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Mathematical Modelling and Argumentation: Designing a Task to Strengthen Variational Thinking by Integrating Data Science

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Abstract: This study explores the role of task design in promoting variational thinking through the PyLVar principles in mathematics education, with a focus on incorporating technological and scientific tools to enhance modelling and argumentation skills among high school students. Interactive mathematical tasks based on the PyLVar principles were developed and implemented, integrating data science and STEAM tools. A qualitative-descriptive approach was used to design, implement, and analyse these tasks, which centred on variational thinking, argumentation, and modelling. The sample consisted of nine 11th-grade students from a public school, selected for convenience. Data were collected through experimental activities, processed with technological tools, and analysed using variational strategies. The students showed significant improvement in their mathematical modelling and argumentation abilities. The tasks, grounded in PyLVar principles, facilitated the identification of variation patterns and promoted critical and analytical thinking. Contextualized interactive tasks not only enhanced variational thinking but also prepared students to solve real-world problems by linking abstract concepts to practical applications, while fostering the integration of technological tools and interdisciplinary approaches.

Keywords: Data science, Mathematics, Mathematical competences, STEAM, Variational thinking

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1. Introduction

In recent years, variational thinking has gained prominence in mathematics education as it enables the understanding of how variables change and relate in everyday life and science (Ospina, 2013). Beyond mere numbers, it involves imagining dynamic systems that reveal patterns and connections between these changes.

In Colombia, the Ministerio de Educación Nacional provides a broader perspective on variational thinking, emphasizing that working with variation and change goes beyond observing numbers. It involves perceiving, modelling, and representing how things relate and evolve in different contexts. This approach not only facilitates analyzing changes in variables but also develops mathematical modelling skills by creating mental models that reflect systems of change and connections, helping identify recurring patterns in the real world ([MEN], 1998).

Similarly, there is a complementary perspective on variational thinking, describing it as a dynamic reasoning process in which mental models are constructed to represent systems of change and dependence among internal variables, enabling the identification of patterns that recur in reality. This process includes several key stages: first, observing the changes and constants of a phenomenon; then identifying recurring patterns, such as temperature fluctuations or the motions involved in free fall and parabolic trajectories (Vasco, 2006). From there, mental models are created to replicate these behaviors and are compared with real phenomena, adjusting them as needed. This type of reasoning is crucial for students to understand the application of mathematics in everyday life, highlighting the importance of educational strategies that directly connect theory with the real world.

Posada et al. (2006) argue that mathematical modelling requires activating fundamental processes such as measuring, recording, and establishing relationships between magnitudes and quantities. However, despite its significance in educational programmes, the development of variational thinking faces significant challenges, such as limited teaching strategies and the predominance of traditional methods focused on algorithms detached from reality. This makes it difficult to meet students' needs within the current educational system (Arciniegas, 2019).

Students entering secondary school in Ecuador face serious challenges in understanding and managing the content of the national curriculum (Giler-Medina & Alcívar-Castro, 2022). These difficulties are particularly evident in their ability to interpret, model, and represent everyday phenomena, both real and simulated. The results of the PISA-D test clearly illustrate this issue, with an average score of 377 points, placing Ecuadorian students at Level two in Mathematics according to the organizations economic co-operation and development and the Instituto Nacional de Evaluación Educativa ([INEVAL] & [OCDE], 2018). This deficit is not an isolated case but is directly linked to the pedagogical strategies used in classrooms and the way students interact with these methodologies.

Colombia obtained an average score of 383 in the PISA tests, ranking sixth out of 13 countries in the region, although it still falls below the global average in key mathematical skills (OECD, 2023). The 2022 PISA tests assessed four subscales: space and shape, quantity, change and relationships, and uncertainty and data.

In the subscale of change and relationships, related to variational thinking, Colombia scored an average of 382, far below the OECD average of 470. This result shows an improvement compared to 2012, when the country scored only 357 points in the same subscale, reflecting a 25-point gain over the decade. However, despite surpassing the Latin American average by 4 points, the progress remains insufficient to close the significant gaps in mathematical competencies (ICFES, 2024).

In the PISA tests, the cycle of mathematical modeling is considered essential for understanding how students actively approach and solve problems (Acebo-Gutiérrez & Rodríguez-Gallegos, 2021). Variational thinking and mathematical modeling are key in primary and secondary education, helping students understand concepts such as functions and derivatives, and promoting meaningful learning (Guarín-Linares, 2015). However few teaching strategies at basic and secondary levels present variation in an applied manner, hindering the development of a proper understanding of the relationships between variables (Torres, 2020). This gap limits the learning of essential concepts for variational thinking and leaves students without the tools needed to analyses complex situations that require managing changes in multiple variables. On the other hand, mathematical modelling involves establishing relationships between the real world and mathematics, facilitating decision-making and understanding phenomena. This process enables students to see how mathematical tools can be applied to everyday situations, equipping them to actively engage with the real world (Bassanezi, 2002).

