Research Article

Rational Type Contraction in Consideration of Fixed-Point Theorems in B-Metric Spaces

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Abstract. In this paper, we prove common fixed-point theorems in complete b-metric spaces using rational type contraction for two self-mappings. Our result improves and extends the results proved by Mlaiki et al. [1] for a single continuous self-mapping in extended complete b-metric space. We improve the results of Mlaiki et al. [1] to complete b-metric spaces for two self-mappings without assuming the continuity of any mapping.

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1. Introduction.

Banach [2] in demonstrated a highly consequential theorem in the context of complete metric spaces, establishing the existence of a unique fixed point. Since then, the fixed-point theory is one of the most important tools in many branches of science, economics, computer science, engineering and the development of nonlinear analysis.

As an extension of metric spaces, the concept of b-metric spaces was introduced by Backhtin [3]. Czerwik [4] first presented a generalization of Banach fixed point theorem in b-metric spaces. Mehmet et al. [5], Boriceanu [6], Bota [7], Pacurar [8] extended used this idea and proved fixed point theorems and its applications in b-metric spaces.

In this paper, we extend the results of Mlaiki et al. [1] and prove a common fixed-point theorem in complete b-metric spaces using rational type contraction for two self-mappings. Our result improves and extends the results proved by Mlaiki et al. [1] for a single continuous self-mapping in extended complete b-metric space. We improve the results of Mlaiki et al. [1] to complete b-metric spaces for two self-mappings without assuming the continuity of any mapping.

2. Preliminaries.

Definition 2.1 [3] Let X be a non empty set and $s \geq 1$ be a given real number.
A function $d_b : X \times X \to [0,\infty)$ is called b-metric if it satisfies the following properties for each $x, y, z \in X$ −
(b_1) $d_b(x,y) = 0 \Leftrightarrow x = y$;
$(b2)$ $d_b(x,y) = d(y,x)$;
$(b3)$ $d_b(x,z) \leq s[d_b(x,y) + d_b(y,z)]$.
The pair (X, d_b) is called a b-metric space.
Example 2.1. Let $X = l_p(R)$ with $0 < p < 1$, where

$$l_p(R) = \{\{x_n\} \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

Define $d_b : X \times X \to R^+$ as-

$$d_b(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$$

where $x = \{x_n\}, y = \{y_n\}$. Then (X, d_b) is a b-metric space with coefficient $s = 2^{\frac{1}{p}}$.

Example 2.2. Let $X = L_p[0,1]$ be the space of all real functions $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$ with $0 < p < 1$. Define $d_b: X \times X \to R^+$ as

$$d_b(x,y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}$$

Then (X, d_b) is a b-metric space with coefficient $s = 2^{\frac{1}{p}}$.

The above examples show that the class of b-metric spaces is larger than the class of metric spaces. When $s = 1$, the concept of b-metric space coincides with the concept of metric space.

Definition 2.2 [9] Let (X, d_b) be a b-metric space. A sequence $\{x_n\}$ in X is said to be:

(I) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;

(II) Convergent if and only if there exist $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$;

(III) The b-metric space (X, d_b) is complete if every Cauchy sequence is convergent.

3. Main Result.

Theorem 3.1. Let $S, T: X \to X$ be self-mappings with (X, d_b) be a complete b-metric space and for all distinct $x, y \in X$-

$$d_b(Sx,Ty) \le \xi_1 d_b(x,y) + \xi_2 \frac{d_b(x,Sx)d_b(y,Sx) + d_b(y,Ty)d_b(x,Ty)}{d_b\,(x,Ty) + d_b(y,Sx)}$$

where $d_b\,(x,Ty) + d_b(y,Sx) \neq 0, 0 < \xi_1 + \xi_2 < 1, \xi_1, \xi_2 \in [0,1)$. Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be a sequence in X such that

