$e^{\#}$-open and *e-open sets via δ-open sets

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Article History: Received: 11 January 2021; Accepted: 27 February 2021; Published online: 5 April 2021

Abstract: The notion of -open sets in a topological space was studied by Velicko. Usha Parmeshwari et.al. and Indira et.al. introduced the concepts of b# and *b open sets respectively. Following this Ekici et. al. studied the notions of e-open and e-closed sets by mixing the closure, interior, -interior and -closure operators. In this paper some new sets namely e#-open and *e-open sets are defined and their relationship with other similar concetps in topological spaces will be investigated.

Keywords: δ-open, e-open, b-open, *b-open, $b^{\#}$-open, $e^{\#}$-open, *e-open.

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1. Introduction

Velicko introduced the notion -open sets in the year 1968. Andrijevic, Indra et. al. and Usha Parameswary et. al. initiated the study of b-open sets, *b-open sets and b#-open sets in the year 1996, 2013 and 2014 respectively. Following this the researchers in point set topology concentrate on investigating the above concepts in topology. Recently Ekici et. al. studied e-open and e-closed sets. In this paper some new sets that are similar to *b-open and b#-open sets are introduced and studied. A brief survey of the basic concepts and results that are needed here is given in Section-2. The Section-3 deals with *e-open and e#-open sets. Throughout this paper (X, τ) is a topological space and A, B are subsets of X.

2. Preliminaries

The interior and closure operators in topological spaces play a vital role in the generalization of open sets and closed sets. The relations on the interior and closure operators motivate the point set topologists to introduce several forms of nearly open sets and nearly closed sets. Some of them are given in the next definition.

2.1. Definition

A subset A of a space X is called

(i) regular open (Stone, 1937) if A= *IntCl*A and regular closed if A=*ClInt*A,
(ii) semiopen(Levine,1963) if A⊆*Cl Int*A and semiclosed if *Int Cl*A ⊆A,
(iii) preopen(Mashhour et,al.,1982) if A⊆ *Int Cl* A and preclosed if *Cl Int* A ⊆A,
(iv) b-open(Andrijevic,1996) if A⊆*Cl Int* A∪*Int Cl*A and b-closed if *Cl Int* A∩ *Int Cl*A⊆A,
(v) *b-open(Indira et.el., 2013) if A⊆*ClInt*A∩ *IntCl*A and *b-closed if *Cl Int*A∪*Int Cl*A⊆A,
(vi) $b^{\#}$-open(Usha Parameswari et.al.,2014) if A= *ClInt*A∪*IntCl*A and $b^{\#}$-closed if *Cl Int* A ∩ *Int Cl* A = A.

For a subset A of a space X, the semiclosure of A, denoted by *sCl*A is the intersection of all semiclosed subsets of X containing A. Analoguesly preclosure of A, denoted by *pCl*A may defined. The semi-interior of A, denoted by *sInt*A is the union of all semiopen subsets of X contained in A. Analoguesly preinterior of A, denoted by *pInt*A may defined.

2.2. Lemma

Let A be a subset of a space X. Then

(i) *sCl* A = A∪*Int Cl* A,
(ii) *sInt* A = A∩*Cl Int* A,
(iii) *pCl* A = A∪*Cl Int* A,
(iv) *pInt* A = A∩*Int Cl* A. (Andrijevic, 1986)

The concept of δ-closure was introduced by Velicko. A point x is in the δ-closure of A if every regular open nbd of x intersects A. Cl_δA denotes the δ-closure of A. A subset A of a space X is δ-closed if A=Cl_δA. The complement of a δ-closed set is δ-open. The collection of all δ-open sets is a topology denoted by τ^δ. This τ^δ is called the semi - regularization of τ. Clearly RO(X,τ) ⊆ τ^δ ⊆ τ. Let Int_δA denote the δ-interior of A.
The next lemma is due to Velicko.

2.3. Lemma

(i) For any open set A, $Cl_\delta A = ClA$

(ii) For any closed set B, $Int_\delta B = IntB$.

(iii) τ^δ , τ have the same family of clopen sets.

2.4. Definition

A subset A of a space (X, τ) is called

(i) e-open (Ekici, 2008) if $A \subseteq Cl\,Int_\delta A \cup Int\,Cl_\delta A$ and e-closed if $Cl\,Int_\delta A \cap Int\,Cl_\delta A \subseteq A$.

(ii) δ-semiopen (Park et.al., 1997) if $A \subseteq Cl\,Int_\delta A$ and δ-semiclosed if $Int\,Cl_\delta A \subseteq A$,

(iii) δ-preopen(Raychaudhuri et. al., 1993) if $A \subseteq Int\,Cl_\delta A$ and δ-preclosed if $Cl\,Int_\delta A \subseteq A$,

The following lemma is due to the authors (2021, February).

2.5. Lemma

For any subset A of a space (X, τ), the following always hold.