Nevertheless, although curricular guidelines promote a theoretical-systemic approach focused on developing mathematical thinking and problem-solving, a direct connection is often lacking between mathematical problems in the classroom and the socio-cultural realities of students (Villareal & Eliécer, 2008). This disconnection affects the authenticity of modelling, as students fail to apply mathematics to situations relevant to their environment. This lack of connection highlights the challenges of teaching modelling, particularly when attempting to integrate examples from standardized populations that incorporate every day, social, and cultural contexts, which are vital for a deep understanding of variational thinking (Cabezas & Mendoza, 2016). Despite the importance of variational thinking in the curriculum, the disconnect between theory and practice in classrooms limits the functional understanding of relationships between variables (Dolores-Flores & Mosquera-García, 2022).

Various studies suggest that, when working on contextualized tasks, students not only improve their mathematical skills but also learn to identify connections between variables, discover patterns, and quantify qualitative information, enabling deeper learning (Salcedo-Talamantes et al., 2021). In the classroom environment, these types of open-ended activities, which do not have a single correct answer, foster a collaborative and discussion-rich setting where students can share ideas and build arguments, thereby strengthening their capacity for mathematical reasoning (Solar & Deulofeu, 2016).

Given the lack of focus on teaching variation, modelling tasks in practical contexts emerge as an effective way to fill this gap. This approach does not solely depend on teacher support but relies on the design of activities that allow students to explore and reflect independently, improving not only their analytical skills but also their autonomy in the learning process. Thus, contextualized modelling tasks serve as an essential bridge for students to understand how to apply variational thinking to real-world situations.

Research supports the importance of this approach, as it shows that contextualized tasks involving modelling and mathematical argumentation not only help students grasp abstract concepts but also develop critical analytical skills. A study aimed to evaluate the impact of a pedagogical intervention to promote variational thinking in primary school students. The results demonstrated that, following the intervention, several students improved their level of abstraction and algebraic modelling through the formulation of equations and the use of graphs and tables (Suarez et al., 2022).

A study at the Agronomy University in Argentina showed that teaching variation and function concepts through practical tasks helped engineering students apply these concepts to real phenomena. Students learned to interpret graphs and coordinate variables, transferring this knowledge to real situations despite initial challenges. This contrast suggests that a task-based approach could provide Colombian students with similar opportunities to develop variational skills, bridging the gap between theory and practice (Vrancken et al., 2016).

In Colombia, at the Colegio Americano de Bogotá with a group of 20 fifth-grade students in the mathematics talent programme demonstrated how the use of an interactive tool allowed them to identify patterns related to variation. As a result, most students were able to formulate conjectures based on what they visualized with the tool, and in some cases, these conjectures were validated and deeply argued mathematically (Acosta-Hernández et al., 2015).

a. Fundamental Components of PyLVar

The research approach to variational thinking (PyLvar) in the Departamento de Matemáticas del Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional in Mexico (Salinas, 2003) emphasizes the importance of studying phenomena and situations where various variation strategies, such as prediction, estimation, and sequencing, are applied. These strategies are integrated into the design of mathematical tasks that address situations with specific variational structures. In this context, the role of the teacher is crucial in creating instructions that link different areas of knowledge to mathematical concepts. Additionally, teachers must organize strategies that facilitate the learning of mathematics and technology, aligning with established educational objectives (Caballero-Pérez & Uriza, 2013; D'Ambrosio, 2008).

A Variational Task [VT] is an activity that is part of a Variational Situation [VS], in which the actions and objectives align with the context and purpose of the task. These tasks utilize Variational Strategies [VS] that enable the exploration and analysis of changes in variables within a given context, whether numerical, graphical, or analytical. These strategies include Comparison: Identifying differences between states of change. Prediction: Anticipating behaviors based on prior analysis. Sequencing: Analyzing consecutive states and establishing relationships. Estimation: Proposing new states based on observed patterns (Caballero-Pérez & Uriza, 2013; Salinas, 2003).

The goal is to organize the study of variation through specific actions. Variational Tasks (VT) are categorized based on the type of analysis they involve Tabulation as numerical variation [TNV] which focuses on generating and analyzing data by assigning values to variables. Analysis of data in numerical tables [ADT] that aims to identify patterns in data, also Graph construction with variation as a reference [GCV] allows for graphical representation of data to model situations and last but not least, graphical analysis with variation as a reference [GAV]: Dedicated to exploring patterns and relationships in graphs (Caballero-Pérez & Uriza, 2013).

MEN (2006) highlights that science, technology, and mathematics are fundamental for developing scientific and critical thinking among Colombian students. In current learning processes, two key aspects are identified: the cognitive and motivational dimensions, the latter being of great relevance as it influences the knowledge students acquire within the educational framework (Causil-Vargas & Rodríguez De la Barrera, 2021). Changes in the teaching of mathematics, science, and technology can transform attitudes toward these subjects through active, inquiry-based didactic strategies that promote a positive attitude toward learning (Quevedo-Benítez et al., 2024).

