Extension of Connecting Formulas on Hypergeometric Function

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ABSTRACT
The Hypergeometric series is the extension of the geometric series and the Confluent Hypergeometric Function is the solution of the Hypergeometric Differential Equation. Kummer has developed six solutions for the differential equation and twenty connecting formulas. The connecting formula consist of a solution expressed as the combination of two other solutions. Further extension was done by Poudel et al. This research work has extended the nine connecting formulas obtained by Poudel et al. to obtain the other nine solutions $w_1(z), w_2(z), w_3(z), w_4(z), w_5(z), and\, w_6(z)$.

KEYWORDS: *Hypergeometric series, Confluent Hypergeometric Function, Kummer's Formula, Connecting Formula*

1. INTRODUCTION

1.1 Hypergeometric series

The hypergeometric series defined by Euler and presently known as Guass hypergeometric series is represented as

$$ {}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots $$

…(1.1.1)

where a, b, and c are the scalar parameters with the variable z. The hypergeometric functions are the rational functions which undergo several transformations [9]. The Guassian hypergeometric series can be represented by the gamma function given as follows;

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \ldots (1.1.2) $$

Later Barne[2] defined the generalized hypergeometric function in terms of many scalar numerator and denominator parameters a_n and denominator parameters b_n as

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, a_3, \ldots a_p \\ b_1, b_2, b_3, \ldots b_q \end{matrix};\quad z\right] = \sum \frac{(a_1)_k (a_2)_k \ldots (a_q)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k}\frac{z^k}{k!} \quad \ldots (1.1.3) $$

On 1812 Guass mentioned that the hypergeometric functions are also the solutions of the second order hypergeometric differential equations. The hypergeometric function is the solution to the hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2}+(c-(a+b+1)z)\frac{dw}{dz}-abw=0 \qquad \ldots (1.1.4)$$

Also from the property of gamma function

$$\Gamma(a)\ \Gamma(1-a)=\frac{\pi}{\sin a\pi} \qquad \ldots(1.1.5)$$

1.2 Connection Formulas by Kummer

Kummer's connection formula is an important result in the theory of confluent hypergeometric functions, particularly the Kummer function (also called the confluent hypergeometric function of the first kind), usually denoted as:

$$M(a,b,z)={}_1F_1\left[\begin{matrix} a \\ b \end{matrix} \quad ;z\right]$$

Kummer's function $M(a,b,z)$ is a solution to the confluent hypergeometric differential equation:

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}-aw=0$$

This equation has two linearly independent solutions, and Kummer's connection formula relates the behavior of these solutions at different regions of the complex plane, particularly near zero and at infinity.

- Near $z=0$, $M(a,b,z)$ is regular and expressed as a power series:

$$M(a,b,z)=1+\frac{a}{b}\frac{z}{1!}+\frac{a(a+1)}{b(b+1)}\frac{z^2}{2!}+\ldots$$

- However, for large |z| (especially z→∞z), this expansion is no longer useful.

To analyze the behavior of the solution at infinity (or connect solutions across regions), the connection formula is needed

Kummer has showed that the Guass ordinary differential equation has three regular singular at 0,1, infinity has one solution namely ${}_2F_1(a,b;c;z)$. It is one of the twenty four solutions. There are altogether 24 solutions and Kummer has obtained 20 connection formulas for the principle branches of Kummer's solution [19] for (1.1.8). They are listed as follows;

$$w_3(z)=\frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_1(z)+\frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)}w_2(z) \qquad \ldots (1.2.1)$$

$$w_4(z)=\frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)}w_1(z)+\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z) \qquad \ldots (1.2.2)$$

$$w_5(z)=\frac{\Gamma(1-c)\Gamma(a-b+1)}{\Gamma(a-c+1)\Gamma(1-b)}w_1(z)+e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)}w_2(z) \qquad \ldots (1.2.3)$$

$$w_6(z)=\frac{\Gamma(1-c)\Gamma(b-a+1)}{\Gamma(b-c+1)\Gamma(1-a)}w_1(z)+e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(b-a+1)}{\Gamma(b)\Gamma(c-a)}w_2(z) \qquad \ldots (1.2.4)$$

