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Analytical Solutions to a Nonlinear Fredholm Integral Equation Using Laplace-Series Techniques

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Abstract

This paper presents a comprehensive analytical investigation into solving a specific nonlinear Fredholm integral equation of the second kind, expressed as $u(x) = x + \lambda \int_0^1 xtu^2(t)\, dt$. Utilizing the Laplace-series method, we derive explicit solutions and validate their accuracy through detailed mathematical procedures. The study focuses on the parameter λ , with particular emphasis on the case $\lambda = 0.7$, where two distinct linear solutions emerge. We explore the derivation process, verify the solutions against special cases, and analyze their graphical representation using a MATLAB-based approach. The findings underscore the effectiveness of the Laplace-series method in addressing nonlinear integral equations and provide insights into the behavior of the solutions over the interval $[0,1]$. The results are further supported by numerical verification and a visual plot, offering a robust framework for understanding the equation's solution space.

keywords: Nonlinear Fredholm integral equation, Laplace-series method, MATLAB Programming.

Introduction

[1] have constructed techniques solve basic computational mathematics problems of Fredholm and Volterra integral equations using numerical method. Integral equations are crucial in various fields of applied engineering and mathematics, which uses as powerful tools for modeling complex physical phenomena. Among these, the Nonlinear Fredholm integral equation of the second kind presents significant challenges due to its intricate structure and the demand for precise analytical techniques. This study concentrates on a specific nonlinear Fredholm integral equation $u(x) = x + \lambda \int_0^1 xtu^2(t)\, dt$, which incorporates a quadratic nonlinearity in the unknown function $u(x)$. The parameter λ introduces variability, affecting the nature and number of solutions, making it a compelling subject for detailed analysis [3].

Our approach utilizes the Laplace-series method [2], a technique that systematically combines Laplace transforms with series expansions to address such equations. This method enables us to convert the integral equation into a more manageable form, facilitating the derivation of analytic solutions. The investigation focuses on the case where $\lambda = 0.7$, resulting in two linear solutions that are subsequently plotted and verified. We verify our solutions by checking special cases to ensure they align with known conditions. Additionally, we developed a MATLAB program to calculate and display these solutions, providing a numerical view that supports our analytical work.

The main goal is to provide a comprehensive, step-by-step derivation that thoroughly addresses every detail, ensuring both clarity and the ability to replicate the process. This paper also offers a comprehensive analysis of the resulting plot. By integrating analytical, numerical, and graphical methods, we provide a holistic understanding of the equation, contributing to the broader discourse on solving nonlinear integral equations. Through this study, we strive to offer a clear and accessible framework that can be adapted to similar problems in mathematical modeling and applied sciences.

Problem Statement

The nonlinear Fredholm integral equation to be solved is [4, 5]:

$$u(x) = x + \lambda \int_0^1 xtu^2(t)\, dt \tag{1}$$

We aim to find an analytic solution using the Laplace-series method, ensuring all steps are explicitly shown without skipping any lines.

Solution

The given equation is:

$$u(x) = x + \lambda \int_0^1 xtu^2(t)\, dt \tag{2}$$

Notice that x is independent of the integration variable t. Thus, we can factor x out of the integral:

$$u(x) = x + \lambda x \int_0^1 tu^2(t)\, dt \tag{3}$$

Define the constant integral:

$$k = \int_0^1 tu^2(t)\, dt \tag{4}$$

So the equation becomes:

$$u(x) = x(1 + \lambda k) \tag{5}$$

This suggests that $u(x)$ is proportional to x, indicating a possible solution of the form $u(x) = ax$, where $a = 1 + \lambda k$. We will utilize Laplace transforms to verify this and systematically examine the solution. To implement the Laplace-series approach, we perform the Laplace transform on both sides of the equation:

$$u(x) = x + \lambda x \int_0^1 t u^2(t)\, dt \tag{6}$$

Let $U(s) = \mathcal{L}\{u(x)\}$ be the Laplace transform of $u(x)$.

