Vol.17 No.01(2026), 17-27
DOI: https://doi.org/10.61841/turcomat.v17i01.15505

APPROXIMATE DYNAMICAL ANALYSIS OF DAMPED NONLINEAR MECHANICAL SYSTEMS WITH TIME-VARYING PARAMETERS USING HYBRID ANALYTICAL METHODS

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Abstract

Nonlinear oscillators characterized by strong damping and time-varying coefficients present notable analytical challenges, primarily attributable to the limitations of classical perturbation techniques. The Krylov–Bogoliubov–Mitropolskii (KBM) method generally applies to weakly nonlinear and weakly damped systems, while the Harmonic Balance (HB) method often encounters convergence difficulties in time-dependent situations. To address these issues, this paper introduces a hybrid analytical framework that integrates the KBM and HB methods with amplitude-phase modulation. This innovative approach effectively approximates strongly damped nonlinear systems with slowly varying parameters by methodically eliminating secular factors and enhancing convergence. The efficacy of the proposed framework is validated using a fourth-order Runge-Kutta scheme on a nonlinear oscillator with varying coefficients. A comparative quantitative error analysis indicates enhanced accuracy over traditional KBM methods, particularly in scenarios involving moderate to large damping. These results suggest that the hybrid KBM-HB methodology significantly expands the applicability of perturbation-based analytical techniques to a broader class of nonlinear dynamical systems, with potential engineering applications in areas such as energy harvesting and vibration control.

Key words: Nonlinear oscillators**;** Strongly damped; Harmonic Balance; Perturbation-based; KBM-HB model.

MSC: 34E05

1. Introduction

Analytical approaches are key to comprehending the behavior of nonlinear oscillatory systems, and among them, perturbation methods remain some of the most powerful and widely adopted techniques for deriving approximate solutions [1]. Following the original works of Krylov and Bogoliubov and, subsequently, of Bogoliubov and Mitropolskii, the study of nonlinear oscillations has found its way into the center of attention of nonlinear science and engineering. Their works provided the bases of asymptotic techniques in solving vibration problems, which gave methodical steps on how to address weakly nonlinear vibration problems and those systems where parameters change very slowly. Over time, scholars have effectively generalized these classical methods to examine oscillators with various forms of nonlinearities, damping, and external forcing [2-3].

To address the intrinsic shortcomings of first-order perturbation methods, scholars have suggested a wide variety of additions and variations that are intended to enhance convergence and accuracy. One of these, the Krylov-Bogoliubov-Mitropolskii (KBM) approach [4-7], has developed into a highly effective analysis tool, with the capabilities to model the higher-order nonlinearities [6-14] multi-degree-of-freedom systems, and time-dependent dynamics [15]. The single KBM-type methods proposed by Shamsul Alam [16] and others are a significant step forward in this connection, allowing the treatment of damped, over-damped, and quasi-linear systems with variable coefficients in a systematic manner [17]. The formulations provide a stable analytical basis for exploring complex nonlinear vibration models beyond the scope of traditional perturbation expansions. In this work, “strongly damped” systems refer to nonlinear oscillators in which the damping coefficient is not asymptotically small (i.e., not of order $\varepsilon \ll 1$), and may significantly influence system dynamics. “Time-dependent systems” refer to systems with coefficients that vary slowly with time, typically expressed as functions of εt, ensuring compatibility with perturbation-based analytical techniques.

The advancement in the harmonic balance (HB) technique [18] has also enhanced the analytical tools available for nonlinear non-resonant vibration analysis. Unlike the conventional perturbation methods, which depend on small parameters, the HB method simply builds periodic solutions by balancing nonlinear and harmonic terms. Its variants with changes, including the multiple-harmonic balance and combined HB-Galerkin methods [19], provide a higher accuracy with highly nonlinear oscillators [20], when there is large amplitude motion or external excitation. Consequently, HB-based methods have found extensive use in nonlinear mechanical systems, fluid-structure interactions, and electromechanical oscillators [21-22].

The removal of secular terms is achieved by imposing solvability conditions on the amplitude and phase evolution equations. Specifically, terms that lead to unbounded growth are eliminated by ensuring orthogonality with the fundamental harmonic components. The harmonic balance component further refines this process by redistributing nonlinear contributions across harmonics, thereby improving convergence and stability compared to standalone KBM approaches.

The study of nonlinear vibrations has grown quickly in the past several years because of the advent of complicated engineering systems, such as micro- and nanoelectromechanical resonators, nonlinear energy harvesters, and high-tech vibration absorbers. These systems tend to be highly nonlinear regarding the stiffness and damping, which require analytical processes capable of well describing amplitude-dependent frequency shifts, resonance dynamics, and dynamic stability properties.

Inspired by these advances, the current study presents an integrated analytical platform, which incorporates the extended KBM technique with the harmonic balance technique. This joint solution methodically addresses nonlinear vibration systems characterized by various damping mechanisms, all of which exhibit high nonlinearity. The framework suggested does not only broaden the range of relevance of classical methods of asymptotic analysis but also has the benefit of predicting complex nonlinear oscillatory systems that are observed in contemporary applied mechanics and engineering design.

The manuscript is structured in the following way. Section 2 presents the key analytical characteristics of the suggested framework and outlines the most important developments made within the context of the study. Section 3 is a formulation of the methodology, which involves the extended KBM process, amplification-phase conversion, and building of the approximate answer. Section 4 provides an example of a nonlinear system and works out the analytical expressions corresponding to using the proposed approach. Section 5 examines the graphical outputs by comparing the perturbation solutions with numerical simulations to prove the reliability of the combined KBM-HB method. Section 6 presents the major contributions and novelty of the work. Section 7 ends the study with a summary of the key findings and a discussion of the general implications of the new framework. To fill out the analytical solution, the formulation of harmonic balance has been presented in an appendix.

