**e\#-open and *e-open sets via δ-open sets**

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**Article History:** Received: 11 January 2021; Accepted: 27 February 2021; Published online: 5 April 2021

**Abstract:** The notion of \( \square \)-open sets in a topological space was studied by Velicko. Usha Parmeshwary et al. and Indira et al. introduced the concepts of \( \# \) and *\( \# \) open sets respectively. Following this Ekici et al. studied the notions of e-open and e-closed sets by mixing the closure, interior, \(-\)-interior and \(-\)-closure operators. In this paper some new sets namely \( e\# \)-open and \( *e\# \)-open sets are introduced and studied. Following this Ekici et al. initiated the study of \( b\# \) and \( *b\# \) open sets respectively. Following this researchers in point set topology concentrate on investigating the above concepts and closed sets. The relations on the interior and closure operators motivate the point set topologists to introduce several forms of nearly open sets and nearly closed sets. Some of them are given in the next definition.

**Keywords:** \( \delta \)-open, e-open, b-open, *b-open, \( \# \)-open, \( \# \)-open, *e-open

1. **Introduction**

Velicko introduced the notion \( \# \)-open sets in the year 1968. Andrijevic, Indra et al. and Usha Parameswary et al. initiated the study of b-open sets, *b-open sets and \( \# \)-open sets in the year 1996, 2013 and 2014 respectively. Following this the researchers in point set topology concentrate on investigating the above concepts in topology. Recently Ekici et al. studied \( e \)-open and \( e \)-closed sets. In this paper some new sets that are similar to \( *b \)-open and \( \# \)-open sets are introduced and studied. A brief survey of the basic concepts and results that are needed here is given in Section-2. The Section-3 deals with \( *e \)-open and \( \# \)-open sets. Throughout this paper \((X, \tau)\) is a topological space and \( A, B \) are subsets of \( X \).

2. **Preliminaries**

The interior and closure operators in topological spaces play a vital role in the generalization of open sets and closed sets. The relations on the interior and closure operators motivate the point set topologists to introduce several forms of nearly open sets and nearly closed sets. Some of them are given in the next definition.

**2.1. Definition**

A subset \( A \) of a space \( X \) is called

(i) regular open (Stone, 1937) if \( A = \text{Int} \text{Cl}A \) and regular closed if \( A = \text{Cl} \text{Int}A \),

(ii) semiopen (Levine, 1963) if \( A \subseteq \text{Cl} \text{Int}A \) and semi-closed if \( \text{Int} \text{Cl}A \subseteq A \),

(iii) pre-open (Mashhour et al, 1982) if \( A \subseteq \text{Int} \text{Cl}A \) and pre-closed if \( \text{Cl} \text{Int}A \subseteq A \),

(iv) b-open (Andrijevic, 1996) if \( A \subseteq \text{Cl} \text{Int}A \subseteq \text{Int} \text{Cl}A \) and b-closed if \( \text{Cl} \text{Int}A \subseteq \text{Int} \text{Cl}A \subseteq A \),

(v) \( *b \)-open (Indira et al., 2013) if \( A \subseteq \text{Cl} \text{Int}A \subseteq \text{Int} \text{Cl}A \) and \( *b \)-closed if \( \text{Cl} \text{Int}A \subseteq \text{Int} \text{Cl}A \subseteq A \),

(vi) \( b\# \)-open (Usha Parameshwari et al., 2014) if \( A \subseteq \text{Cl} \text{Int}A \subseteq \text{Int} \text{Cl}A \) and \( b\# \)-closed if \( \text{Cl} \text{Int}A \subseteq \text{Int} \text{Cl}A = A \).

For a subset \( A \) of a space \( X \), the semiclosure of \( A \), denoted by \( s \text{Cl}A \) is the intersection of all semiclosed subsets of \( X \) containing \( A \). Analogously preclosure of \( A \), denoted by \( p \text{Cl}A \) may defined. The semi-interior of \( A \), denoted by \( s \text{Int}A \) is the union of all semiopen subsets of \( X \) contained in \( A \). Analoguesly preinterior of \( A \), denoted by \( p \text{Int}A \) may defined.