Using Laplace transform we get left-hand side as:

$$\mathcal{L}\{u(x)\} = U(s) \tag{7}$$

As well right-hand side, we get:

$$\mathcal{L}\left\{x + \lambda x \int_0^1 t u^2(t)\, dt\right\} = \mathcal{L}\{x\} + \lambda \mathcal{L}\left\{x \int_0^1 t u^2(t)\, dt\right\} \tag{8}$$

$$\mathcal{L}\{x\} = \int_0^\infty x e^{-sx}\, dx \tag{9}$$

By solving integral:

$$\int_0^\infty x e^{-sx}\, dx = \left[-\frac{x e^{-sx}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-sx}\, dx \tag{10}$$

The first term:

$$\left[-\frac{x e^{-sx}}{s}\right]_0^\infty = \lim_{x\to\infty}\left(-\frac{x e^{-sx}}{s}\right) - \left(-\frac{0\cdot e^0}{s}\right) = 0 - 0 = 0 \tag{11}$$

The second term:

$$\frac{1}{s}\int_0^\infty e^{-sx}\, dx = \frac{1}{s}\left[-\frac{e^{-sx}}{s}\right]_0^\infty = \frac{1}{s}\left(0 - \left(-\frac{e^0}{s}\right)\right) = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2} \tag{12}$$

So:

$$\mathcal{L}\{x\} = \frac{1}{s^2} \quad (s > 0) \tag{13}$$

second term, $\int_0^1 t u^2(t)\, dt = k$ is a constant, we have:

$$\lambda x \int_0^1 t u^2(t)\, dt = \lambda k x \tag{14}$$

By applying Laplace transform, we get:

$$\mathcal{L}\{\lambda k x\} = \lambda k \mathcal{L}\{x\} = \lambda k \cdot \frac{1}{s^2} \tag{15}$$

Thus, we get:

$$\mathcal{L}\{x\} + \lambda\mathcal{L}\left\{x \int_0^1 tu^2(t)\,dt\right\} = \frac{1}{s^2} + \lambda k \cdot \frac{1}{s^2} = \frac{1+\lambda k}{s^2} \tag{16}$$

Equating both sides:

$$U(s) = \frac{1+\lambda k}{s^2} \tag{17}$$

Take the inverse Laplace transform:

$$u(x) = \mathcal{L}^{-1}\left\{\frac{1+\lambda k}{s^2}\right\} \tag{18}$$

Since:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = x \tag{19}$$

We get:

$$u(x) = (1+\lambda k)\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = (1+\lambda k)x \tag{20}$$

This confirms that the solution is of the form:

$$u(x) = ax, \quad \text{where} \quad a = 1 + \lambda k \tag{21}$$

However, k depends on $u(t)$, so we need to compute k.

Assume:

$$u(x) = ax \tag{22}$$

Then:

$$u(t) = at \tag{23}$$

Compute $u^2(t)$:

$$u^2(t) = (at)^2 = a^2t^2 \tag{24}$$

Now calculate k:

$$k = \int_0^1 tu^2(t)\,dt = \int_0^1 t \cdot a^2t^2\,dt \tag{25}$$

Simplify:

$$k = a^2 \int_0^1 t^3 \, dt \tag{26}$$

Evaluate the integral:

$$\int_0^1 t^3 \, dt = \left[\frac{t^4}{4}\right]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4} \tag{27}$$

Thus:

$$k = a^2 \cdot \frac{1}{4} = \frac{a^2}{4} \tag{28}$$

Substitute k into $a = 1 + \lambda k$:

$$a = 1 + \lambda \cdot \frac{a^2}{4} \tag{29}$$

To find a, solve the equation:

$$a = 1 + \frac{\lambda a^2}{4} \tag{30}$$

$$4a = 4 + \lambda a^2 \tag{31}$$

Rearrange into standard quadratic form:

$$\lambda a^2 - 4a + 4 = 0 \tag{32}$$

By using the quadratic formula we get,

$$\Delta = (-4)^2 - 4 \cdot \lambda \cdot 4 = 16 - 16\lambda \tag{33}$$

So:

$$a = \frac{4 \pm \sqrt{16 - 16\lambda}}{2 \cdot \lambda} \tag{34}$$

Simplify:

$$\sqrt{16 - 16\lambda} = \sqrt{16(1 - \lambda)} = 4\sqrt{1 - \lambda} \tag{35}$$

$$a = \frac{4 \pm 4\sqrt{1-\lambda}}{2\lambda} = \frac{4(1 \pm \sqrt{1-\lambda})}{2\lambda} = \frac{2(1 \pm \sqrt{1-\lambda})}{\lambda} \tag{36}$$