$$x_{n+1} = Sx_n, x_{n+2} = Tx_{n+1}.$$

Then

$$\begin{aligned} d_b(x_{n+1}, x_{n+2}) &= d_b(Sx_n, Tx_{n+1}) \\ &\le \xi_1 d_b(x_n, x_{n+1}) \\ &+ \xi_2 \frac{d_b(x_n, Sx_n)d_b(x_{n+1}, Sx_n) + d_b(x_{n+1}, Tx_{n+1})d_b(x_n, Tx_{n+1})}{d_b\,(x_n, Tx_{n+1}) + d_b(x_{n+1}, Sx_n)} \\ &= \xi_1 d_b(x_n, x_{n+1}) \\ &+ \xi_2 \frac{d_b(x_n, x_{n+1})d_b(x_{n+1}, x_{n+1}) + d_b(x_{n+1}, x_{n+2})d_b(x_n, x_{n+2})}{d_b(x_n, x_{n+2}) + d_b(x_{n+1}, x_{n+1})} \\ &= \xi_1 d_b(x_n, x_{n+1}) + \xi_2 d_b(x_{n+1}, x_{n+2}) \end{aligned}$$

which implies

$$d_b(x_{n+1}, x_{n+2}) \le \frac{\xi_1}{1-\xi_2} d_b(x_n, x_{n+1}) = \xi d_b(x_n, x_{n+1})$$

where $\xi = \frac{\xi_1}{1-\xi_2} \in [0,1)$.

Applying it recursively, we get

$$d_b(x_{n+1}, x_{n+2}) \le \xi^n d_b(x_0, x_1).$$

Since $\xi \in [0,1)$, we have

$$\lim_{n\to\infty} d_b(x_{n+1}, x_{n+2}) = 0$$

Or

$$\lim_{n\to\infty} d_b(x_n, x_{n+1}) = 0.$$

By triangular inequality, for any $m \geq 1$

$$\begin{aligned} d_b(x_n, x_{n+m}) &\leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+m})] \\ &\leq bd_b(x_n, x_{n+1}) + b^2[d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_{n+m})] \\ &\leq bd_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) \\ &\quad + b^3[d_b(x_{n+2}, x_{n+3}) + d_b(x_{n+3}, x_{n+m})] \ldots\ldots \end{aligned}$$

Therefore

$$\begin{aligned} d_b(x_n, x_{n+m}) &\leq [b\xi^n + b^2\xi^{n+1} + b^3\xi^{n+2} + \cdots \ldots\ldots]d_b(x_0, x_1) \\ &= b\xi^n[1 + (b\xi)^2 + (b\xi)^3 + \cdots \ldots]d_b(x_0, x_1) = \frac{b\xi^n}{1 - b\xi} d_b(x_0, x_1) \end{aligned}$$

Therefore, we have

$$d_b(x_n, x_{n+m}) \leq \frac{b\xi^n}{1 - b\xi} d_b(x_0, x_1)$$

As $n \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence in complete b-metric space (X, d_b). Hence there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Now to show that

$$Sx^* = x^*.$$

We have

$$\begin{aligned} d_b(Sx^*, x^*) &\leq b[d_b(Sx^*, Tx_{n+1}) + d_b(Tx_{n+1}, x^*)] \\ &\leq bd_b(x_{n+2}, x^*) \\ &\quad + b\Bigg[\xi_1 d_b(x^*, x_{n+1}) \\ &\quad + \xi_2 \frac{d_b(x^*, Sx^*)d_b(x_{n+1}, Sx^*) + d_b(x_{n+1}, Tx_{n+1})d_b(x^*, Tx_{n+1})}{d_b(x^*, Tx_{n+1}) + d_b(x_{n+1}, Sx^*)}\Bigg] \\ &\leq bd_b(x_{n+2}, x^*) \\ &\quad + b\Bigg[\xi_1 d_b(x^*, x_{n+1}) \\ &\quad + \xi_2 \frac{d_b(x^*, Sx^*)d_b(x_{n+1}, Sx^*) + d_b(x_{n+1}, x_{n+2})d_b(x^*, x_{n+2})}{d_b(x^*, x_{n+2}) + d_b(x_{n+1}, Sx^*)}\Bigg] \end{aligned}$$

As $n \to \infty$,

$$d_b(Sx^*, x^*) \leq b\xi d_b(Sx^*, x^*)$$

which is a contradiction as $\xi \in [0,1)$. Hence

$$Sx^* = x^*.$$

Similarly, we can show

$$Tx^* = x^*.$$

Therefore S and T have a common fixed point in X i.e.