**2.2. Lemma**

Let \( A \) be a subset of a space \( X \). Then

\begin{align*}
\text{(i)} & \quad s \text{Cl}A = A \cup \text{Int} \text{Cl}A, \\
\text{(ii)} & \quad s \text{Int}A = A \cup \text{Cl} \text{Int}A, \\
\text{(iii)} & \quad p \text{Cl}A = A \cup \text{Cl} \text{Int}A, \\
\text{(iv)} & \quad p \text{Int}A = A \cup \text{Int} \text{Cl}A. \quad \text{(Andrijevic, 1986)}
\end{align*}

The concept of \( \delta \)-closure was introduced by Velicko. A point \( x \) is in the \( \delta \)-closure of \( A \) if every regular open nbd of \( x \) intersects \( A \). \( \text{Cl} \delta A \) denotes the \( \delta \)-closure of \( A \). A subset \( A \) of a space \( X \) is \( \delta \)-closed if \( A = \text{Cl} \delta A \). The complement of a \( \delta \)-closed set is \( \delta \)-open. The collection of all \( \delta \)-open sets is a topology denoted by \( \tau^{\delta} \). This \( \tau^{\delta} \) is called the semi-regularization of \( \tau \). Clearly \( \text{RO}(X, \tau) \subseteq \tau^{\delta} \subseteq \tau \). Let \( \text{Int} \delta A \) denote the \( \delta \)-interior of \( A \). The next lemma is due to Velicko.
2.3. Lemma
(i) For any open set A, \( Cl_0 A = C I A \)
(ii) For any closed set B, \( Int_0 B = Int B \).
(iii) \( \emptyset \), \( \tau \) have the same family of clopen sets.

2.4. Definition
A subset A of a space \((X, \tau)\) is called
(i) *e-open (Ekici, 2008) if \( A \subseteq Cl_0 Int A \cup Int Cl_0 A \) and *e-closed if \( Cl Int_0 A \cap Int Cl_0 A \subseteq A \).
(ii) \( \delta \)-semiopen (Park et al., 1997) if \( A \subseteq Cl_0 Int A \) and \( \delta \)-semiclosed if \( Int Cl_0 A \subseteq A \).
(iii) \( \delta \)-preopen (Raychaudhuri et al., 1993) if \( A \subseteq Int Cl_0 A \) and \( \delta \)-preclosed if \( Cl Int_0 A \subseteq A \).
The following lemma is due to the authors (2021, February).

2.5. Lemma
For any subset A of a space \((X, \tau)\), the following always hold.
(i) \( Int Cl_0 A = Int_0 Cl_0 A \cap Int Cl_0 A \).
(ii) \( Cl_0 Int_0 A \subseteq Cl_0 A = Cl_0 Int A \).
The next definition and the subsequent lemma are due to the authors (2021).

2.6. Definition
A subset A of a space \((X, \tau)\) is an \( r^* \)-set if
\( Int Cl_0 A = Int_0 Cl_0 A \) and an \( r^* \)-set if \( Cl_0 Int A = Cl_0 Int_0 A \).

2.7. Lemma
(i) A is an \( r^* \)-set \( \iff \) \( Int_0 Cl_0 A = Int_0 Cl_0 A = Int Cl_0 A \).
(ii) A is an \( r^* \)-set \( \iff \) \( Cl_0 Int_0 A = Cl_0 Int_0 A = Cl_0 Int_0 A \).

3. \( e^* \)-open and \( *e \)-open sets

3.1. Definition
A subset A of a space \((X, \tau)\) is
(i) \( e^* \)-open if \( A \subseteq Cl_0 Int A \cup Int Cl_0 A \)
(ii) \( e^* \)-closed if \( Cl_0 Int A \cap Int Cl_0 A = A \).

3.2. Definition
A subset A of a space \((X, \tau)\) is
(i) \( *e \)-open if \( A \subseteq Cl_0 Int A \cap Int Cl_0 A \) and
(ii) \( *e \)-closed if \( Cl_0 Int A \cap Int Cl_0 A \subseteq A \).

It is noted worthy to see that every \( e^* \)-open set is \( e \)-open and every \( *e \)-open set is \( e \)-open. However the converse implications are not true. The following proposition is an easy consequence of the definitions.

3.3. Proposition
(i) A is \( e^* \)-open \( \iff \) \( X A \) is \( e^* \)-closed.
(ii) A is \( *e \)-open \( \iff \) \( X A \) is \( *e \)-closed.