2. Highlights

- A systematic analytical framework is developed for resilient nonlinear differential systems with progressively increasing coefficients.
- The approach enhances solution accuracy by employing both the expanded Krylov–Bogoliubov–Mitropolskii (KBM) technique and the Harmonic Balance (HB) method.
- The approach eliminates secular terms, ensuring the stability of the first-order approximation solutions.
- We employ amplitude-phase transformation to facilitate the development of solutions while maintaining their generality.
- A comparison with numerical solutions demonstrates a high degree of similarity, thereby validating the robustness of the technique.

3. Methodology

The nonlinear differential system can be explained

$$\ddot{p} + 2c_1(\tau) + (c_2^2 + c_3 cos\tau + c_4^2 sin2\tau)p = -\varepsilon f(p,\tau), \qquad \tau = \varepsilon t \qquad (1)$$

when f is a specified nonlinear function, and the over-dots indicate differentiation with respect to t, ε is a tiny parameter, c_1 , c_2 , c_3 and c_4 are constants, $c_2 = c_3 = c_4 = O(\varepsilon)$. Frequency is referred to as $\omega(\tau)$, setting $\omega = \omega_0\sqrt{c_2^2 + c_3 cos\tau + c_4^2 sin2\tau}.$

The unperturbed solution of (1) can be obtained by setting $\varepsilon = 0$ and $\tau = \tau_0$ = constant, in Eq. (1), which takes the form

$$p(t,0) = p_{1,0}e^{\lambda_1(\tau_0)t} + p_{-1,0}e^{\lambda_2(\tau_0)t} . \qquad (2)$$

When $\varepsilon \neq 0$, We pursue a resolution of Equation (1) in the following format.

$$p(t,\varepsilon) = p_1(t,\tau) + p_{-1}(t,\tau) + \varepsilon u_1(p_1,p_{-1},t,\tau) + \varepsilon^2 u_2(p_1,p_{-1},t,\tau) + \cdots \qquad (3)$$

where p_1 and p_{-1} satisfy the given equations

$$\dot{p}_1 = \lambda_1(\tau)p_1 + \varepsilon\, P_1(p_1,p_{-1},\tau) + \varepsilon^2 P_1(\,p_1,p_{-1},\tau) + \cdots$$

$$\dot{p}_{-1} = \lambda_2(\tau)p_{-1} + \varepsilon\, P_{-1}(p_1,p_{-1},\tau) + \varepsilon^2 P_{-1}(\,p_1,p_{-1},\tau) + \cdots \qquad (4)$$

By differentiating $p(t,\varepsilon)$ twice with respect to t, substituting the second derivative $\ddot{p}$ and p into the original equation (1), and equating the coefficients, we arrive

$$(\lambda_1 p_1 \Omega\, p_1 + \lambda_2 p_{-1} \Omega p_{-1}) P_1 + (\lambda_1 p_1 \Omega\, p_1 + \lambda_2 p_{-1} \Omega p_{-1}) P_{-1} + \lambda_1'\, p_1 + \lambda_2' p_{-1} - \lambda_2\, P_1 - \lambda_1 P_{-1} + (\lambda_1 p_1 \Omega\, p_1 + \lambda_2 p_{-1} \Omega p_{-1} - \lambda_1)(\lambda_1 p_1 \Omega\, p_1 + \lambda_2 p_{-1} \Omega p_{-1} - \lambda_2) u_1 = -f^{(0)}(p_1, p_{-1}, \tau) \quad . \tag{5}$$

where $\lambda_1' = \frac{d\lambda_1}{d\tau}$, $\lambda_2' = \frac{d\lambda_2}{d\tau}$, $\Omega\, p_1 = \frac{\partial}{\partial p_1}$, $\Omega\, p_{-1} = \frac{\partial}{\partial p_{-1}}$, $f^{(0)} = f(\, p_0, \dot{p}_0, \tau)$.

It is presumed here that $f^{(0)}$ can be expanded in Taylor's series

$$f^{(0)} = \sum_{r_1, r_2 = 0}^{\infty} F_{r_1, r_2}(\tau) p_1^{r_1}\, p_{-1}^{r_2}. \tag{6}$$

We establish the restriction that $u_1 \ldots$ eliminates the terms$p_1^{i_1} p_{-1}^{i_2}$, $i_1 - i_2 = \pm 1 \quad i_1, i_2 = 0,1,2 \ldots$ in order to arrive at a solution of (1). The assumption guarantees that $u_1 \ldots$ there are no secular type terms $te^{-\lambda_1 t}$. We may transform (3) into the appropriate form of the KBM. [1,2,4] solution by changing $p_1 = \rho e^{i\emptyset}/2$ and $p_{-1} = \rho e^{-i\emptyset}/2$ and $\lambda_1 = -\zeta + i\omega$, $\lambda_2 = -\zeta - i\omega$. In this case, ρ and ϕ are respectively amplitude and phase variables (see Shamsul [6]) then first order solution has been fully determined in (1).