$$w_1(z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}w_3(z)+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}w_4(z) \qquad \ldots (1.2.5)$$

$$w_2(z)=\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}w_3(z)+\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_4(z) \qquad \ldots (1.2.6)$$

$$w_5(z)=e^{a\pi i}\frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)}w_3(z)+e^{(c-b)\pi i}\frac{\Gamma(a-b+1)\Gamma(a+b-1)}{\Gamma(a)\Gamma(a-c+1)}w_4(z) \qquad \ldots (1.2.7)$$

$$w_6(z)=e^{b\pi i}\frac{\Gamma(b-a+1)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(c-a)}w_3(z)+e^{(c-a)\pi i}\frac{\Gamma(b-a+1)\Gamma(a+b-c)}{\Gamma(b)\Gamma(b-c+1)}w_4(z) \qquad \ldots (1.2.8)$$

$$w_1(z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}w_5(z)+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}w_6(z) \qquad \ldots (1.2.9)$$

$$w_2(z)=e^{(1-c)\pi i}\frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(1-a)\Gamma(b-c+1)}w_5(z)+e^{(1-c)\pi i}\frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(1-b)\Gamma(a-c+1)}w_6(z) \qquad \ldots (1.2.10)$$

$$w_3(z)=e^{-a\pi i}\frac{\Gamma(a+b-c+1)\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)}w_5(z)+e^{-b\pi i}\frac{\Gamma(a+b-c+1)\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}w_6(z) \qquad \ldots (1.2.11)$$

$$w_4(z) = e^{(b-c)\pi i}\frac{\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z) + e^{(a-c)\pi i}\frac{\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)}w_6(z)$$

… (1.2.12)

$$w_1(z) = e^{b\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)}w_3(z) + e^{(b-c)\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)}w_5(z) \qquad \text{… (1.2.13)}$$

$$w_2(z) = e^{a\pi i}\frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a+b-c+1)\Gamma(c-a)}w_3(z) + e^{(a-c)\pi i}\frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b-a+1)}w_6(z) \qquad \text{… (1.2.14)}$$

$$w_2(z) = e^{(b-c+1)\pi i}\frac{\Gamma(2-c)\Gamma(a)}{\Gamma(a+b-c+1)\Gamma(1-b)}w_3(z) + e^{(b-c)\pi i}\frac{\Gamma(2-c)\Gamma(a)}{\Gamma(a-b+1)\Gamma(b-c+1)}w_5(z)$$

… (1.2.15)

$$w_2(z) = e^{(a-c+1)\pi i}\frac{\Gamma(2-c)\Gamma(b)}{\Gamma(a+b-c+1)\Gamma(1-a)}w_3(z) + e^{(a-c)\pi i}\frac{\Gamma(2-c)\Gamma(b)}{\Gamma(b-a+1)\Gamma(a-c+1)}w_6(z)$$

… (1.2.16)

$$w_1(z) = e^{(c-a)\pi i}\frac{\Gamma(c)\Gamma(1-b)}{\Gamma(a)\Gamma(c-a-b+1)}w_4(z) + e^{-a\pi i}\frac{\Gamma(c)\Gamma(1-b)}{\Gamma(a-b+1)\Gamma(c-a)}w_5(z)$$

… (1.2.17)

$$w_1(z) = e^{(c-b)\pi i}\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(b)\Gamma(c-a-b+1)}w_4(z) + e^{-b\pi i}\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(b-a+1)\Gamma(c-b)}w_6(z)$$

… (1.2.18)

$$w_2(z) = e^{(1-a)\pi i}\frac{\Gamma(2-c)\Gamma(c-b)}{\Gamma(a-c+1)\Gamma(c-a-b+1)}w_4(z) + e^{-a\pi i}\frac{\Gamma(2-c)\Gamma(c-b)}{\Gamma(a-b+1)\Gamma(a-1)}w_5(z)$$

… (1.2.19)

$$w_2(z) = e^{(1-b)\pi i}\frac{\Gamma(2-c)\Gamma(c-a)}{\Gamma(b-c+1)\Gamma(c-a-b+1)}w_4(z) + e^{-b\pi i}\frac{\Gamma(2-c)\Gamma(c-a)}{\Gamma(b-a+1)\Gamma(1-b)}w_6(z)$$