3.4. Proposition
For a subset A of a space X,
(i) A is \( e^* \)-open \( \iff \) A is \( *e \)-closed and \( e \)-open.
(ii) A is \( e^* \)-closed \( \iff \) A is \( *e \)-open and \( e \)-closed.

Proof
A is \( e^* \)-open \( \iff \) \( Cl Int_0 A \cap Int Cl_0 A = A \)
\( \iff \) \( Cl Int_0 A \cap Int Cl_0 A \subseteq A \) and \( A \subseteq Cl_0 Int A \cap Int Cl_0 A \). Then it follows that A is \( e^* \)-open if and only if A is \( *e \)-closed and \( e \)-open. This proves (i) and the proof for (ii) is analogous.

3.5. Proposition
The following are equivalent.
(i) A is \( e \)-closed.
(ii) A is preclosed and semiclosed in \((X, \tau^0)\).
(iii) A is \( \delta \)-preclosed and \( \delta \)-semiclosed in \((X, \tau)\).

Proof
Let A be \( *e \)-closed. Since A is \( *e \)-closed \( Cl_0 Int_0 A \cap Int Cl_0 A \subseteq A \), it follows that \( Cl_0 Int_0 A \subseteq A \) and \( Int Cl_0 A \subseteq A \). Then \( Cl_0 Int_0 A = Cl_0 Int_0 A \) and \( Int Cl_0 A = Int Cl_0 A \). Now let A be preclosed and semiclosed in \((X, \tau^0)\). Then \( Cl_0 Int_0 A \subseteq A \) and \( Int Cl_0 A \subseteq A \) that implies by using the same lemma we have \( Cl_0 Int_0 A \subseteq A \) and \( Int Cl_0 A \subseteq A \) which further implies A is \( *e \)-closed. This proves (ii) \( \Rightarrow \) (i).

The proof for the next proposition is analogous to the above proposition.

3.6. Proposition
The following are equivalent.
3.12. Theorem

(i) A is *e-open.
(ii) A is preopen and semiopen in \((X, \tau^0)\).
(iii) A is \(\delta\)-preopen and \(\delta\)-semiopen in \((X, \tau)\).

3.11. Theorem

(i) If A is \(e^\delta\)-open then A is \(\delta\)-preclosed, and \(\delta\)-semiclosed.
(ii) If A is \(e^\delta\)-closed then A is \(\delta\)-preopen and \(\delta\)-semiopen.

**Proof**

Let A be \(e^\delta\)-closed. Since A is \(e^\delta\)-closed, \(Cl\{A\} \cap Int\{Cl\{A\}\} = A\). It follows that \(A \subseteq Cl\{A\}\) and \(A \subseteq Int\{Cl\{A\}\}\).

Thus A is \(\delta\)-preopen and \(\delta\)-semiopen. This proves (ii) and the proof for (i) is analogous.

3.10. Theorem

(i) A is \(\delta\)-preopen and \(\delta\)-preclosed in \((X, \tau^0)\).
(ii) A is \(\delta\)-semiopen and \(\delta\)-semiopen in \((X, \tau)\).

**Proof**

Let A be \(e^\delta\)-closed. Since A is \(e^\delta\)-closed, \(Cl\{A\} \cap Int\{Cl\{A\}\} = A\). It follows that \(A \subseteq Cl\{A\}\) and \(A \subseteq Int\{Cl\{A\}\}\).

Thus A is \(\delta\)-preopen and \(\delta\)-semiopen. This proves (ii) and the proof for (i) is analogous.

3.9. Proposition

(i) A is \(e^\delta\)-open \(\iff\) it is \(b^\delta\)-open in \((X, \tau^0)\).
(ii) A is \(e^\delta\)-closed \(\iff\) it is \(b^\delta\)-closed in \((X, \tau^0)\).
(iii) A is \(e\)-open \(\iff\) A is \(b\)-open in \((X, \tau^0)\).
(iv) A is \(e\)-closed \(\iff\) A is \(b\)-closed in \((X, \tau^0)\).
(v) A is \(e\)-open \(\iff\) it is \(b^*\)-open in \((X, \tau^0)\).
(vi) A is \(e\)-closed \(\iff\) it is \(b^*\)-closed in \((X, \tau^0)\).