(i) $IntClA = Int_\delta ClA \subseteq Int_\delta Cl_\delta A = IntCl_\delta A$.

(ii) $Cl_\delta Int_\delta A = ClInt_\delta A \subseteq ClIntA = Cl_\delta IntA$

The next definition and the subsequent lemma are due the authors (2021).

2.6. Definition

A subset A of a space (X, τ) is an r-set if

$IntCl_\delta A = IntClA$ and an r*-set if $Cl\,Int_\delta A = Cl\,IntA$.

2.7. Lemma

(i) A is an r-set $\Leftrightarrow$ $Int_\delta ClA = Int_\delta Cl_\delta A = Int\,ClA = Int\,Cl_\delta A$.

(ii) A is an r*-set $\Leftrightarrow$ $Cl\,Int_\delta A = Cl\,IntA = Cl_\delta Int_\delta A = Cl_\delta\,IntA$.

3. $e^\#$-open and *e-open sets

3.1. Definition

A subset A of a space (X, τ) is

(i) $e^\#$-open if $A = Cl\,Int_\delta A \cup Int\,Cl_\delta A$

(ii) $e^\#$-closed if $Cl\,Int_\delta A \cap Int\,Cl_\delta A = A$.

3.2. Definition

A subset A of a space (X, τ) is

(i) *e-open if $A \subseteq Cl\,Int_\delta A \cap Int\,Cl_\delta A$ and

(ii) *e-closed *if* $Cl\,Int_\delta A \cup Int\,Cl_\delta A \subseteq A$.

It is note worthy to see that every $e^\#$-open set is e-open and every *e-open set is e-open. However the converse implications are not true. The following proposition is an easy consequence of the definitions.

3.3. **Proposition**

(i) A is $e^\#$-open $\Leftrightarrow$ X\A is $e^\#$-closed.

(ii) A is *e-open $\Leftrightarrow$ X\A is *e-closed.

3.4. Proposition

For a subset A of a space X,

(i) A is $e^\#$-open $\Leftrightarrow$ A is *e-closed and e-open.

(ii) A is $e^\#$-closed $\Leftrightarrow$ A is *e-open and e-closed .

Proof

A is $e^\#$-open $\Leftrightarrow Cl\,Int_\delta A \cup IntCl_\delta A = A$

$\Leftrightarrow ClInt_\delta A \cup IntCl_\delta A \subseteq A$ and $A \subseteq Cl\,Int_\delta A \cup IntCl_\delta A$. Then it follows that

A is $e^\#$-open if and only if A is *e-closed and e-open. This proves (i) and the proof for (ii) is analog.

3.5. Proposition

The following are equivalent.

(i) A is *e-closed.

(ii) A is preclosed and semiclosed in (X, τ^δ).

(iii) A is δ-preclosed and δ-semiclosed in (X, τ).

Proof

Let A be *e-closed. Since A is *e-closed $ClInt_\delta A \cup IntCl_\delta A \subseteq A$. It follows that $Cl\,Int_\delta A \subseteq A$ and $Int\,Cl_\delta A \subseteq A$ that implies A is both δ-preclosed, and δ-semiclosed. Then $Cl_\delta Int_\delta A = ClInt_\delta A \subseteq A$ and $Int_\delta Cl_\delta A = IntCl_\delta A \subseteq A$ that implies A is preclosed and semiclosed in (X,τ^δ). This proves (i)$\Rightarrow$(ii), (i)$\Rightarrow$(iii) and (ii)$\Leftrightarrow$(iii). Now let A be preclosed and semiclosed in (X, τ^δ). Then $Cl_\delta Int_\delta A \subseteq A$ and $Int_\delta Cl_\delta A \subseteq A$ that implies by using the same lemma we have $ClInt_\delta A \subseteq A$ and $IntCl_\delta A \subseteq A$ so that $ClInt_\delta A \cup IntCl_\delta A \subseteq A$ which further implies A is *e-closed. This proves (ii)$\Rightarrow$(i).

The proof for the next proposition is analogous to the above proposition.

3.6. Proposition

The following are equivalent.

(i) A is *e-open.
(ii) A is preopen and semiopen in (X, τ^{δ}).
(iii) A is δ-preopen and δ-semiopen in (X, τ).

3.7. Proposition

(i) If A is $e^{\#}$-open then A is δ-preclosed, and δ-semiclosed.
(ii) If A is $e^{\#}$-closed then A is δ-preopen and δ-semiopen.

Proof

Let A be $e^{\#}$-closed. Since A is $e^{\#}$-closed, $ClInt_{\delta}A \cap IntCl_{\delta}A = A$. It follows that $A \subseteq ClInt_{\delta}A$ and $A \subseteq IntCl_{\delta}A$. Thus A is δ-preopen and δ-semiopen, This proves (ii) and the proof for (i) is analog.