Despite limitations such as a lack of resources, computer equipment, and connectivity, especially among more experienced teachers incorporating educational technology offers long-term benefits (Blanco-Iturralde et al., 2024). Teaching strategies that include competencies in data science and mathematical modelling strengthen variational thinking and prepare students for the digital and professional challenges of the 21st century (Horton & Hardin, 2021).

Recently, artificial intelligence [AI] and data science [DS] have experienced significant advancements driven by big data, large-scale models, and innovative technological applications (Cao, 2023). This progress has not only been technical but has also spurred a renewal of paradigms, theories, and methodologies, focusing on more integrative and flexible approaches. Data-driven research, regarded as the fourth scientific paradigm, has revolutionized the way problems are addressed in AI and DS, facilitating evidence-based investigations (Ma et al., 2019). However, challenges such as data quality, complexity, and bias persist, emphasizing the need to combine data-driven analysis with independent perspectives to achieve more robust research outcomes (Alves et al., 2021).

Transdisciplinary thinking is emerging as a key solution to addressing current complexities in AI and DS, integrating various disciplines and offering innovative approaches to contemporary challenges (Cao, 2023).

Similarly, practical strategies foster a positive attitude toward mathematical learning and enhance students' confidence in their analytical abilities, preparing them for environments where interpretation and informed decision-making are increasingly crucial (Vera & Vera, 2021). Therefore, investing in these strategies not only benefits students in the present but also helps them develop essential skills for the future.

Data science involves not only analyzing large volumes of information but also organizing, visualizing, and interpreting patterns and relationships between variables—skills directly tied to variational thinking (VanderPlas, 2016; Zheng, 2017). By integrating this knowledge, students not only gain a deeper understanding of mathematics but also improve their ability to make informed decisions in complex situations, such as those encountered in STEAM disciplines. These skills will be fundamental for tackling 21st-century challenges, as demonstrated by advancements in data science and other technological fields (Bosch et al., 2015; Cruz, 2017; Rodríguez-de la Barrera & Genes-Quintero, 2024).

Variational thinking is recognized as a fundamental aspect of mathematics education, serving as a critical foundation for understanding change across diverse contexts (Naranjo-Barahona & Caro-Roldan, 2023). In Colombia, the Ministry of National Education highlights the need to integrate variation into mathematics teaching, especially in everyday scenarios. Despite this, many educational practices remain theoretical, lacking practical applications. Research shows that context-based tasks enhance the learning of variation, equipping students with abstract reasoning and critical skills essential for addressing challenges.

This research emphasises the need for designed tasks based on PyLVar principles to develop variational thinking, modelling, and argumentation skills in 11th-grade students. These tasks incorporate data science and physics tools, encouraging dynamic reasoning to identify patterns of change. They focus on practical scenarios, involving analysis of variables, relationships, and changes. The research aim is to strengthen students' mathematical, modelling, and critical thinking skills, essential for their academic and professional development.

2. Methodology

A qualitative-descriptive approach was used to interpret mathematics teaching phenomena, focusing on developing mathematical concepts like derivatives and physical laws. Empirical findings, guided by observations and prior studies, validated the thesis through identified patterns (Hernández-Sampieri & Mendoza, 2018).

This study highlights the importance of connecting mathematics to problem-solving in 11th-grade students by integrating physics and data science tools. This flexible approach uses results to guide both theoretical development and educational strategies.

The methodology aimed to integrate tasks that connect mathematics with disciplines like physics, using data science tools. The focus was on fostering variational thinking, modelling, and argumentation skills in high school students.

An analysis of the STEAM approach was conducted, narrowing from a broad perspective to a specific case in Barranquilla, Colombia. Despite good infrastructure, the school lacked internet access, requiring alternative strategies like local applications.

The study focused on nine 11th-grade students, aged 16, selected based on their interest in mathematics and availability, following accessibility criteria (Otzen & Manterola, 2017).

Task design within the STEAM framework fosters mathematical argumentation, modelling, and variational thinking, as noted by (Cervantes-Barraza & Aroca-raujo, 2023; Zheng, 2017).

This project-based learning approach connects theoretical concepts to practical applications. It aims to strengthen analytical and creative skills while building confidence, promoting decision-making, and developing key abilities.

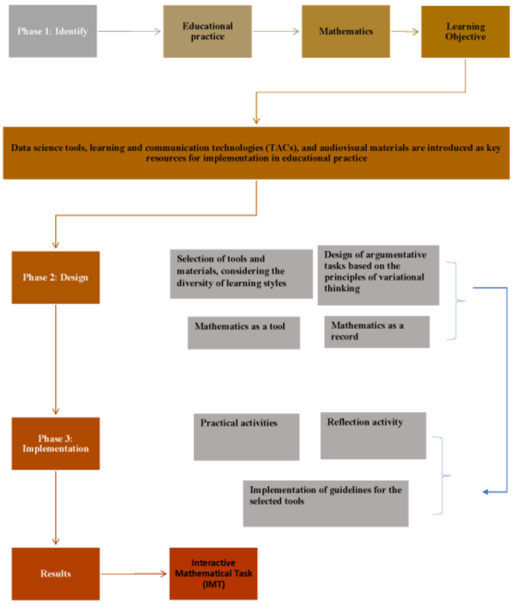


Fig. 1 Adaptation of an Analytical Framework for Task Design Proposed by Cervantes-Barraza and Araujo.