**Proof**

We have \(Int\{A\} = Int\{Cl\{A\}\} \subseteq Int\{Cl\{A\}\} = Int\{Cl\{A\}\}\) and \(Cl\{Int\{A\}\} = Cl\{Int\{A\}\} \subseteq Cl\{Int\{A\}\}\). Therefore we get

\[
Int\{Cl\{A\}\} \cup Cl\{Int\{A\}\} = Int\{Cl\{A\}\} \cup Cl\{Int\{A\}\}\]  \hspace{1cm} (Exp. 2.1)

and

\[
Int\{Cl\{A\}\} \cap Cl\{Int\{A\}\} = Int\{Cl\{A\}\} \cap Cl\{Int\{A\}\}\]  \hspace{1cm} (Exp. 2.2)

Then the proposition follows from Exp.2.1 and Exp.2.2.

Let A be an r-set and an \(r^*\)-set in the next six theorems whose proof follow from the lemma on r-sets and \(r^*\)-sets.

3.8. Theorem

The following are equivalent.

(i) A is \(b\)-open
(ii) A is \(b\)-open in \((X, \tau^0)\)
(iii) A is \(e\)-open
(iv) \(A \subseteq Int\{Cl\{A\}\} \cup Cl\{Int\{A\}\}\)
(v) \(A \subseteq Int\{Cl\{A\}\} \cap Cl\{Int\{A\}\}\)

**Proof**

Suppose A is an r-set and \(r^*\)-set. Then we have

\[
Int\{Cl\{A\}\} = Int\{Cl\{A\}\} = Int\{Cl\{A\}\} = Int\{Cl\{A\}\}\]  \hspace{1cm} (Exp. 2.1)

and

\[
Cl\{Int\{A\}\} = Cl\{Int\{A\}\} = Cl\{Int\{A\}\} = Cl\{Int\{A\}\}\]  \hspace{1cm} (Exp. 2.2)

that implies the theorem.

The next five theorems whose proof is analogous to the above theorem and characterize some nearly open and nearly closed sets.

3.7. Theorem

The following are equivalent.

(i) A is \(b\)-open
(ii) A is \(b\)-open in \((X, \tau^0)\)
(iii) A is \(e\)-open
(iv) \(A \subseteq Int\{Cl\{A\}\} \cup Cl\{Int\{A\}\}\)
(v) \(A \subseteq Int\{Cl\{A\}\} \cap Cl\{Int\{A\}\}\)

3.11. Theorem

The following are equivalent.

(i) A is \(b\)-open
(ii) A is \(b\)-open in \((X, \tau^0)\)
(iii) A is \(e\)-open
(iv) \(A \subseteq Int\{Cl\{A\}\} \cup Cl\{Int\{A\}\}\)
(v) \(A \subseteq Int\{Cl\{A\}\} \cap Cl\{Int\{A\}\}\)

3.12. Theorem

The following are equivalent.

(i) A is \(b\)-open
(ii) A is \(b\)-open in \((X, \tau^0)\)
(iii) A is \(e\)-open
(iv) $\text{Int}_\delta \text{Cl}_b \cap \text{Cls} \text{Int} A \subseteq A$
(v) $\text{Int}_\delta \text{Cl} \text{Cl}_b \text{Int} A \subseteq A$

3.13. Theorem
The following are equivalent.
(i) $A$ is $b^\delta$-closed
(ii) $A$ is $b^\delta$-closed in $(X, \tau^\delta)$
(iii) $A$ is $e^\delta$-closed
(iv) $\text{Int}_\delta \text{Cl}_b \cap \text{Cls} \text{Int} A = A$
(v) $\text{Int}_\delta \text{Cl} \text{Cl}_b \text{Int} A = A$

3.14. Theorem
The following are equivalent.
(i) $A$ is $^*b$-closed
(ii) $A$ is $^*b$-closed in $(X, \tau^\delta)$
(iii) $A$ is $e^\delta$-closed
(iv) $\text{Int}_\delta \text{Cl}_b \cup \text{Cls} \text{Int} A \subseteq A$
(v) $\text{Int}_\delta \text{Cl} \cup \text{Cl}_b \text{Int} A \subseteq A$

Conclusion
The two-level operators in topology namely $\text{IntCl}_b A$ and $\text{ClInt}_\delta A$ are used to define new sets in topology namely $e^\delta$-open set and $^*e$-open set. Some existing sets in topology are characterized using these sets.

References