4. Illustration

Let's look at a nonlinear vibrating system [23] with slowly varying coefficients as an example of the following process

$$\ddot{p} + 2c_1(\tau) + (c_2^2 + c_3 cos\tau + c_4^2 sin2\tau) p = -\varepsilon p^3. \tag{7}$$

Differentiation with respect to $t$is indicated here by over dots, where $p_0 = p_1 + p_{-1}$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(p_1^3 + 3p_1^2 p_{-1} + 3\, p_1 p_{-1}^2 + p_{-1}^3). \tag{8}$$

In accordance with the assumption (excluded in section 2) we replace in (5) and divide it into two parts as

$$(\lambda_1 p_1 \Omega\, p_1 + \lambda_2 p_{-1} \Omega p_{-1}) P_1 + (\lambda_1 p_1 \Omega\, p_1 + \lambda_2 p_{-1} \Omega p_{-1}) P_{-1} + \lambda_1'\, p_1 + \lambda_2' p_{-1} - \lambda_2\, P_1 - \lambda_1 P_{-1} = -3(p_1^2 p_{-1} + p_1 p_{-1}^2). \tag{9}$$

and

$$-(p_1^3 + p_{-1}^3) = (\lambda_1 p_1 \Omega\, p_1 + \lambda_2 p_{-1} \Omega p_{-1} - \lambda_1)(\lambda_1 p_1 \Omega\, p_1 + \lambda_2 p_{-1} \Omega p_{-1} - \lambda_2) u_1. \tag{10}$$

The particular solution of (10) is

$$u_1 = -p_1^3/2\lambda_1(3\lambda_1 - \lambda_2) - p_{-1}^3/2\lambda_2(3\lambda_2 - \lambda_1). \tag{11}$$

The particular solutions of (9) is

$$P_1 = -\frac{\lambda_1' p_1}{\lambda_1 - \lambda_2} - \frac{3p_1^2 p_{-1}}{2\lambda_1}. \quad \text{and}$$

$$P_{-1} = -\frac{\lambda_2' p_{-1}}{\lambda_1 - \lambda_2} - 3\frac{p_1 p_{-1}^2}{2\lambda_2}. \tag{12}$$

After arranging and substituting the functional values of P_1 and P_{-1} from (12) into (4), we get

$$\dot{p}_1 = \lambda_1 p_1 + \varepsilon\,(-\lambda_1' p_1/(\lambda_1 - \lambda_2) - 3p_1^2 p_{-1}/2\lambda_1). \quad \text{and}$$

$$\dot{p}_{-1} = \lambda_2 p_{-1} + \varepsilon\,(\lambda_2' p_{-1}/(\lambda_1 - \lambda_2) - 3p_1 p_{-1}^2/2\lambda_2). \tag{13}$$

The variational equations of the amplitude ρ and the phase ϕ in the real form which convert (13) to

$$\dot{\rho} = -\zeta p - \varepsilon p \omega'/2\omega + 3\varepsilon p^3 \zeta/8(\zeta^2 + \omega^2) \quad \text{and}$$

$$\dot{\phi} = \omega + \varepsilon\zeta/2\omega + 3\varepsilon p^2 \omega/8(\zeta^2 + \omega^2). \tag{14}$$

where $\omega^2 = (c_2 + c_3 \cos\tau + c_4^2 \sin 2\tau)$.

The variational equation (14) resembles the expanded KBM solution. The variational equations for amplitude and phase are typically expressed as a system of first-order differential equations and resolved using numerical methods (see to Shamsul [6]).

Thus, the improved solution of problem (7) is

$$p(t,\varepsilon) = \rho cos\phi + \varepsilon u_1 + \cdots \qquad (15)$$

where $\varphi = \omega t + \phi$ (see **Appendix A (HB method)**) and ρ , ϕ are the solutions of the equations (14) and u_1 is provided by Eq. (11).

By appropriately reformulating amplitude evolution equations, the traditional KBM technique can be applied to systems that are moderate or severely damped, even though it is mainly designed for weakly nonlinear systems. By stabilizing higher-order harmonic contributions and enhancing convergence behavior, the harmonic balancing component's inclusion in this study makes up for this restriction. The KBM framework can continue to be useful outside of its conventional range of validity thanks to this hybridization.

5. Graphical Representation

The extended KBM-HB method, which is the composite of the modified Krylov-Bogoliubov-Mitropolsky (KBM) method and the Harmonic Balance (HB) method, is an efficient methodological analysis that can be used to obtain approximate solutions of damped nonlinear differential equations with constant and time-dependent coefficients at the first order. The approach is only theoretically able to obtain solutions of higher order but is usually limited to a first-order accuracy because of the exponentially growing algebraic complexity. It has been seen that the validity and accuracy of the given method is checked by comparing the analytical perturbation solutions with the numerical simulations, which are perfect benchmarks. Specifically, in the case of the Duffing oscillator, the analytical results, produced by the KBM and HB model, show a high level of correlation with the results produced by the fourth-order Runge-Kutta model, which proves the accuracy and stability of the offered scheme. Additionally, physical insight has also been improved by plotting the dynamical behaviors and stability. These plots, generated at different settings of the temporal parameter, are quite useful in demonstrating how the system varies and oscillates with time, which justifies the validity of the analytical method as well as the physical relevance of the proposed method. A comparison is conducted between the perturbation technique approximation and a numerical solution, which establishes the standard for accuracy. This study compares the improved KBM and HB method approach to the unified KBM method for different damping effects presented in the article. MATLAB is utilized to create all visual representations as two-dimensional graphs.