… (1.2.20)

1.3 Extension of the above Connection formulas

From the above listed connection formulas (1.2.1)-(1.2.20), the combination of the three solutions for a given particular solution of the principle branches of Kummer's solution were derived by Poudel et.al [10,11]. There are altogether nine linear combination of three solutions for a single solutions. They are given as follows

$$w_1(z)=\frac{1}{\Gamma(1-c)}\left[e^{(b-c)\pi i}\frac{\Gamma(b-a)\Gamma(1-b)}{\Gamma(c-a)}w_5(z)+e^{(a-c)\pi i}\frac{\Gamma(a-b)\Gamma(1-a)}{\Gamma(c-b)}w_6(z)-\frac{\Gamma(c-1)\Gamma(1-a)\Gamma(1-b)}{\Gamma(c-a)\Gamma(c-b)}w_2(z)\right] \quad ...(1.3.1)$$

$$w_2(z)=\frac{1}{\Gamma(c-1)}\left[e^{-a\pi i}\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b-c+1)}w_5(z)+e^{-b\pi i}\frac{\Gamma(a-b)\Gamma(b)}{\Gamma(a-c+1)}w_6(z)-\frac{\Gamma(1-c)\Gamma(a)\Gamma(b)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_1(z)\right] \quad ...(1.3.2)$$

$$w_3(z)=\frac{e^{-a\pi i}}{\Gamma(c-a-b)}\left[\frac{\Gamma(1-c)\Gamma(c-b)}{\Gamma(a-c+1)}w_1(z)+e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(1-b)}{\Gamma(a)}w_2(z)-e^{(c-b)\pi i}\frac{\Gamma(a+b-1)\Gamma(1-b)\Gamma(c-b)}{\Gamma(a)\Gamma(a-c+1)}w_4(z)\right] \quad ...(1.3.3)$$

$$w_4(z)=\frac{e^{(a-c)\pi i}}{\Gamma(a+b-c)}\left[\frac{\Gamma(1-c)\Gamma(b)}{\Gamma(1-a)}w_1(z)+e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(b-c+1)}{\Gamma(c-a)}w_2(z)-e^{b\pi i}\frac{\Gamma(c-a-b)\Gamma(b)\Gamma(b-c+1)}{\Gamma(1-a)\Gamma(c-a)}w_3(z)\right] \quad ....(1.3.4)$$

$$w_5(z)=\frac{1}{\Gamma(b-a)}\left[\frac{\Gamma(c-a-b)\Gamma(b)}{\Gamma(c-b)}w_3(z)+\frac{\Gamma(a+b-c)\Gamma(c-a)}{\Gamma(a)}w_4(z)-\frac{\Gamma(a-b)\Gamma(b)\Gamma(c-a)}{\Gamma(a)\Gamma(c-b)}w_6(z)\right] \quad ....(1.3.5)$$

$$w_6(z)=e^{(c-a)\pi i}\frac{\Gamma(1-b)\Gamma(c-b)}{\Gamma(a-b)}\left[\frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}w_1(z)+\frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z)-e^{(b-c)\pi i}\frac{\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z)\right] \quad ....(1.3.6)$$

$$w_3(z)=\frac{\Gamma(a+b-c+1)\Gamma(c-b)}{\Gamma(a-c+1)}\left[\frac{e^{(c-2b)\pi i}\Gamma(1-a)}{\Gamma(b)\Gamma(c-a-b+1)}w_4(z)+\frac{e^{-2b\pi i}\Gamma(1-a)}{\Gamma(b-a+1)\Gamma(c-b)}w_6(z)-\frac{e^{-c\pi i}\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)}w_5(z)\right] \quad ..(1.3.7)$$