The fundamentals principles of the research line on variational thinking (PyLVar) and task design lie in an approach that promotes a deep understanding of mathematical and scientific concepts. The following outlines the necessary stages for designing educational tasks within this framework:

In stage 1, the teacher begins by identifying a scientific phenomenon to support the educational task. This phenomenon, such as free fall, is analyzed in relation to mathematical knowledge and learning objectives, establishing meaningful connections between disciplines (Cervantes-Barraza & Aroca-Araujo, 2023; Huru et al., 2018).

Moreover, in stage 2 "task design" includes four key aspects:

Variational Situation (VS): Refers to the proposed instructions that require variational strategies for analysis. Specific Variational Structure [SVS]: Represents the mathematical and scientific tools and procedures employed. In this project, tools such as Tracker and Anaconda (Python 3 offline) were used to explore concepts of change and variation. Variational Codes [VC]: Correspond to the responses, whether oral or written, that demonstrate the analysis of variation. Variational Argument [VA]: Refers to conclusions derived from analyzing the VS, reflecting a deep understanding of change. Variational Strategies [VS] are implemented to support the [VA] (Caballero-Pérez & Uriza, 2013; Salinas, 2003).

The teacher acts as a guide, linking stages 1 and 2 so that in the next stage, students can achieve meaningful learning. And last but not least, in the stage of stage 3 "task implementation", students and teachers collaborate using the selected tools. With the STEAM approach as a foundation, the teacher fosters ownership of learning by connecting mathematical tasks with specific scientific phenomena (Cervantes-Barraza & Aroca-Araujo, 2023; Huru et al., 2018).

b. Fundamental Components of Educational Tasks

Cognitive Demand Level (P1):

Determines the level of thinking required. High-demand tasks promote complex analysis and the use of multiple representations, while low-demand tasks focus on basic rules and formulas (Boston & Smith, 2009).

Task Formulation (P2):

Tasks can be presented through narratives or questions, either open-ended or closed, adapting to the educational context (Cabañas-Sánchez & Cervantes-Barraza, 2019).

Argument Confrontation Management (P3):

Tasks should stimulate the confrontation of ideas, fostering critical thinking and solid argumentation (Rumsey & Langrall, 2016; Solar et al., 2023).

In the next section, specific tasks are described, highlighting their structure, objectives, principles, and application in an educational context. These tasks are framed as an educational project, supported by real data and codes necessary for implementation. The Python programming language codes were designed and developed by the advisor in collaboration with preservice teachers and artificial intelligence tools and were subsequently provided to 11th-grade students at Betania Norte.

c. Design of interactive mathematical tasks

Task 1: Preliminary Steps

Table 1. Initial stage: Tabulation as Numerical Variation (TNV)

Learning objective:	To understand mathematical concepts related to algebra and precalculus through the experimental modelling of physical phenomena, such as free fall, using Tracker, Anaconda: Jupyter Notebook, and Python language, grounding the analysis on variational reasoning
Mathematical concept:	Limit of a function
Data base	Real data based on a real experiment
Data base link:	DATOS CAIDA LIBRE 200CM.xlsx
Task	As an introduction to a diagnostic test, the following questionnaire is jointly developed in Canvas. Afterward, watch the following video presented in the Infographic The goal is to give students an understanding of the concept of derivatives. Next, students must watch a guided YouTube video to carry out a free-fall experiment with a golf ball dropped from two metre. They will use the Tracker tool to model the motion, recording time and height. These data points will be entered into Excel, and with the help of specific codes, students will model their graph by entering the data into the ANACONDA platform using the Jupyter Notebook server (VS). **VS Questions:** 1. What height does the ball reach when the time approaches 0.6 seconds? What mathematical concept applies in this situation? (VS: Estimation) 2. Observing the data for time and height, both in the table and graph, what would

happen if the ball exceeded the maximum height?

(VS: Prediction)

Assessment Criteria:

P1 (Cognitive Demand): Medium-high level, as it requires tabulating data and recognising initial patterns of variation.

P2 (Formulation): Presented through technological tools like Tracker and Python, with open-ended instructions related to free fall.

P3 (Argument Confrontation): Allows initial discussions to substantiate basic findings using experimental data.