The first figure 1(a) represents Present perturbation solution (7) (dotted line) corresponding numerical solution (solid line) they are drawn with initial conditions $x(0) = 0.7000, \dot{x}(0) = 0.000$ or $x_1 = 0.70000, y_1 = -0.006228$ for $c_1 = 0.01, \varepsilon = 0.06$,

$\omega = \omega_0\sqrt{c_2^2 + c_3cos\tau + c_4^2sin2\tau}$. and figure 1(b) shows unified KBM solutions (A.1) (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $x(0) = 0.7000, \dot{x}(0) = 0.000$ or $x_1 = 0.7000, y_1 = -0.008014$ for $c_1 = 0.01, \varepsilon = 0.06$, $\omega = \omega_0\sqrt{c_2^2 + c_3cos\tau + c_4^2sin2\tau}$. Wherein figure 1(c) represents the comparison of the Present perturbation solution, numerical solution and unified KBM solution. 1(d) line of Time vs error 1(a) for the damping coefficient $c_1 = 0.01$. 1(e) line of Time vs error 1(b) for the damping coefficient $c_1 = 0.01$. Figure 1(f) shows the comparison of the line of error 1(a) vs error 1(b) for $c_1 = 0.01$.

The next figure 2(a) represents Present perturbation solution (7) (dotted line) corresponding numerical solution (solid line) they are drawn with initial conditions $x(0) = 0.7000, \dot{x}(0) = 0.000$ or $x_1 = 0.70000, y_1 = -0.012457$ for $c_1 = 0.02, \varepsilon = 0.06$,

$\omega = \omega_0\sqrt{c_2^2 + c_3cos\tau + c_4^2sin2\tau}$. and figure 2(b) shows unified KBM solutions (A.1) (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $x(0) = 0.7000, \dot{x}(0) = 0.000$ or $x_1 = 0.7000, y_1 = -0.016029$ for $c_1 = 0.02, \varepsilon = 0.06$, $\omega = \omega_0\sqrt{c_2^2 + c_3cos\tau + c_4^2sin2\tau}$. Wherein figure 2(c) represent the comparison of the Present perturbation solution, numerical solution and unified KBM solution. 2(d) line of Time vs error 2(a) for the damping coefficient $c_1 = 0.02$. 2(e) line of Time vs error 2(b) for the damping coefficient $c_1 = 0.02$. Figure 2(f) shows the comparison of the line of error 2(a) vs error 2(b) for $c_1 = 0.02$.

The figure 3(a) represents Present perturbation solution (7) (dotted line) corresponding numerical solution (solid line) they are drawn with initial conditions $x(0) = 0.7000, \dot{x}(0) = 0.000$ or $x_1 = 0.70000, y_1 = -0.031151$ for $c_1 = 0.05, \varepsilon = 0.06$,

$\omega = \omega_0\sqrt{c_2^2 + c_3cos\tau + c_4^2sin2\tau}$. and figure 3(b) shows unified KBM solutions (A.1) (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $x(0) = 0.7000, \dot{x}(0) = 0.000$ or

$x_1 = 0.7000, y_1 = -\dot{0}.040092$ for $c_1 = 0.05, \varepsilon = 0.06$, $\omega = \omega_0\sqrt{c_2^2 + c_3 cos\tau + c_4^2 sin2\tau}$. Wherein figure 3(c) represent the comparison of the Present perturbation solution, numerical solution and unified KBM solution. 3(d) line of Time vs error 3(a) for the damping coefficient $c_1 = 0.05$. 3(e) line of Time vs error 3(b) for the damping coefficient $c_1 = 0.05$. Figure 3(f) shows the comparison of the line of error 3(a) vs error 3(b) for $c_1 = 0.05$.

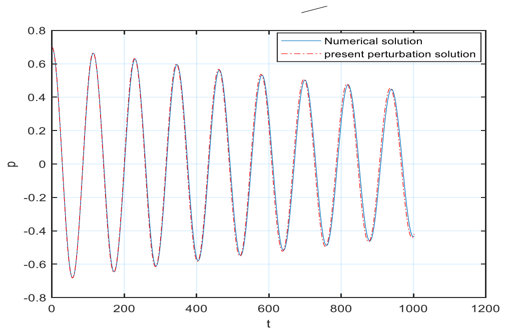
And finally figure 4(a) represents Present perturbation solution (7) (dotted line) corresponding numerical solution (solid line) they are drawn with initial conditions $x(0) = 0.7000, x(0) \doteq 0.000$ or $x_1 = 0.70000, y_1 = -\dot{0}.043624$ for $c_1 = 0.07, \varepsilon = 0.06$,

$\omega = \omega_0\sqrt{c_2^2 + c_3 cos\tau + c_4^2 sin2\tau}$. and figure 4(b) shows unified KBM solutions (A.1) (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $x(0) = 0.7000, x(0) \doteq 0.000$ or $x_1 = 0.7000, y_1 = -\dot{0}.040092$ for $c_1 = 0.05, \varepsilon = 0.06$, $\omega = \omega_0\sqrt{c_2^2 + c_3 cos\tau + c_4^2 sin2\tau}$. Wherein figure 4(c) represent the comparison of the Present perturbation solution, numerical solution and unified KBM solution. 4(d) line of Time vs error 4(a) for the damping coefficient $c_1 = 0.05$. 4(e) line of Time vs error 4(b) for the damping coefficient $c_1 = 0.07$. Figure 4(f) shows the comparison of the line of error 4(a) vs error 4(b) for $c_1 = 0.07$.