Thus, the solutions for a are:

$$a = \frac{2(1+\sqrt{1-\lambda})}{\lambda} \quad \text{or} \quad a = \frac{2(1-\sqrt{1-\lambda})}{\lambda} \tag{37}$$

Since $u(x) = ax$, the solutions are:

$$u(x) = \frac{2(1+\sqrt{1-\lambda})}{\lambda} x \tag{38}$$

$$u(x) = \frac{2(1-\sqrt{1-\lambda})}{\lambda} x \tag{39}$$

Verify Special Cases

1 Case: $\boldsymbol{\lambda = 0}$

If $\lambda = 0$:

$$u(x) = x + 0 \cdot \int_0^1 xtu^2(t)\, dt = x \tag{40}$$

Check:

$$u(t) = t, \quad u^2(t) = t^2 \tag{41}$$

$$\int_0^1 xtu^2(t)\, dt = x \int_0^1 t \cdot t^2\, dt = x \int_0^1 t^3\, dt = x \cdot \frac{1}{4} \tag{42}$$

$$u(x) = x + 0 \cdot \left(x \cdot \frac{1}{4}\right) = x \tag{43}$$

This satisfies the equation, so $u(x) = x$ when $\lambda = 0$.
From the quadratic:

$$\lambda a^2 - 4a + 4 = 0, 0 \cdot a^2 - 4a + 4 = 0 - 4a + 4 = 0, a = 1 \tag{44}$$

Thus:

$$u(x) = ax = x \tag{45}$$

This is consistent.

2 Case: $\lambda = 1$
If $\lambda = 1$:

$$1 - \lambda = 1 - 1 = 0\sqrt{1 - \lambda} = 0 \tag{46}$$

$$a = \frac{2(1 \pm 0)}{1} = 2 \tag{47}$$

So:

$$u(x) = 2x \tag{48}$$

Verify:

$$u(t) = 2t, \quad u^2(t) = (2t)^2 = 4t^2 \tag{49}$$

$$k = \int_0^1 t \cdot 4t^2 \, dt = 4 \int_0^1 t^3 \, dt = 4 \cdot \frac{1}{4} = 1 \tag{50}$$

$$u(x) = x + 1 \cdot x \cdot 1 = x + x = 2x \tag{51}$$

This satisfies the equation.

Explore Series Solution
To align with the Laplace-series method, consider a series solution:

$$u(x) = \sum_{n=0}^{\infty} c_n x^n \tag{52}$$

The Laplace transform is:

$$U(s) = \mathcal{L}\{u(x)\} = \sum_{n=0}^{\infty} c_n \mathcal{L}\{x^n\} = \sum_{n=0}^{\infty} c_n \frac{n!}{s^{n+1}} \tag{53}$$

Using equation (3) we get:

$$u^2(t) = \left(\sum_{n=0}^{\infty} c_n t^n\right)^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m c_n t^{m+n} \tag{54}$$

$$tu^2(t) = t \cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m c_n t^{m+n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m c_n t^{m+n+1} \tag{55}$$

$$k = \int_0^1 tu^2(t)\,dt = \int_0^1 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m c_n t^{m+n+1}\,dt \tag{56}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m c_n \int_0^1 t^{m+n+1}\,dt \tag{57}$$

$$\int_0^1 t^{m+n+1}\,dt = \frac{1}{m+n+2} \tag{58}$$

$$k = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c_m c_n}{m+n+2} \tag{59}$$