Python Codes

SVS:

```
#Step 1: Import the necessary libraries
import pandas as pd
import matplotlib.pyplot as plt
#Step 2: Read data from the Excel file
data = pd. read_excel ('/content/DATOS CAIDA LIBRE
200CM.xlsx')
#Step 3: Assign data columns to variables (adjust column
names as per the file)
x_data = data['t'].values  # Time
y_data = data['y'].values  # Height
#Step 4: Print the Excel table data for verification
print("Experimental data:")
print(data)
#Step 5: Plot the experimental points without fitting
plt.scatter(x_data, y_data, label='Experimental data',
color='blue')
plt.xlabel('Time (s)')
plt.ylabel('Height (m)')
plt.title('Free fall of an object - Experimental data')
plt. legend ()
```

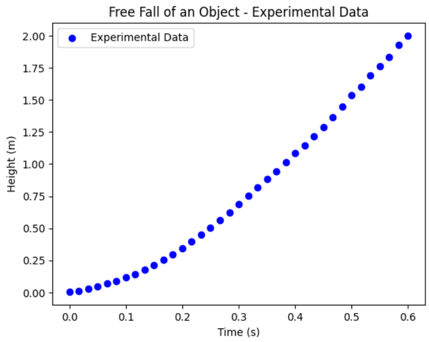


Fig. 2 Table of Experimental Data and Function Modelling of the Experiment through Jupyter Notebook

Task 2: Analysis of the Behavior of a Golf Ball's Free Fall

Table 2. Development stage: Principle of Graph Construction with Variation as a Reference (GCV)

Learning objective:	Use programming tools, such as Jupyter Notebook and the Python language, to graphically represent experimental data, with the aim of fitting a quadratic function that describes the trajectory of the motion and verifying some of the physics laws involved, supported by a variational argument
Mathematical concept:	First Derivative: By Definition
Data base	Real data based on a real experiment
Data base link:	DATOS CAIDA LIBRE 200CM.xlsx
Task	Using the experimental data from the free fall of a golf ball: **VS Questions:** 1. What type of function represents the behaviour of the graph? (VS: Serialisation) 2. Select five points from the x-axis in ascending order, substitute them into the first derivative equation, and compare them. What can you deduce from this? (VS: Prediction) **Assessment Criteria:** P1 (Cognitive Demand): High, as it involves constructing graphs and fitting quadratic functions that model trajectories. P2 (Formulation): A technological task in Jupyter Notebook, where graphs are explored using variation as a reference using open-ended questions. P3 (Argument Confrontation): Encourages students to argue about the models created, supporting their analyses with evidence.

Python Codes	**SVS:**
	`#Step 1: Import the additional required libraries`
	`import numpy as np`
	`from scipy.optimize import curve_fit`
	`import sympy as sp`
	`import matplotlib.pyplot as plt`
	`#Step 2: Define the quadratic function`
	`def quadratic_function(x, a, b, c):`
	`    return a * x**2 + b * x + c`
	`#Step 3: Perform the quadratic fit`
	`params, covariance = curve_fit(quadratic_function, x_data, y_data)`
	`a, b, c = params  # Parameters of the quadratic function`
	`#Step 4: Create a time range for plotting the fit`
	`x_fit = np.linspace(min(x_data), max(x_data), 100)`
	`y_fit = quadratic_function(x_fit, *params)`
	`#Step 5: Print the equation of the fitted quadratic`

```
function
print(f'Fitted quadratic function parameters: a={a:.4f},
b={b:.4f}, c={c:.4f}')
print(f'Equation of the quadratic function: y = {a:.4f}x^2
+ {b:.4f}x + {c:.4f}')
#Step 6: Define the symbolic variable and the fitted
quadratic function for differentiation
x = sp.symbols('x')
quadratic_expr = a * x**2 + b * x + c
#Step 7: Calculate the first derivative of the quadratic
function
first_derivative_expr = sp.diff(quadratic_expr, x)  # First
derivative
print(f'First derivative of the quadratic function:
{first_derivative_expr}')
#Step 8: Convert the symbolic derivative into a lambda
function
first_derivative_func = sp.lambdify(x,
first_derivative_expr, 'numpy')
#Step 9: Evaluate the derivative over the time range
y_derivative = first_derivative_func(x_fit)
#Step 10: Plot the experimental data, fitted function, and
derivative
plt.scatter(x_data, y_data, label='Experimental data',
color='blue')
plt.plot(x_fit, y_fit, label='Quadratic fit (Height)',
color='red')
plt.plot(x_fit, y_derivative, label='First derivative',
color='green', linestyle='--')
plt.xlabel('Time (s)')
plt.ylabel('Height (m) / Velocity (m/s)')
plt.title('Free fall: Quadratic fit and its derivative')
plt.legend()
plt.show()
```

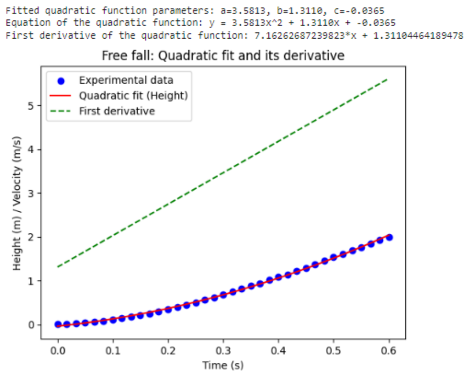


Fig. 3 Graph table of modelling functions for the fitted quadratic function and its first derivative (Rate of change of velocity)