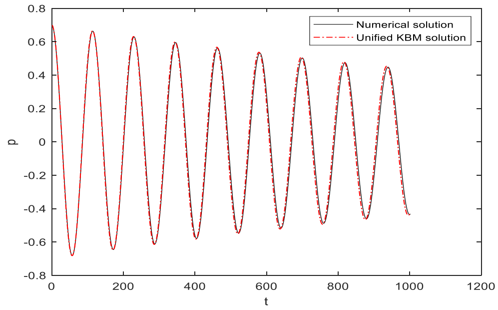
Because it avoids recursive expansions and minimizes algebraic complexity, the suggested method has a substantially lower computing cost than higher-order perturbation techniques. It offers analytical insight with less computing overhead than numerical solvers.

The resulting mismatch clearly demonstrates the reduced applicability of the unified perturbation scheme at moderate-to-large parameter strengths. The figure thus reinforces a central conclusion: the unified perturbation method is highly effective in the small-parameter regime but loses accuracy as the perturbation intensity grows.

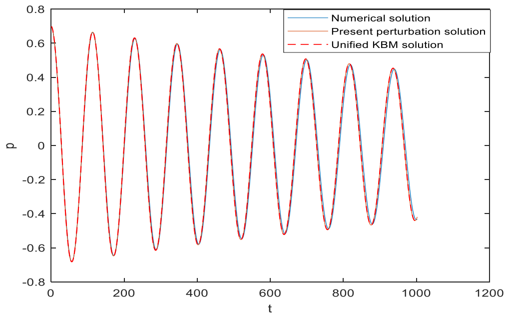
Comparison of the present KBM–HB perturbation solution, unified KBM solution, and numerical solution of Eq. (7) for a fixed damping coefficient.



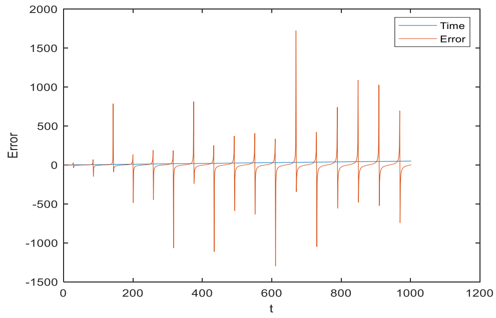
1(a). Present perturbation solution vs numerical solution



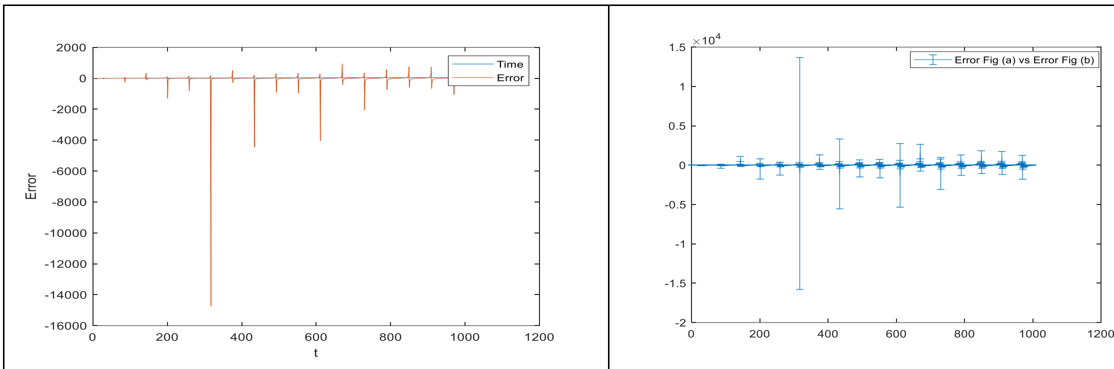
1(b). Unified KBM solution vs numerical solution