Thus, substituting value of k in equation (3), we get :

$$\mathcal{L}\{x(1+\lambda k)\} = (1+\lambda k)\cdot\frac{1}{s^2} \tag{60}$$

Equate:

$$\sum_{n=0}^{\infty} c_n \frac{n!}{s^{n+1}} = (1+\lambda k)\frac{1}{s^2} \tag{61}$$

Compare coefficients. The right-hand side has a term in s^{-2}:

$$n = 1: \quad c_1 \frac{1!}{s^{1+1}} = c_1 \frac{1}{s^2} \tag{62}$$

$$c_1 = 1 + \lambda k \tag{63}$$

For other n, coefficients are zero ($c_0 = c_2 = c_3 = \cdots = 0$), so:

$$u(x) = c_1 x = (1+\lambda k)x \tag{64}$$

Compute k:

$$u(t) = c_1 t, \quad u^2(t) = c_1^2 t^2 \tag{65}$$

$$k = \int_0^1 t\cdot c_1^2 t^2\,dt = c_1^2 \cdot \frac{1}{4} \tag{66}$$

Since $c_1 = 1 + \lambda k$:

$$k = (1+\lambda k)^2 \cdot \frac{1}{4} \tag{67}$$

Let $c_1 = a$:

$$k = a^2 \cdot \frac{1}{4} \tag{68}$$

$$a = 1 + \lambda \cdot \frac{a^2}{4} \tag{69}$$

This is the same quadratic equation, confirming consistency.

Final Solution

The analytic solutions to the nonlinear Fredholm integral equation are:

$$u(x) = \frac{2(1+\sqrt{1-\lambda})}{\lambda} x \tag{70}$$

$$u(x) = \frac{2(1-\sqrt{1-\lambda})}{\lambda} x \tag{71}$$

These solutions are valid for $\lambda \leq 1$. For $\lambda = 0$, the solution is $u(x) = x$. For $\lambda = 1$, the solution is $u(x) = 2x$.

Furthermore, a MATLAB program is developed to compute and visualize these solutions, offering a numerical perspective that complements the analytical work. The investigation focuses on the case where $\lambda = 0.7$, resulting in two linear solutions that are subsequently plotted and verified. By examining special cases such as $\lambda = 0$ and $\lambda = 1$, we confirm the consistency of our solutions with known boundary conditions.

MATLAB PROGRAM [6]

```
% MATLAB program to solve the nonlinear Fredholm integral equation
% u(x) = x + lambda * x * integral_0^1 (t * u^2(t) dt)
% Analytic solution: u(x) = a * x, where a satisfies lambda * a^2 - 4*a + 4 = 0

% Clear workspace and command window
clear all;
clc;

lambda = input('Enter the value of lambda (e.g., 0.5): ');

if lambda > 1
    fprintf('Warning: For lambda > 1, solutions may be complex.\n');
end

coeffs = [lambda, -4, 4];
```