$$Sx^* = Tx^* = x^*.$$

To show uniqueness of the fixed point, let $z \neq x^*$ be another fixed point of S and T i.e.

$$Sz = Tz = z; Sx^* = Tx^* = x^*.$$

Then

$$\begin{aligned} d_b(z, x^*) = d_b(Sz, Tx^*) &\leq \xi_1 d_b(z, x^*) + \xi_2 \frac{d_b(z, Sz)d_b(x^*, Sz) + d_b(x^*, Tx^*)d_b(z, Tx^*)}{d_b(z, Tx^*) + d_b(x^*, Sz)} \\ &= \xi_1 d_b(z, x^*) + \xi_2 \frac{d_b(z, z)d_b(x^*, z) + d_b(x^*, x^*)d_b(z, x^*)}{d_b(z, x^*) + d_b(x^*, z)} = \xi_1 d_b(z, x^*). \end{aligned}$$

Since $\xi_1 \in [0,1)$, we have $d_t(z, x^*) = 0$ i.e. $z = x^*$.

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This completes the proof.

Theorem 3.2. Let (X, d_b) be a complete b-metric space and $S, T : X \to X$ be self-mappings satisfying:

$$d_b(Sx,Ty) \le \alpha d_b(x,y) + \beta[d_b(Sx,y) + d_b(x,Ty)] + \gamma \frac{d_b(Sx,y)d_b(y,Ty) + d_b(Sx,x)d_b(x,Ty)}{d_b(x,Ty) + d_b(Sx,y)}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma \in [0,1)$ such that $0 < \alpha + \beta + \gamma < 1$. Then S, T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X such that

$$x_{2k+1} = Sx_{2k}, x_{2k+2} = Tx_{2k+1}.$$

Then

$$\begin{aligned} d_b(x_{2k+1}, x_{2k+2}) &= d_b(Sx_{2k}, Tx_{2k+1}) \\ &\le \alpha d_b(x_{2k}, x_{2k+1}) + \beta[d_b(Sx_{2k}, x_{2k+1}) + d_b(x_{2k}, Tx_{2k+1})] \\ &\quad + \gamma \frac{d_b(Sx_{2k}, x_{2k+1})d_b(x_{2k+1}, Tx_{2k+1}) + d_b(Sx_{2k}, x_{2k})d_b(x_{2k}, Tx_{2k+1})}{d_b(x_{2k}, Tx_{2k+1}) + d_b(Sx_{2k}, x_{2k+1})} \\ &= \alpha d_b(x_{2k}, x_{2k+1}) + \beta[d_b(x_{2k+1}, x_{2k+1}) + d_b(x_{2k}, x_{2k+2})] \\ &\quad + \gamma \frac{d_b(x_{2k+1}, x_{2k+1})d_b(x_{2k}, x_{2k+2}) + d_b(x_{2k+1}, x_{2k})d_b(x_{2k}, x_{2k+2})}{d_b(x_{2k}, x_{2k+2}) + d_b(x_{2k+1}, x_{2k+1})} \\ &= (\alpha + \gamma)d_b(x_{2k}, x_{2k+1}) + \beta d_b(x_{2k}, x_{2k+2}) \end{aligned}$$

Therefore

$$d_b(x_{2k+1}, x_{2k+2}) \le (\alpha + \gamma)d_b(x_{2k}, x_{2k+1}) + \beta b[d_b(x_{2k}, x_{2k+1}) + d_b(x_{2k+1}, x_{2k+2})]$$

Which implies

$$d_b(x_{2k+1}, x_{2k+2}) \le \frac{\alpha + \beta b + \gamma}{1 - \beta b} d_b(x_{2k}, x_{2k+1}) = h d_b(x_{2k}, x_{2k+1})$$

where $h = \frac{\alpha+\beta b+\gamma}{1-\beta b} \in [0,1)$.