1(c). Combined comparison of all three solutions



1(d). Time vs error for present perturbation solution

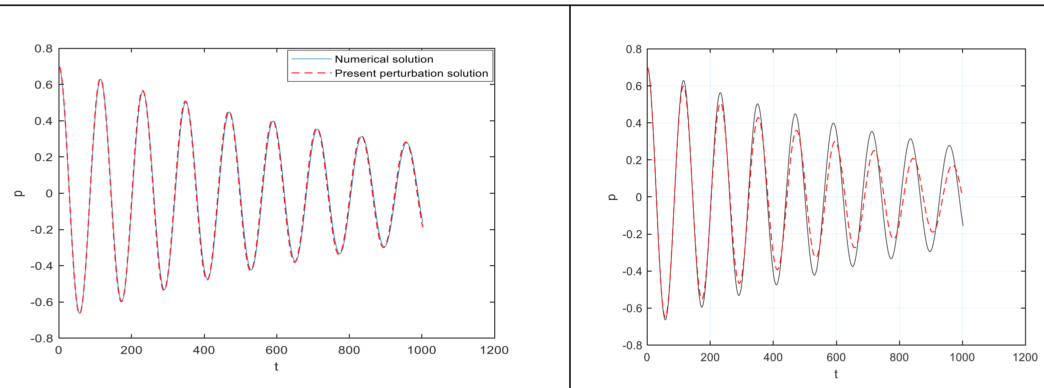


1(e). Time vs error for unified KBM solution

1(f). Error comparison between present and unified methods

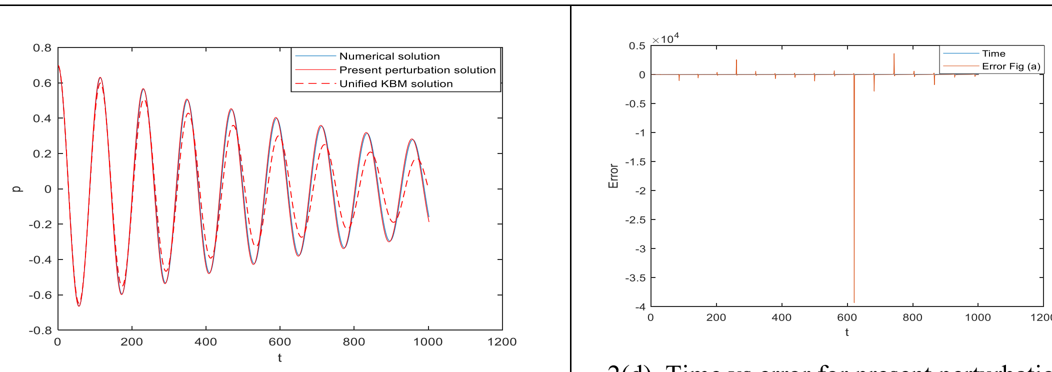
Figure 1: A diagram of the perturbation solution (dotted line) and its corresponding numerical solution (solid line) derived from Eq. (7) with initial conditions $\rho = 0.7000,\ \phi = -0.006228\ \ [p(0) = 0.7000,\ \dot{p}(0) = 0.0000]$ for $e = 0.6$, with damping coefficient is $c_1 = .01$.

Comparison of analytical and numerical solutions of Eq. (7) under a different damping coefficient.



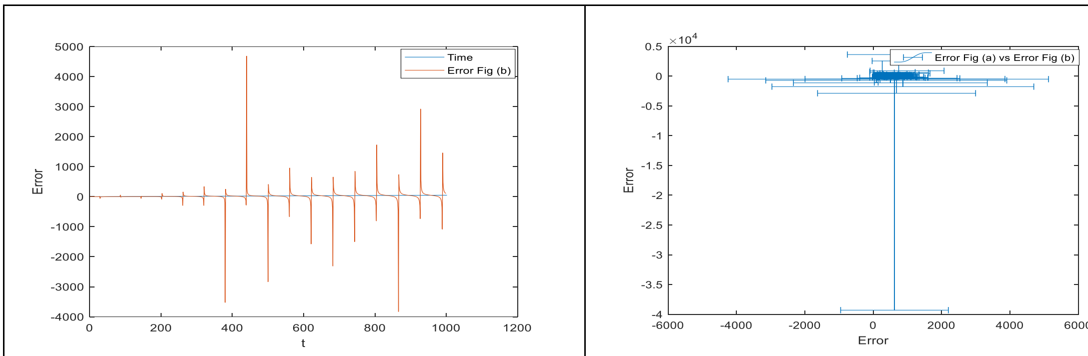
2(a). Present perturbation solution vs numerical solution

2(b). Unified KBM solution vs numerical solution



2(c). Combined comparison of all three solutions

2(d). Time vs error for present perturbation solution

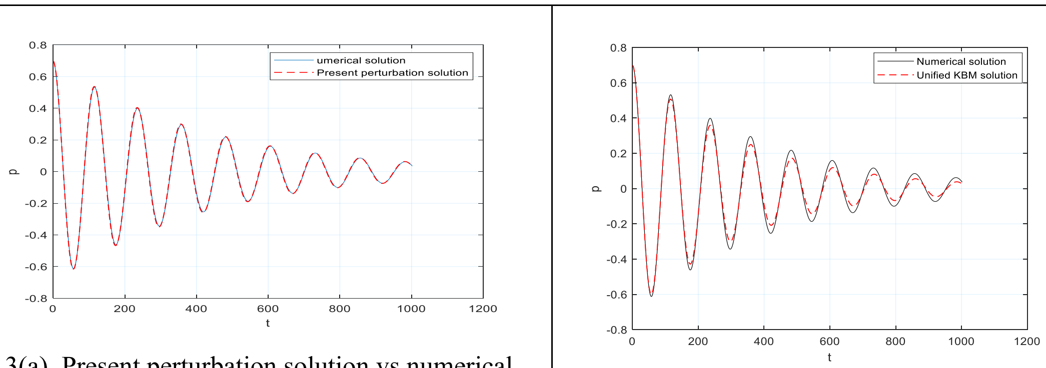


2(e). Time vs error for unified KBM solution

2(f). Error comparison between present and unified methods

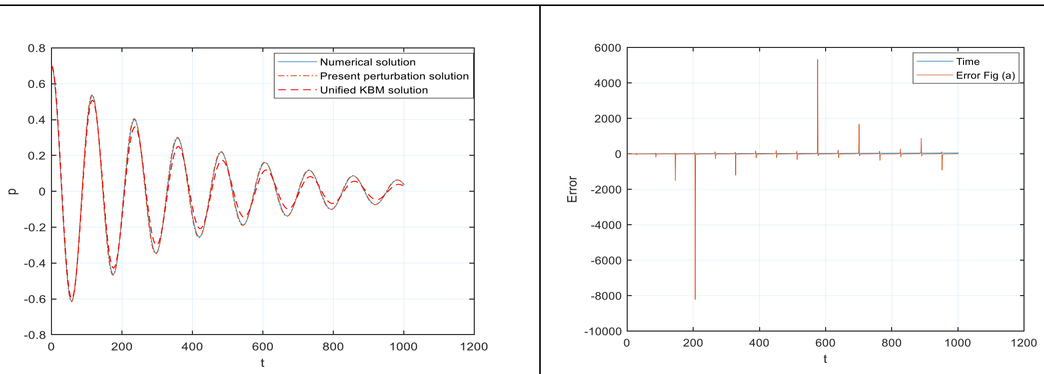
Figure 2: Initial conditions are presented alongside the perturbation solution (dotted line) and the matching numerical solution (solid line) derived from Eq. (7) $\rho = 0.7000,\ \phi = -0.012457\ [p(0) = 0.7000,\ \dot{p}(0) = 0.0000]$ for $e = 0.6$, with damping coefficient is $c_1 = .02$.