3.8. Proposition

(i) A is $e^{\#}$-open ⇔ it is $b^{\#}$-open in (X, τ^{δ}).
(ii) A is $e^{\#}$-closed ⇔ it is $b^{\#}$-closed in (X, τ^{δ}).
(iii) A is e-open ⇔ A is b-open in (X, τ^{δ}).
(iv) A is e-closed ⇔ it is b-closed in (X, τ^{δ})
(v) A is *e-open ⇔it is *b-open in (X, τ^{δ}).
(vi) A is *e-closed ⇔ it is *b-closed in (X, τ^{δ}).

Proof

We have $IntClA = Int_{\delta}ClA \subseteq Int_{\delta}Cl_{\delta}A = IntCl_{\delta}A$ and $Cl_{\delta}Int_{\delta}A = Cl\ Int_{\delta}A \subseteq Cl\ IntA = Cl_{\delta}\ IntA$. Therefore we get

$$Int_{\delta}Cl_{\delta}A \cup Cl_{\delta}Int_{\delta}A = Int\ Cl_{\delta}A \cup Cl\ Int_{\delta}A \qquad \text{(Exp. 2.1)}$$

and

$$Int_{\delta}Cl_{\delta}A \cap Cl_{\delta}Int_{\delta}A = Int\ Cl_{\delta}A \cap Cl\ Int_{\delta}A \qquad \text{(Exp. 2.2)}$$

Then the proposition follows from Exp.2.1 and Exp.2.2.

Let A be an r-set and an r*-set in the next six theorems whose proof follow from the lemma on r-sets and r*-sets.

3.9. Theorem

The following ae equivalent.

(i) A is b-open
(ii) A is b-open in (X, τ^{δ})
(iii) A is e-open
(iv) $A \subseteq Int_{\delta}ClA \cup Cl_{\delta}\ IntA$
(v) $A \subseteq Int_{\delta}ClA \cup Cl\ Int_{\delta}A$

Proof

Suppose A is an r-set and r*-set. Then we have $Int_{\delta}ClA = Int_{\delta}Cl_{\delta}A = Int\ ClA = Int\ Cl_{\delta}A$ and $ClInt_{\delta}A = Cl\ IntA = Cl_{\delta}Int_{\delta}A = Cl_{\delta}\ IntA$ from which it follows that $Int\ ClA \cup Cl\ IntA = Int_{\delta}Cl_{\delta}A \cup Cl_{\delta}Int_{\delta}A = IntCl_{\delta}A \cup ClInt_{\delta}A$

$$= Int_{\delta}ClA \cup Cl_{\delta}IntA = IntCl_{\delta}A \cup Cl_{\delta}IntA$$

that implies the theorem.

The next five theorems whose proof is analogous to the above theorem and characterize some nearly open and nearly closed sets.

3.10. Theorem

The following ae equivalent.

(i) A is $b^{\#}$-open
(ii) A is $b^{\#}$-open in (X, τ^{δ})
(iii) A is $e^{\#}$-open
(iv) $A = Int_{\delta}ClA \cup Cl_{\delta}\ IntA$
(v) $A = Int_{\delta}ClA \cup Cl\ Int_{\delta}A$

3.11.Theorem

The following ae equivalent.

(i) A is *b-open
(ii) A is *b-open in (X, τ^{δ})
(iii) A is *e-open
(iv) $A \subseteq Int_{\delta}ClA \cap Cl_{\delta}\ IntA$
(v) $A \subseteq Int_{\delta}ClA \cap Cl\ Int_{\delta}A$

3.12.Theorem

The following ae equivalent.

(i) A is b-closed
(ii) A is b-closed in (X, τ^{δ})
(iii) A is e-closed

(iv) $Int_{\delta}ClA \cap Cl_{\delta} IntA \subseteq A$

(v) $Int_{\delta}ClA \cap Cl\, Int_{\delta}A \subseteq A$

3.13.Theorem

The following ae equivalent.

(i) A is $b^{\#}$-closed

(ii) A is $b^{\#}$-closed in (X, τ^{δ})

(iii) A is e#-closed

(iv) $Int_{\delta}ClA \cap Cl_{\delta} IntA = A$

(v) $Int_{\delta}ClA \cap Cl\, Int_{\delta}A = A$

3.14. Theorem

The following ae equivalent.

(i) A is *b-closed

(ii) A is *b-closed in (X, τ^{δ})

(iii) A is *e-closed

(iv) $Int_{\delta}ClA \cup Cl_{\delta} IntA \subseteq A$

(v) $Int_{\delta}ClA \cup Cl\, Int_{\delta}A \subseteq A$

Conclusion

The two level operators in topology namely $IntCl_{\delta}A$ and $ClInt_{\delta}A$ are used to define new sets in topology namely $e^{\#}$-open set and *e-open set. Some existing sets in topology are characterized using these sets.

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