 \\
 & \text{if k+4 is even:} & \\
 & f(v'_i) = & f(v'_{d_{k+4}})+2(i-d_{k+4})\text{where} \\
 & & d_{k+4}+1 \leq i \leq n-1 \\
 & f(v_i) = & f(v_{d_{k+3}})+2(i-d_{k+3})\text{where} \\
 & & d_{k+3}+1 \leq i \leq n-1 \\
(vi) & f(v'_i) = & f(v'_{n-1})+2i : 1 \leq i < d_2 \\
(vii) & f(v_i) = & f(v_{n-1})+2(i-1) : 2 \leq i < d_3 \\
(viii) & f(v'_i) = & f(v'_{((d_l)-1)})+2(i-d_l)\text{where} \\
 & & d_l < i < d_{l+2}, l = 2,4,\cdots, l \leq k+4 \\
(ix) & f(v_i) = & f(v_{((d_l)-1)})+2(i-d_l)\text{where} \\
 & & d_l < i < d_{l+2}, l = 3,5,\cdots, l \leq k+4
\end{array}
$$

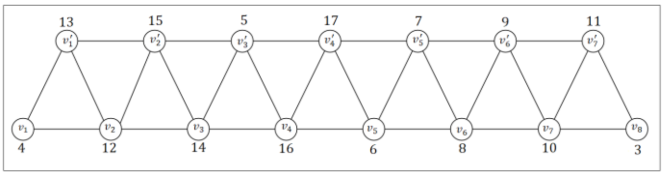
Clearly, *f* is an injective function. We can also verify that *f* satisfies *(2)* and so it follows that *f* satisfies Radio mean condition for all pairs of vertices of $T(P_n)$ where n ∈ $[8 + S_k, 12 + S_k + k]$ where $S_k = 5 + 6 + 7 + \cdots + (5 + k - 1)$ and $k = 1, 2, 3, \cdots$. The largest integer used under this mapping is $3n - k - 7$. Hence the radio mean number of *f*, and the *rmn(f)* = $3n - k - 7$.

Theorem: III-C.2

For $T(P_n)$, $n \in [8 + S_k, 12 + S_k + k]$ where $S_k = 5 + 6 + \cdots + (5 + k - 1)$ and $k = 1, 2, 3, \cdots$ as in previous lemma, the radio mean number $rmn(T(P_n))$s $3n - k - 7$.

Proof: Proof follows from the preceding lemma.

Fig:2 Total graph of Path on 8 vertices



IV. CONCLUSION

In this paper, authors have obtained Radio mean labelings of Path graph and its total graph with minimum span. The radio mean number of P_n is given by

$$
rmn(P_n) = \begin{cases} n & \text{if } n = 2,3, \\ 2n-4 & \text{if } n = 4,5,6, \\ 2n-k-4 & \text{if } n \in [4+S_k, 6+S_k+k], \\ & k \in \mathbb{N}, \\ & S_k = 3+4+\cdots+(3+k-1). \end{cases}
$$

And the radio mean number of $T(P_n)$ is given by

$$rmn(\mathcal{T}(P_n)) = \begin{cases} 2n-1 & \text{if } n = 2,3,4, \\ 3n-6 & \text{if } n = 6,7, \\ 3n-7 & \text{if } n = 8,9,10,11,12, \\ 3n-k-7 & \text{if } n \in [8+S_k, 12+S_k+k], \\ & k \in \mathbb{N}, \\ & S_k = 5+6+\cdots+(5+k-1) \end{cases}$$

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