$$w_4(z)=\frac{\Gamma(a)\Gamma(c-a-b+1)}{\Gamma(1-b)}\left[\frac{e^{(2a-c)\pi i}\Gamma(b-c+1)}{\Gamma(a+b-c+1)\Gamma(c-a)}w_3(z)+\frac{e^{2(a-c)\pi i}\Gamma(b-c+1)}{\Gamma(a)\Gamma(b-a+1)}w_6(z)-\frac{e^{-c\pi i}\Gamma(1-b)}{\Gamma(a-b)\Gamma(a-b+1)}w_5(z)\right] \quad ....(1.3.8)$$

$$w_6(z)=\frac{e^{(c-1)\pi i}\Gamma(1-b)\Gamma(a-c+1)}{\Gamma(a-b)}\left[\frac{\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}w_3(z)+\frac{\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_4(z)-\frac{e^{(1-c)\pi i}\Gamma(b-a)}{\Gamma(1-a)\Gamma(b-c+1)}w_5(z)\right] \quad ....(1.3.9)$$

2. MAIN RESULTS

In this section, the results given above (1.3.1 – 1.3.9) are extended for a single solution to be expressed as the linear combination of other four solutions. For this, the following formulas holds.

1. $$w_1=\frac{1}{\Gamma(1-c)}\left[\begin{array}{l}\frac{e^{(b-c)\pi i}\Gamma(b-a)\Gamma(1-b)}{\Gamma(c-a)}w_5(z)+\frac{e^{(a-c)\pi i}\Gamma(a-b)\Gamma(1-a)}{\Gamma(c-b)}w_6(z)\\ -\frac{-\pi\Gamma(c-a-b)}{\sin(c-1)\pi\Gamma(c-a)\Gamma(c-b)}w_3(z)\\ -\frac{\sin(c-a)\pi\sin(c-b)\pi\Gamma(1-a)\Gamma(1-b)\Gamma(a+b-c)}{\pi\sin(c-1)\pi}w_4(z)\end{array}\right] \quad ...(2.1)$$

2. $$w_2=\frac{1}{\Gamma(c-1)}\left[\begin{array}{l}\frac{e^{-a\pi i}\Gamma(b-a)\Gamma(a)}{\Gamma(b-c+1)}w_5(z)+\frac{e^{-b\pi i}\Gamma(a-b)\Gamma(b)}{\Gamma(a-c+1)}w_6(z)\\ -\frac{\sin(c-a)\pi\sin(c-b)\pi\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\pi\sin(1-c)\pi}w_3(z)\\ -\frac{\pi\Gamma(a+b-c)}{\sin c\pi\Gamma(a-c+1)\Gamma(b-c+1)}w_4(z)\end{array}\right] \quad ...(2.2)$$

3. $$w_3=\frac{e^{-a\pi i}}{\Gamma(c-a-b)}\left[\begin{array}{l}\frac{\Gamma(1-c)\Gamma(c-b)}{\Gamma(a-c+1)}w_1(z)+e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(1-b)}{\Gamma(a)}w_2(z)\\ -\frac{e^{(a-b)\pi i}\Gamma(a+b-1)\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(a)\Gamma(b-c+1)}w_4(z)\\ -\frac{\sin a\pi\sin(c-a)\pi\Gamma(a+b-1)\Gamma(1-b)\Gamma(c-b)\Gamma(c-a-b+1)\Gamma(b-a)}{\pi^2}w_5(z)\end{array}\right] \quad ...(2.3)$$

4. $$w_4=\frac{e^{(a-c)\pi i}}{\Gamma(a+b-c)}\left[\begin{array}{l}\frac{\Gamma(1-c)\Gamma(b)}{\Gamma(1-a)}w_1(z)+e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(b-c+1)}{\Gamma(c-a)}w_2(z)\\ -e^{(b-a)\pi i}\frac{\Gamma(c-a-b)\Gamma(c-b+1)\Gamma(a+b-c+1)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)\Gamma(b-c+1)}w_5(z)\\ -b^{-2b\pi i}\frac{\sin a\pi\sin(c-a)\pi\Gamma(c-a-b)\Gamma(b)\Gamma(c-b+1)\Gamma(a-b)\Gamma(a+b-c+1)}{\pi^2}w_6(z)\end{array}\right] \quad \ldots(2.4)$$