Dynamic response comparison for Eq. (7) with another set of damping parameters.



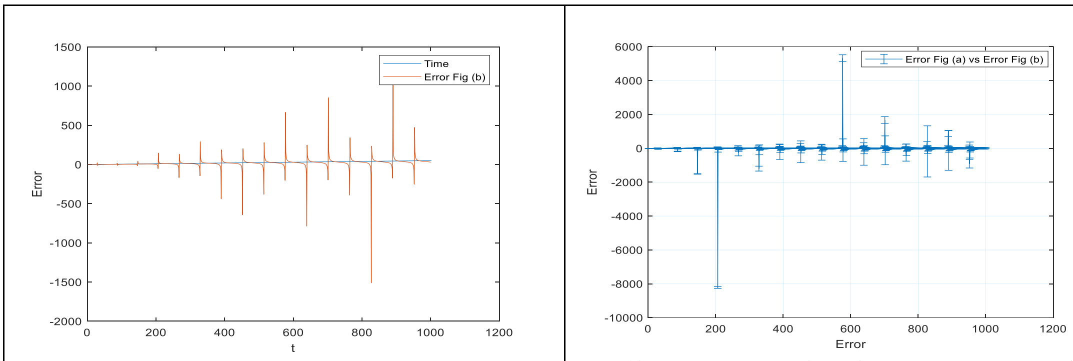
3(a). Present perturbation solution vs numerical solution

3(b). Unified KBM solution vs numerical solution



3(c). Combined comparison of all three solutions

3(d). Time vs error for present perturbation solution

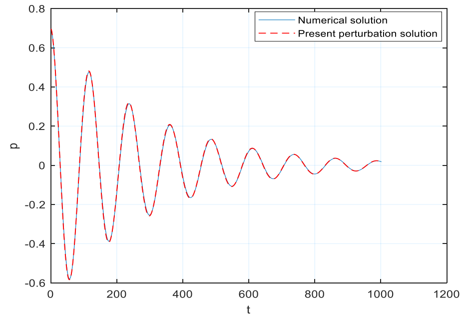


3(e). Time vs error for unified KBM solution

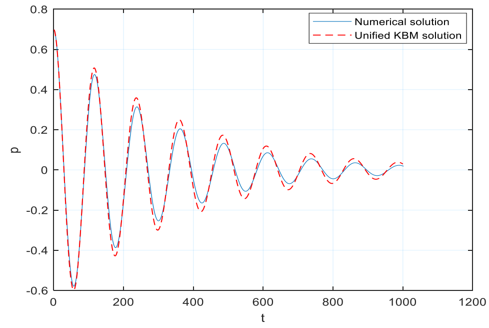
3(f). Error comparison between present and unified methods

Figure 3: Initial conditions are presented alongside the perturbation solution (dotted line) and the matching numerical solution (solid line) derived from Eq. (7) $\rho = 0.7000,\ \phi = -0.031151\ [p(0) = 0.7000,\ \dot{p}(0) = 0.0000]$ for $e = 0.6$, with damping coefficient is $c_1 = .05$.

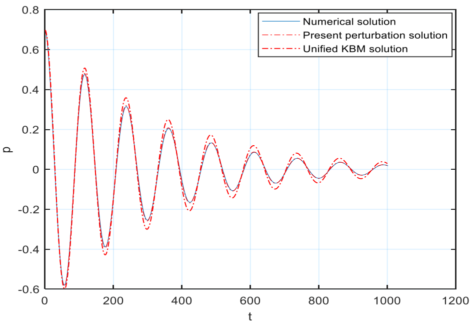
Performance evaluation of analytical methods under varying damp conditions for Eq. (7).



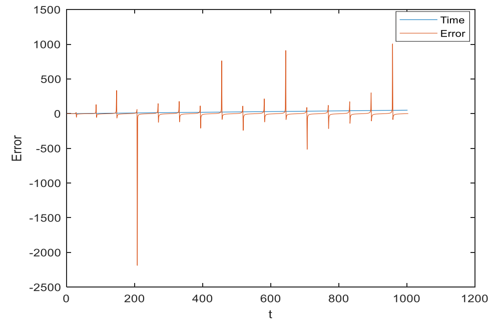
4(a). Present perturbation solution vs numerical solution



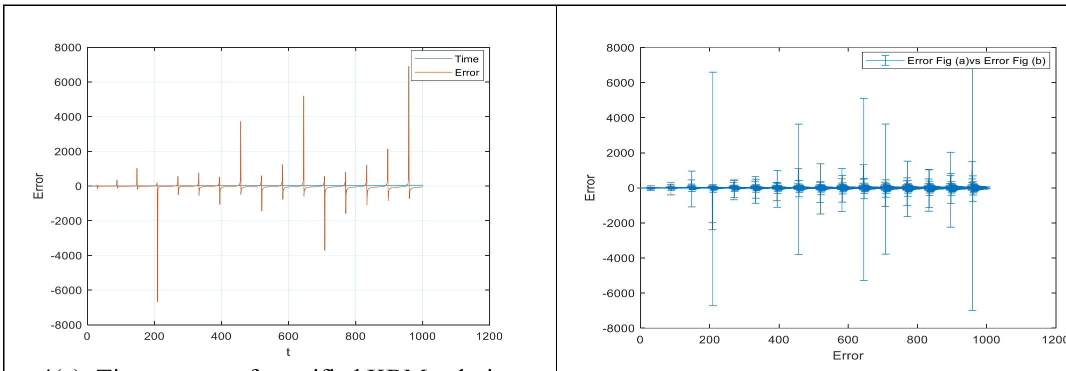
4(b). Unified KBM solution vs numerical solution