Task 3: Reflect and respond

Table 3. Closing Stage: Principle of Graphical Analysis with Variation as Reference (GAV)

Learning objective:	Explore the relationship between the variation in height and time in a free-fall experiment, using Jupyter Notebook to analyse the limits and derivatives of the obtained quadratic function. This will allow the identification of the object's instantaneous velocity and acceleration at different points in its trajectory, supporting the analysis with a variational argument
Mathematical concept:	Second derivate: By definition
Data base	Real data based on a real experiment
Data base link:	DATOS CAIDA LIBRE 200CM.xlsx
Task	In this section, we analyse how the motion of the ball changes as it falls by reflecting on the results of the experiment in Code task three and the debate between Carlos and Ana. At Betania Norte School in Barranquilla, Carlos and Ana are analysing the results of a free-fall experiment. They dropped an object from a height of 200 cm and calculated the function that describes the height as a function of time. Carlos observes that, according to the data and the first derivative, the velocity changes predictably, but the second derivative shows a different behaviour. Ana argues that the second derivative behaves differently because it represents the average velocity. **VS Question:** Based on the previous hypothesis, do you agree that this represents the second derivative of the height function with respect to time? Why? (EV: Prediction)

	Assessment Criteria: P1 (Cognitive Demand): Very high, as it involves using limits and derivatives to analyse velocity and acceleration in the experiment. P2 (Formulation): Designed in a technological environment (Jupyter Notebook) to perform detailed graphical analysis. P3 (Argument Confrontation): Requires justifying findings and contrasting them with physical laws, strengthening critical thinking skills.
Python Codes	**SVS:**

```
#Step 1: Import the required libraries
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit
import sympy as sp
#Step 2: Read data from the Excel file
data = pd. read_excel ('/content/DATOS CAIDA LIBRE
200CM.xlsx')
#Step 3: Assign data columns to variables
x_data = data['t'].values  # Time
y_data = data['y'].values  # Height
#Step 4: Plot experimental points without fitting
plt.scatter(x_data, y_data, label='Experimental Data',
color='blue')
#Step 5: Define the quadratic function
def quadratic_function(x, a, b, c):
    return a * x**2 + b * x + c
#Step 6: Perform quadratic fitting
params, _ = curve_fit(quadratic_function, x_data, y_data)
a, b, c = params
#Step 7: Create symbolic expressions for the derivative
x = sp.symbols('x')
quadratic_expr = a * x**2 + b * x + c
first_derivative_expr = sp.diff(quadratic_expr, x)
second_derivative_expr = sp.diff(first_derivative_expr, x)
#Convert to NumPy evaluable functions
first_derivative_func = sp.lambdify(x, first_derivative_expr,
'numpy')
second_derivative_value = sp.lambdify(x,
second_derivative_expr, 'numpy')(0)  # Second derivative
constant
#Step 8: Create a range of time values for plotting
x_fit = np.linspace(min(x_data), max(x_data), 100)
y_fit = quadratic_function(x_fit, a, b, c)
```

```
first_derivative_values = first_derivative_func(x_fit)
second_derivative_values = np.full_like(x_fit,
second_derivative_value)
#Step 9: Select five specific points for evaluation in the
derivatives
selected_points = [x_data[0], x_data[5], x_data[10],
x_data[15], x_data[20]]
selected_first_derivatives = [first_derivative_func(t) for t
in selected_points]
selected_second_derivatives = [second_derivative_value] *
len(selected_points)
#Different colors for each tangent
tangent colors = ['blue', 'green', 'red', 'purple', 'orange']
tangent_labels_function = [f'Tangent at t={t:.2f}s' for t in
selected_points]
#Step 10: Plot the quadratic function and its derivatives
plt.plot(x_fit, y_fit, label='Quadratic Function (Height vs.
Time)', color='blue')
plt.plot(x_fit, first_derivative_values, label='First
Derivative (Velocity)', color='green')
plt.plot(x_fit, second_derivative_values, label='Second
Derivative', color='red')
# Plot tangents on the quadratic function (only for the
quadratic function)
for i, t in enumerate(selected_points):
    #Tangent on the quadratic function (different color for
each point)
    tangent_y = quadratic_function(t, a, b, c)
    tangent_slope = selected_first_derivatives[i]
    tangent_line = tangent_slope * (x_fit - t) + tangent_y
    plt.plot(x_fit, tangent_line, color=tangent_colours[i],
linestyle='--', alpha=0.7, label=tangent_labels_function[i])
#Mark the selected points on the plot
plt.scatter(selected_points,
quadratic_function(np.array(selected_points), a, b, c),
color='blue', marker='o', label='Points on Function')
plt.scatter(selected_points, selected_first_derivatives,
color='green', marker='x', label='Points on First
Derivative')
plt.scatter(selected_points, selected_second_derivatives,
color='red', marker='s', label='Points on Second Derivative')
#Configure the plot
plt.xlabel('Time (s)')
```

```
plt.ylabel('Height (m)')  # Ensure the y-axis is in metres
plt.title('Quadratic Function and Its Derivatives with
Tangents at Selected Points')
#Place the legend outside the plot
plt.legend(bbox_to_anchor=(1.5, 1), loc='upper left')
plt.grid(True)
plt.show()
#Step 11: Print the derivative expressions
print(f'First Derivative of the Quadratic Function:
{first_derivative_expr}')
print(f'Second Derivative of the Quadratic Function:
{second_derivative_expr}')
# Step 12: Print the derivatives evaluated at the selected
points
print("\nDerivatives Evaluated at Selected Time Points:")
for i, t in enumerate(selected_points):
    print(f"Time: {t:.6f} s")
    print(f"  Height (Quadratic Function):
{quadratic_function(t, a, b, c):.6f} m")
    print(f"  First Derivative (Velocity):
{selected_first_derivatives[i]:.6f} m/s")
    print(f"  Second Derivative:
{selected_second_derivatives[i]:.6f} m/s²")
```