5. $$w_5=\frac{1}{\Gamma(b-a)}\left[\begin{array}{l}\frac{\Gamma(c-a-b)\Gamma(b)}{\Gamma(c-b)}w_3(z)+\frac{\Gamma(a+b-c)\Gamma(c-a)}{\Gamma(a)}w_4(z)\\ -\frac{\sin\pi a\sin(c-b)\pi\Gamma(b)\Gamma(a-b)\Gamma(c-a)}{\pi\sin(a-b)\pi}w_1(z)\\ -\frac{\pi e^{(c-1)\pi i}}{\sin(a-b)\pi\Gamma(a)\Gamma(c-b)}w_2(z)\end{array}\right] \quad \ldots(2.5)$$

6. $$w_6=\frac{e^{(c-a)\pi i}\Gamma(1-b)\Gamma(c-b)}{\Gamma(a-b)}\left[\begin{array}{l}\frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}w_1(z)+\frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z)\\ \frac{-\pi e^{(a+b-c)\pi i}\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(c-a)\Gamma(c-b)}w_3(z)\\ -\frac{\sin a\pi\sin(c-a)\pi\Gamma(a+b-1)}{\pi\sin(b-a)\pi}w_4(z)\end{array}\right] \quad \ldots(2.6)$$

7. $$w_3=\frac{\Gamma(a+b-c+1)\Gamma(c-b)}{\Gamma(a-c+1)}\left[\begin{array}{l}\frac{e^{(c-2b)\pi i}\Gamma(1-a)}{\Gamma(b)\Gamma(c-a-b+1)}w_4(z)+\frac{e^{-2b\pi i}\Gamma(1-a)}{\Gamma(b-a+1)\Gamma(c-b)}w_6(z)\\ -\frac{e^{c\pi i}\Gamma(1-c)\sin b\pi}{\pi}w_1(z)-e^{(2c-1)\pi i}\frac{\Gamma(c-1)\Gamma(a-c+1)}{\Gamma(a)\Gamma(b)\Gamma(c-b)}w_2(z)\end{array}\right] \quad \ldots(2.7)$$

8. $$w_4=\frac{\Gamma(a)\Gamma(c-a-b+1)}{\Gamma(1-b)}\left[\begin{array}{l}\frac{e^{(2a-c)\pi i}\Gamma(b-c+1)}{\Gamma(a+b-c+1)\Gamma(c-a)}w_3(z)+\frac{e^{2(a-c)\pi i}\Gamma(b-c+1)}{\Gamma(a)\Gamma(b-a+1)}w_6(z)\\ -\frac{e^{-c\pi i}\Gamma(1-c)\sin(c-a)\pi}{\pi}w_1(z)-\frac{e^{-\pi i}\Gamma(1-b)\Gamma(c-1)}{\Gamma(a)\Gamma(c-b)\Gamma(c-a)}w_2(z)\end{array}\right] \quad \ldots(2.8)$$

9. $$w_6(z)=\frac{e^{(c-1)\pi i}\Gamma(1-b)\Gamma(a-c+1)}{\Gamma(a-b)}\left[\begin{array}{l}\frac{\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}w_3(z)+\frac{\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_4(z)\\ -\frac{e^{(1-c)\pi i}\Gamma(1-c)\pi}{\Gamma(1-a)\Gamma(b-c+1)\Gamma(a-c+1)\Gamma(1-b)\sin(b-a)\pi}w_1(z)\\ -\frac{\Gamma(c-1)\sin(c-b)\pi}{\Gamma(a)\Gamma(1-b)\sin(b-a)\pi}w_2(z)\end{array}\right]\ldots(2.9)$$

Proof:

The proofs of (2.1) – (2.9) are given as follows;