4(c). Combined comparison of all three solutions



4(d). Time vs error for present perturbation solution



4(e). Time vs error for unified KBM solution

4(f). Error comparison between present and unified methods

Figure 4: Initial conditions are presented alongside the perturbation solution (dotted line) and the matching numerical solution (solid line) derived from Eq. (7) $\rho = 0.7000,\ \phi = -0.043624\ [p(0) = 0.7000,\ \dot{p}(0) = 0.0000]$ for $e = 0.6$, with damping coefficient is $c_1 = .07$.

6. Novelty and Contributions

This paper introduces a hybrid analytical framework known as KBM–HB, which integrates harmonic balancing with amplitude–phase modulation to address the complexities of nonlinear systems. This innovative approach allows for simultaneous analysis of nonlinearity, damping, and time-varying coefficients within a unified formulation. Unlike traditional KBM methods, which are primarily suited to weakly damped systems, this framework enhances the amplitude evolution equations, making it applicable even in moderately and severely damped scenarios. A key feature of the proposed method is its systematic approach to eliminate secular components, which significantly enhances the stability and convergence of solutions compared to traditional perturbation techniques. Moreover, the framework's reduced dependency on higher-order expansions results in improved computational efficiency, allowing for more rapid processing of complex systems. Through a detailed comparative error analysis, the KBM–HB method demonstrates superior accuracy relative to the unified KBM method, particularly in contexts involving higher nonlinear parameters. This advancement positions the KBM–HB approach as both reliable and practical for comprehensive analysis of intricate nonlinear systems.

7. Conclusion

A hybrid KBM-HB analysis methodology has been developed to derive credible approximate dynamic solutions to damp nonlinear mechanical systems having slowly varying coefficients. The comparative studies have proved that the suggested approach provides results in a superb comparison with numerical simulations, which proves the accuracy and efficiency of the methodology. The combination of amplitude-phase transformation and harmonic balancing increases the ability of the method to effectively model the key nonlinear oscillatory system characteristics in the presence of strong damping. Due to its stability, simplicity, and wide applicability, the framework design provides a useful analysis tool for exploring complex engineering systems such as vibration isolation, energy harvesting, and nonlinear control applications. Despite its advantages, the method has limitations. Its accuracy may degrade for extremely strong nonlinearities or rapidly varying coefficients. Additionally, higher-order approximations increase algebraic complexity, limiting practical implementation.

Credit Authorship Contribution Statement: All authors contributed to the study conception and design. Material preparation, data analysis, validation, visualization, software, methodology was performed by **Ramjan Ali Akanda, Md. Antajul Islam**, & **Nasrin Nahar Rimu**. The first draft of the manuscript was written by **Md. Antajul Islam, Nasrin Nahar Rimu** and all authors commented on previous versions of the manuscript. **Rezaul Karim, Nasir Uddin, and Pinakee Dey** contributed to Writing – review and editing this manuscript, with serving as the supervisor. All authors read and approved of the final manuscript.

Declaration of Competing Interest: The authors affirm that they do not possess any identifiable competing interests or personal ties that could have potentially influenced the findings presented in this paper.

Data Statement: This manuscript has no associated data.

Funding: We didn’t receive any funding for this study.

Acknowledgement: The anonymous referees provided valuable feedback and suggestions to enhance the article's quality, for which the authors are grateful. The Mawlana Bhashani Science and Technology University authorities are also appreciated by the authors for their assistance in their work.

Declaration of generative AI in scientific writing: During the preparation of this manuscript, the author(s) employed QuillBot to enhance the English language in specific portions. Subsequent to utilizing the tool, the author(s) meticulously evaluated, modified, and sanctioned all content, assuming all accountability for the final iteration of the work.

Appendix -A

The nonlinear differential equation can be examined as

$$\ddot{p} + 2c_1(\tau) + (c_2 + c_3 cos\tau + c_4^2 sin2\tau)p = -\varepsilon p^3 \qquad \text{(A.1)}$$

According to [2], [7], [18], The **Harmonic Balance (HB)** method approaches Eq. (A.1) has a periodic solution in the form,

$$p(t,0) = acos\varphi + a^3 c_3 cos3\varphi + \cdots \qquad \text{(A.2)}$$

where a and c_3 are constants.

By replacing Eq. (A.2) into Eq. (A.1) and equating the coefficient of $\cos\varphi$, yields the following

$$-\dot{\varphi}^2 + \omega^2 = \frac{3\varepsilon a^2}{4}(1 + a^2 c_3) \qquad \text{(A.3)}$$

In this case $c_3 = 0$ then the Eq. (A.3) becomes

$$-\dot{\varphi}^2 + \omega^2 = \frac{3\varepsilon a^2}{4} \qquad \text{(A.4)}$$

Make this equation (A.4) simplify, we obtain

$$\dot{\varphi} = 4\omega(\sqrt{(1 - 3\varepsilon a^2/4\omega^2)}\ ) \qquad \text{(A.5)}$$

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