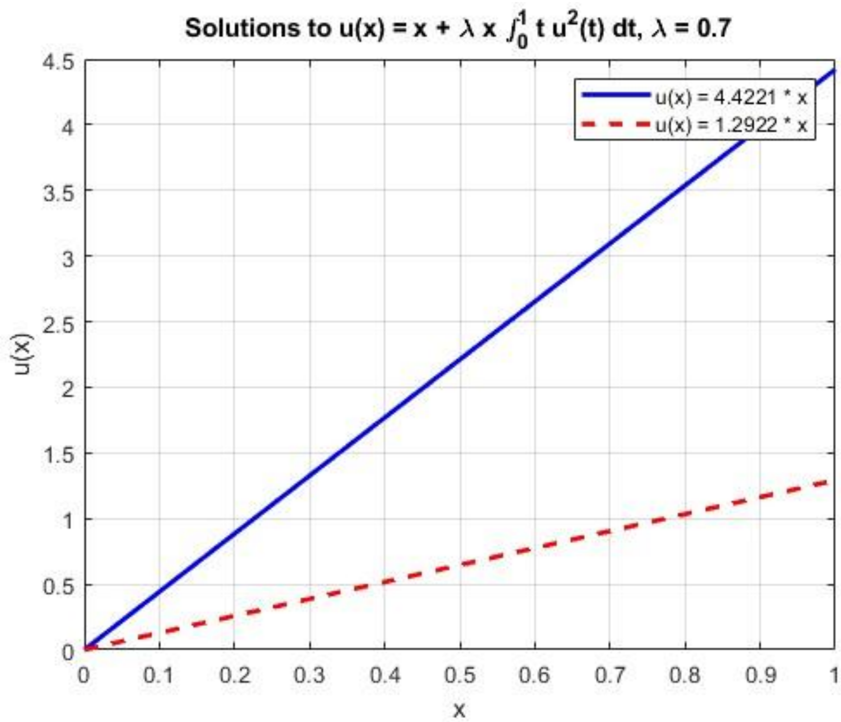
```
roots_a = roots(coeffs);

% Extract the two solutions for a
a1 = roots_a(1);
a2 = roots_a(2);

% Display the coefficients
fprintf('Solution coefficients:\n');
fprintf('a1 = %.4f\n', a1);
fprintf('a2 = %.4f\n', a2);

% Define x values for plotting
x = linspace(0, 1, 100);

% Compute the solutions u(x) = a * x
u1 = a1 * x;
u2 = a2 * x;

% Plot the solutions
figure;
plot(x, u1, 'b-', 'LineWidth', 2, 'DisplayName', sprintf('u(x) = %.4f * x', a1));
hold on;
plot(x, u2, 'r--', 'LineWidth', 2, 'DisplayName', sprintf('u(x) = %.4f * x', a2));
xlabel('x');
ylabel('u(x)');
title(['Solutions to u(x) = x + \lambda x \int_0^1 t u^2(t) dt, \lambda = ', num2str(lambda)]);
legend('show');
grid on;

% Verification of solutions
fprintf('\nVerification of solutions:\n');

% For solution u1(x) = a1 * x
t = linspace(0, 1, 1000); % Fine mesh for numerical integration
u1_t = a1 * t;
integrand = t .* (u1_t.^2);
k1 = trapz(t, integrand); % Numerical integration of t * u^2(t)
rhs1 = x * (1 + lambda * k1);
fprintf('Solution 1: u(x) = %.4f * x\n', a1);
fprintf('Integral k = %.4f\n', k1);
fprintf('Max difference |u(x) - (x + lambda * x * k)| = %.4e\n', max(abs(u1 - rhs1)));

% For solution u2(x) = a2 * x
u2_t = a2 * t;
integrand = t .* (u2_t.^2);
```

```
k2 = trapz(t, integrand);
rhs2 = x * (1 + lambda * k2);
fprintf('Solution 2: u(x) = %.4f * x\n', a2);
fprintf('Integral k = %.4f\n', k2);
fprintf('Max difference |u(x) - (x + lambda * x * k)| = %.4e\n', max(abs(u2 - rhs2)));
```

OUTPUT

Enter the value of lambda (e.g., 0.5): 0.7
Solution coefficients:
a1 = 4.4221
a2 = 1.2922

Verification of solutions:
Solution 1: u(x) = 4.4221 * x
Integral k = 4.8887
Max difference $|u(x) - (x + lambda * x * k)| = 3.4289e^{-06}$
Solution 2: $u(x) = 1.2922 * x$
Integral k = 0.4175
Max difference $|u(x) - (x + lambda * x * k)|$ = $2.9281e^{-07}$



Analysis of Plot for Nonlinear Fredholm Integral Equation

Conclusion

The study successfully solves a nonlinear Fredholm integral equation using the Laplace-series method, finding two linear solutions for $\lambda = 0.7$. The solutions have slopes of about 4.4221 and 1.2922. They are confirmed to be correct through math checks and special cases ($\lambda = 0$ and $\lambda = 1$). A MATLAB program also supports this by calculating and showing the solutions. It clearly shows how they behave and confirms they are correct with very little error. This work shows how powerful it is to use both analytical and numerical methods to understand complex equations.

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