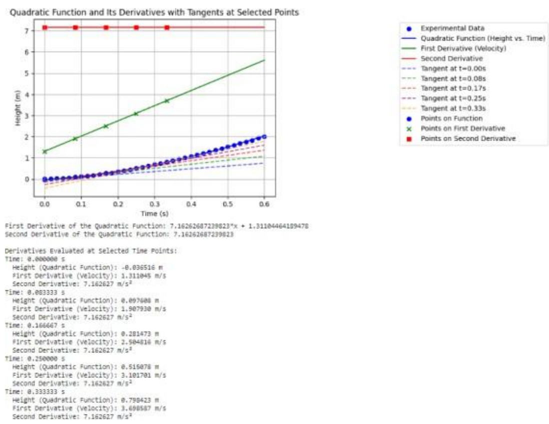


Fig. 4 Graph table of modelling functions for the fitted quadratic function and its first derivative (Rate of change of velocity)

3. Results and discussions

Nine 11th-grade high school students from the Betania Norte school participated in this project. They were asked to solve three tasks through a presentation, with the tasks described in the Genially application (see reference). Due to various limitations related to internet connectivity in the computer lab where this project originated, the tasks were completed using Word and saved individually onto a USB drive.

Seven students completed the tasks individually, labelled as E1, E2, E3, E4, E5, E6, and E7. A group of two students completed the tasks collaboratively and were designated as E89.

The results of this project come from thematic analysis of the Variational Task (VT) designs presented by these future high school graduates. The tasks were developed in the context of fostering argumentation and modelling competencies through variational thinking. To identify the types of variational tasks these students developed, the following categories (C) were established:

C1: Tabulation as numerical variation,

C2: Construction of graphs with variation as a reference

C3: Graphical analysis with variation as a reference.

To analyze the fundamental components of educational tasks, the following categories (P) were established:

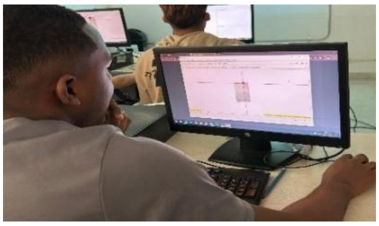
Tasks with high and low cognitive demand (P1).

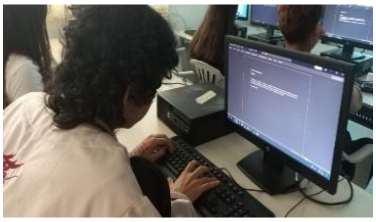
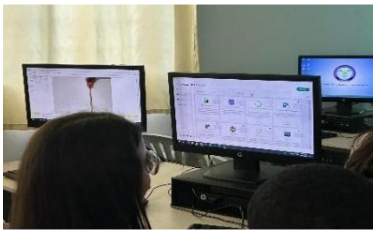
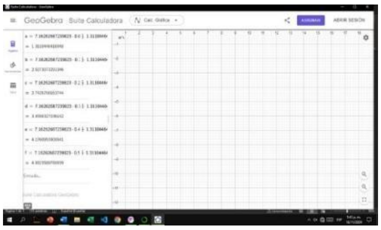
Narrative tasks or questions, open or closed (P2).

Tasks that stimulate the confrontation of ideas (P3).

A table of evidence summarizes results from the student sample.