Therefore, we have for all n,

$$d_b(x_{n+1}, x_n) \le h d_b(x_n, x_{n-1}) \le h^2 d_b(x_{n-1}, x_{n-2}) \le \cdots \ldots\ldots \le h^n d_b(x_1, x_0).$$

By triangular inequality, for any $m \ge 1$

$$\begin{aligned} d_b(x_n, x_{n+m}) &\le b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+m})] \\ &\le b d_b(x_n, x_{n+1}) + b^2[d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_{n+m})] \\ &\le b d_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) \\ &\quad + b^3[d_b(x_{n+2}, x_{n+3}) + d_b(x_{n+3}, x_{n+m})] \ldots\ldots \end{aligned}$$

Therefore

$$\begin{aligned} d_b(x_n, x_{n+m}) &\le [bh^n + b^2h^{n+1} + b^3h^{n+2} + \cdots \ldots]d_b(x_0, x_1) \\ &= bh^n[1 + bh + (bh)^2 + (bh)^3 + \cdots \ldots]d_b(x_0, x_1) = \frac{bh^n}{1 - bh} d_b(x_0, x_1) \end{aligned}$$

Therefore, we have

$$d_b(x_n, x_{n+m}) \le \frac{bh^n}{1 - bh} d_b(x_0, x_1)$$

As $n \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence in complete b-metric space (X, d_b). Hence there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Now to show that

$$Sx^* = x^*.$$

We have

$$
\begin{aligned}
d_b(Sx^*,x^*) &\le b[d_b(Sx^*,Tx_{2k+1}) + d_b(Tx_{2k+1},x^*]\\
&= b[d_b\,(x_{2k+2},x^*) + d_b(Sx^*,Tx_{2k+1})]\\
&\le b\Bigg[d_b\,(x_{2k+2},x^*) + \alpha d_b(x^*,x_{2k+1}) + \beta[d_b(Sx^*,x_{2k+1}) + d_b(x^*,Tx_{2k+1})]\\
&\quad + \gamma\frac{d_b(Sx^*,x_{2k+1})d_b(x_{2k+1},Tx_{2k+1}) + d_b(Sx^*,x^*)d_b(x^*,Tx_{2k+1})}{d_b(x^*,Tx_{2k+1}) + d_b(Sx^*,x_{2k+1})}\Bigg]\\
&= b\Bigg[d_b\,(x_{2k+2},x^*) + \alpha d_b(x^*,x_{2k+1}) + \beta[d_b(Sx^*,x_{2k+1}) + d_b(x^*,x_{2k+2})]\\
&\quad + \gamma\frac{d_b(Sx^*,x_{2k+1})d_b(x_{2k+1},x_{2k+2}) + d_b(Sx^*,x^*)d_b(x^*,x_{2k+2})}{d_b(x^*,x_{2k+2}) + d_b(Sx^*,x_{2k+1})}\Bigg]
\end{aligned}
$$

As $k \to \infty$, we have

$$d_b(Sx^*,x^*) \le b\beta d_b(Sx^*,x^*)$$

which is a contradiction. Hence $Sx^* = x^*$. Similarly, we can show $Tx^* = x^*$.
Thus

$$Sx^* = Tx^* = x^*.$$

Therefore x^* is the common fixed point of S and T.
To show uniqueness, let u^* be another fixed point of S and T such that $u^* \ne x^*$ i.e.

$$Sx^* = Tx^* = x^*,$$
$$Su^* = Tu^* = u^*.$$

We have

$$
\begin{aligned}
d_b(u^*,x^*) &= d_b(Su^*,Tx^*)\\
&\le \alpha d_b\,(u^*,x^*) + \beta[d_b(Su^*,x^*) + d_b(u^*,Tx^*)]\\
&\quad + \gamma\frac{d_b(Su^*,x^*)d_b(x^*,Tx^*) + d_b(Su^*,u^*)d_b(u^*,Tx^*)}{d_b(u^*,Tx^*) + d_b(Su^*,x^*)}\\
&= \alpha d_b(u^*,x^*) + \beta[d_b(u^*,x^*) + d_b(u^*,x^*)]\\
&\quad + \gamma\frac{d_b(u^*,x^*)d_b(x^*,x^*) + d_b(u^*,u^*)d_b(u^*,x^*)}{d_b(u^*,x^*) + d_b(u^*,x^*)}
\end{aligned}
$$

which implies

$$d_b(u^*,x^*) \le (\alpha + 2\beta)d_b(u^*,x^*)$$

Which is a contradiction. Therefore

$$u^* = x^*.$$

This proves the theorem.

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