1. From,(1.2.6) and (1.3.1),

$$w_1(z)=\frac{1}{\Gamma(1-c)}\left[\begin{array}{l}\frac{e^{(b-c)\pi i}\Gamma(b-a)\Gamma(1-b)}{\Gamma(c-a)}w_5(z)+\frac{e^{(a-c)\pi i}\Gamma(a-b)\Gamma(1-a)}{\Gamma(c-b)}w_6(z)\\ -\frac{\Gamma(c-1)\Gamma(1-a)\Gamma(1-b)}{\Gamma(c-a)\Gamma(c-b)}\times\left[\begin{array}{l}\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}w_3(z)\\ +\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_4(z)\end{array}\right]\end{array}\right]$$

$$=\frac{1}{\Gamma(1-c)}\left[\begin{array}{l}\frac{e^{(b-c)\pi i}\Gamma(b-a)\Gamma(1-b)}{\Gamma(c-a)}w_5(z)+\frac{e^{(a-c)\pi i}\Gamma(a-b)\Gamma(1-a)}{\Gamma(c-b)}w_6(z)\\ -\frac{\Gamma(c-1)\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}w_3(z)\\ -\frac{\Gamma(c-1)\Gamma(1-a)\Gamma(1-b)\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a-c+1)\Gamma(b-c+1)}w_4(z)\end{array}\right]$$

$$=\frac{1}{\Gamma(1-c)}\left[\begin{array}{l}\frac{e^{(b-c)\pi i}\Gamma(b-a)\Gamma(1-b)}{\Gamma(c-a)}w_5(z)+\frac{e^{(a-c)\pi i}\Gamma(a-b)\Gamma(1-a)}{\Gamma(c-b)}w_6(z)\\ -\frac{\Gamma(c-1)\Gamma(1-(c-1))\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}w_3(z)\\ -\frac{\Gamma(c-1)\Gamma(1-(c-1))\Gamma(1-a)\Gamma(1-b)\Gamma(a+b-c)}{\Gamma(c-a)\Gamma(1-(c-a))\Gamma(c-b)\Gamma(1-(c-b))}w_4(z)\end{array}\right]$$

Now using (1.1.5), we get

$$=\frac{1}{\Gamma(1-c)}\left[\begin{array}{l}\frac{e^{(b-c)\pi i}\Gamma(b-a)\Gamma(1-b)}{\Gamma(c-a)}w_5(z)+\frac{e^{(a-c)\pi i}\Gamma(a-b)\Gamma(1-a)}{\Gamma(c-b)}w_6(z)\\ -\frac{-\pi\Gamma(c-a-b)}{\sin(c-1)\pi\Gamma(c-a)\Gamma(c-b)}w_3(z)\\ -\frac{\sin(c-a)\pi\sin(c-b)\pi\Gamma(1-a)\Gamma(1-b)\Gamma(a+b-c)}{\pi\sin(c-1)\pi}w_4(z)\end{array}\right]$$

2. Using (1.2.8) , (1.3.2) and from (1.1.5)

$$W_2(z)=\frac{1}{\Gamma(c-1)}\left[\begin{array}{l}\frac{e^{-a\pi i}\Gamma(b-a)\Gamma(a)}{\Gamma(b-c+1)}w_5(z)+\frac{e^{-b\pi i}\Gamma(a-b)\Gamma(b)}{\Gamma(a-c+1)}w_6(z)\\ -\frac{\Gamma(1-c)\Gamma(a)\Gamma(b)}{\Gamma(a-c+1)\Gamma(b-c+1)}\\ \times\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}w_3(z)+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}w_4(z)\right]\end{array}\right]$$

$$=\frac{1}{\Gamma(c-1)}\left[\begin{array}{l}\frac{e^{-a\pi i}\Gamma(b-a)\Gamma(a)}{\Gamma(b-c+1)}w_5(z)+\frac{e^{-b\pi i}\Gamma(a-b)\Gamma(b)}{\Gamma(a-c+1)}w_6(z)\\ -\frac{\Gamma(1-c)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a-c+1)\Gamma(b-c+1)}w_3(z)\\ -\frac{\Gamma(c)\Gamma(1-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_4(z)\end{array}\right]$$

$$=\frac{1}{\Gamma(c-1)}\left[\begin{array}{l}\frac{e^{-a\pi i}\Gamma(b-a)\Gamma(a)}{\Gamma(b-c+1)}w_5(z)+\frac{e^{-b\pi i}\Gamma(a-b)\Gamma(b)}{\Gamma(a-c+1)}w_6(z)\\ -\frac{\sin(c-a)\pi\sin(c-b)\pi\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\pi\sin(1-c)\pi}w_3(z)\\ -\frac{\pi\Gamma(a+b-c)}{\sin c\pi\Gamma(a-c+1)\Gamma(b-c+1)}w_4(z)\end{array}\right]$$