Table 4. A synthesis of the evidence sample from the students

Task	Objectives	Characterisation of PyLVar and Task Design	Interactive Mathematical Tasks
Task 1: Preliminary Steps (Initial stage)	Understanding mathematical concepts related to algebra and precalculus through experimental modelling of physical phenomena, such as free fall, using tracker, anaconda: Jupyter notebook, and python, grounded in analysis through a variational argument.	**Question 1:** Estimation **Question 2:** Prediction **Task Type:** C1 **Cognitive Demand Level:** High **Task Formulation:** Open-ended question **Argument Management:** Stimulates confrontation of ideas	E1: TAREA E2: TAREA E3: TAREA E4: TAREA E5: TAREA E6: TAREA E7: TAREA E8: TAREA E89: TAREA  **Fig. 5** A student from Betania Norte modelling the physics phenomenon of "free fall"

			 **Fig. 6** A student from Betania Norte carrying out the task using Word
Task 2: Analysis of the Behaviour of a Golf Ball's Free Fall (Development stage)	Using programming tools, such as Jupyter notebook and python, to graphically represent experimental data, with the aim of fitting a quadratic function that describes the motion's trajectory and verifying certain physical laws, grounded in a variational argument.	**Question 1:** Seriation **Question 2:** Prediction **Task Type:** C2 **Cognitive Demand Level:** High **Task Formulation:** Open-ended question **Argument Management:** Fosters critical thinking and confrontation of ideas	E1: TAREA E2: TAREA E3: TAREA E4: TAREA E5: TAREA E6: TAREA E7: TAREA E8: TAREA E89: TAREA  **Fig. 7** A student using local applications such as “Anaconda” and “Jupyter” for the task codes  **Fig. 8** Screenshot of a Student from Betania

			Norte Performing Calculations for Question 1
Task 3: Reflect (Closing stage)	Explore the relationship between height variation and time in a free-fall experiment using Jupyter notebook to analyse the limits and derivatives of the quadratic function obtained. this will allow identifying the instantaneous velocity and acceleration of the object at different points along its trajectory, grounded in a variational argument.	**Question 1:** Prediction **Task Type:** C3 **Cognitive Demand Level:** High **Task Formulation:** Open-ended narrative question **Argument Management:** Stimulates critical thinking and solid arguments	E1: TAREA E2: TAREA E3: TAREA E4: TAREA E5: TAREA E6: TAREA E7: TAREA E8: TAREA E89: TAREA  **Fig. 9** A Student using Python Language Code for the Development of Task 3

This research examined how interactive mathematical tasks, grounded in variational thinking and the STEAM approach, foster modelling and mathematical argumentation skills in high school students. Students effectively modelled physical phenomena like free fall using technological tools, enhancing their understanding of abstract concepts such as derivatives and variation. These findings support (Parra-Arenales et al., 2021), who argue that mathematical modelling bridges theory and practice, promoting critical reasoning. Visualizing and analyzing real data through graphs further strengthened students' argumentation and decision-making skills.

Aligned with the fourth scientific paradigm, where data analysis is central to knowledge generation (Ma et al., 2019), this study employed tools like Python and Tracker to construct dynamic models and explore variable relationships (Pinargote-Zambrano et al., 2024). However, data quality and complexity posed challenges, underscoring the importance of transdisciplinary thinking (Cao, 2023) to address complex educational problems.

Technological tools within the STEAM framework enhanced learning by promoting autonomy and exploration of complex concepts (Cervantes-Barraza & Aroca-Araujo, 2023). Students identified patterns of change and built models reflecting variable relationships, supported by the view that interactive tools are vital for understanding variation (Salcedo-Talamantes et al., 2021). Despite challenges like limited connectivity, offline tools such as Jupyter Notebook showed how adapting strategies to contextual constraints can still be effective (Blanco-Iturralde et al., 2024). The teacher's role was crucial, underlining the need for specialized training in technology and interdisciplinary methods to foster students' modelling skills (Horton & Hardin, 2021).

Interactive tasks facilitated meaningful, transferable learning, consistent with (Solar & Deulofeu, 2016), who advocate for open-ended, multidisciplinary tasks to connect abstract concepts to real-world problems. Students analyzed data, identified variation patterns, and made predictions, gaining skills essential for academic and professional success.

Overall, the study underscores the transformative potential of integrating emerging technologies, data analysis, and transdisciplinary approaches in mathematical education (Cao, 2023; Ma et al., 2019).

8. Conclusion

The design of interactive mathematical tasks aligned with the PyLVar approach is essential for strengthening students' understanding of variation and change. This approach integrates technological and scientific tools within an interdisciplinary framework that combines data science, technology, and STEAM methodologies, promoting the development of analytical and argumentative skills. Tasks such as Numerical Variational Tabulation (TNV), Graphical Analysis with Variation (GAV), and Construction of Graphs with Variation (CGV) allow students to identify patterns in numerical data, model situations using graphs, and analyse trends using derivatives, bridging theory with practical applications. Additionally, the three components of task design in education, including cognitive demand level, appropriate task formulation, and managing argument confrontation, are crucial for fostering deep learning. High-demand tasks encourage complex thinking, while well-designed tasks and managing argumentation help develop critical thinking and strong argumentative skills. These activities not only strengthen variational thinking but also prepare students to solve real-world problems, developing the analytical and critical skills needed to face challenges in diverse contexts.

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Author contributions

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Declaration of interest

No conflict of interest is declared by authors.

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