3. Using (1.2.1) ,(1.3.3) and from (1.1.5)

$$w_3(z)=\frac{e^{-a\pi i}}{\Gamma(c-a-b)}\left[\begin{array}{l}\frac{\Gamma(1-c)\Gamma(c-b)}{\Gamma(a-c+1)}w_1(z)+\frac{e^{(c-1)\pi i}\Gamma(c-1)\Gamma(1-b)}{\Gamma(a)}w_2(z)\\ -\frac{e^{(c-b)\pi i}\Gamma(a+b-1)\Gamma(1-b)\Gamma(c-b)}{\Gamma(a)\Gamma(a-c+1)}\\ \times\left[\frac{e^{(b-c)\pi i}\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z)+\frac{e^{(a-c)\pi i}\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)}w_6(z)\right]\end{array}\right]$$

$$=\frac{e^{-a\pi i}}{\Gamma(c-a-b)}\left[\begin{array}{l}\frac{\Gamma(1-c)\Gamma(c-b)}{\Gamma(a-c+1)}w_1(z)+\frac{e^{(c-1)\pi i}\Gamma(c-1)\Gamma(1-b)}{\Gamma(a)}w_2(z)\\ -\frac{\Gamma(a+b-1)\Gamma(1-b)\Gamma(c-b)\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(a)\Gamma(a-c+1)\Gamma(1-a)(c-a)}w_5(z)\\ -\frac{e^{(a-b)\pi i}\Gamma(a+b-1)\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}w_6(z)\end{array}\right]$$

$$=\frac{e^{-a\pi i}}{\Gamma(c-a-b)}\left[\begin{array}{l}\frac{\Gamma(1-c)\Gamma(c-b)}{\Gamma(a-c+1)}w_1(z)+\frac{e^{(c-1)\pi i}\Gamma(c-1)\Gamma(1-b)}{\Gamma(a)}w_2(z)\\ -\frac{\Gamma(a+b-1)\Gamma(1-b)\Gamma(c-b)\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(a)\Gamma(1-a)\Gamma(c-a)\Gamma(1-(c-a))}w_5(z)\\ -\frac{e^{(a-b)\pi i}\Gamma(a+b-1)\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(a)\Gamma(b-c+1)}w_6(z)\end{array}\right].$$

$$=\frac{e^{-a\pi i}}{\Gamma(c-a-b)}\left[\begin{array}{l}\frac{\Gamma(1-c)\Gamma(c-b)}{\Gamma(a-c+1)}w_1(z)+\frac{e^{(c-1)\pi i}\Gamma(c-1)\Gamma(1-b)}{\Gamma(a)}w_2(z)\\ -\frac{e^{(a-b)\pi i}\Gamma(a+b-1)\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(a)\Gamma(b-c+1)}w_4(z)\\ -\frac{\sin a\pi\sin(c-a)\pi\Gamma(a+b-1)\Gamma(1-b)\Gamma(c-b)\Gamma(c-a-b+1)\Gamma(b-a)}{\pi^2}w_5(z)\end{array}\right]$$

Similarly the other results holds.

3. CONCLUSION

The hypergeometric function is the solution of the second order differential equation. Kummer has twenty connecting formulas for the second order hypergeometric differential equation. Through these formulas listed in (1.2.1-1.2.20) and (1.3.1-1.3.9) respectively, for the hypergeometric differential equation, I have obtained six extensions [(2.1.1)-(2.1.9)] of the connecting formulas for all six solutions. Every solution are expressed as the linear combination of other four solutions. Kummer's connection formula provides a bridge between solutions of the confluent hypergeometric equation near the origin and those valid at infinity. It plays a key role in understanding the full analytic behavior of special functions across the complex plane, in quantum mechanics for Hydrogen atom solutions, in mathematical physics specially in the field of perturbation theory, wave functions, numerical analysis for accurate computation of hypergeometric functions and finally on asymptotic analysis for Matching solutions in different regions.

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