

Fuzzy Simple Partially Ordered Γ - Semigroups

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Article History: Received: 11 January 2021; Accepted: 27 February 2021; Published online: 5 April 2021

Abstract: In this paper, it is shown that $f_{(s\Gamma a]}$ and $f_{(a\Gamma s]}$ are respectively fuzzy left and fuzzy right ideals of S. S is a fuzzy left(right) simple po- Γ -semigroup $\Leftrightarrow f_{(s\Gamma a]} = f_S = S$ ($f_{(a\Gamma s]} = f_S = S$) $\forall a \in S$.

It is proved that for any semi group S “TFAE”

(1) S is left(right) simple po- Γ -semigroup.

(2) S is a fuzzy left(right) simple Γ -semigroup.

The union of all proper fuzzy ideals of S is the only fuzzy maximal ideal of S ,where S is a **PO Γ -Semigroup** with 'e', unity.

MATHEMATICAL SUBJECT CLASSIFICATION (2010): 20M12; 20M17; 20N25.

Keywords: Fuzzy semisimple, globally idempotent, fuzzy globally idempotent, left simple po- Γ -semigroup ,fuzzy left simple po- Γ - semigroup, right simple po- Γ -semigroup ,fuzzy right simple po- Γ - semigroup, semisimple and idempotent.

1. INTRODUCTION:

CLIFFORD [2],[3] and LJAPIN [4] have thoroughly explored the algebraic theory of semigroups. Anjaneyulu [1] puts forth the ideal theory in general semigroups. The definitions of fuzzy simple Γ -semigroup, fuzzy left simple Γ -semigroup, and fuzzy right simple Γ -semigroup are presented. In 1965, LAZADEH suggested the concept of a fuzzy subset of a set. A number of experiments on fuzzy sets then resulted in fuzzy logic, the theory of fuzzy sets, fuzzy algebra, etc. ROSENFELD was the founder of fuzzy abstract algebra. The fuzzy theory of semigroups was advanced by Kuroki and Xie.

2. PRELIMINARIES:

In the present paper “**FLSPOGS**” denotes **Fuzzy left simple po- Γ -semigroup**,

“**LSPOGS**” denotes **left simple po- Γ -semigroup**,

“**FRSPOGS**” denotes **Fuzzy Right simple po- Γ -semigroup**,

“**RSPOGS**” denotes **Right simple po- Γ -semigroup**,

“**TFAE**” denotes “The following are equal”.

‘S’ denotes po- Γ -semigroup unless otherwise specified

DEFINITION 2.1: S is a Γ -semigroup with ordered relation ‘ \leq ’ is said to be **PO- Γ -Semigroup** if S is a poset such that $m \leq n \Rightarrow m\gamma p \leq n\gamma p$ and $p\gamma m \leq p\gamma n \forall m, n, p \in S$ and $\gamma \in \Gamma$.

DEFINITION 2.2: A function $f: S \rightarrow [0,1]$ is known as **fuzzy subset of S**.

DEFINITION 2.3: Let $X(\neq \emptyset) \subseteq S$. We define $f_X: S \rightarrow [0, 1]$ by $f_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases}$. Then f_X is a fuzzy subset of S.

DEFINITION 2.4: For p & q , two fuzzy subsets of S, the **inclusion relation** $p \subseteq q$ is defined by $p(z) \leq q(z), \forall z \in S$ and $p \cup q, p \cap q$ are defined by

$$(p \cup q)(z) = \max\{p(z), q(z)\} = p(z) \vee q(z), \text{ for all } z \in S,$$

$$(p \cap q)(z) = \min\{p(z), q(z)\} = p(z) \wedge q(z), \text{ for all } z \in S.$$

DEFINITION 2.5: ([13]) For $x \in S, f$ & g (two fuzzy subsets of S), the product $(f \Gamma g)$ is stated as

$$(f \Gamma g)(x) = \begin{cases} \bigvee_{x \leq y\gamma z} f(y) \wedge g(z), & \text{if } x \leq y\gamma z \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION 2.6: For any subset P of S, **(P) defined as (P) = { z \in S / z \leq p for some p \in P }**. For $P = \{k\}$ we define $(k) = (\{k\}) = \{z \in S / z \leq k\}$.

DEFINITION 2.7: A fuzzy subset ‘ τ ’ of S is known as (A) fuzzy Γ -subsemigroup ([14]) of S if $\tau(m\gamma p) \geq \tau(m) \wedge \tau(p), \forall m, p \in S$ and $\gamma \in \Gamma$. (B) fuzzy Γ -subsemigroup ([15]) of S $\Leftrightarrow \tau \Gamma \tau \subseteq \tau$. (C) fuzzy po- Γ -subsemigroup of S if (i) $u \leq v$ then $\tau(u) \geq \tau(v)$

(ii) $\tau(u\gamma v) \geq \tau(u) \wedge \tau(v), \forall u, v \in S, \forall \gamma \in \Gamma$. (D) fuzzy left ideal [16] of S if

(i) $u \leq v$ then $\tau(u) \geq \tau(v)$ (ii) $\tau(u\gamma v) \geq \tau(v), \forall u, v \in S, \forall \gamma \in \Gamma$.

(E) fuzzy left ideal [16] of S \Leftrightarrow ‘ τ ’ satisfies that (i) $u \leq v$ then $\tau(u) \geq \tau(v), \forall u, v \in S$ (ii) $s \Gamma \tau \subseteq \tau$.

(F) fuzzy right ideal [16] of S if (i) $u \leq v$ then $\tau(u) \geq \tau(v)$ (ii) $\tau(u\gamma v) \geq \tau(u), \forall u, v \in S, \forall \gamma \in \Gamma$.

(G) *fuzzy right ideal* [16] of $S \Leftrightarrow \tau$ satisfies that (1) $u \leq v$ then $\tau(u) \geq \tau(v), \forall u, v \in S$ (2) $\tau \Gamma s \subseteq \tau$. (H) *fuzzy ideal* [16] of S if (1) $u \leq v$ then $\tau(u) \geq \tau(v)$ (2) $\tau(u\gamma v) \geq \tau(v), \tau(u\gamma v) \geq \tau(u), \forall u, v \in S, \forall \gamma \in \Gamma$. (I) *fuzzy ideal* [16] of S iff τ satisfies that (i) $u \leq v$ then $\tau(u) \geq \tau(v), \forall u, v \in S$ (ii) $\tau \Gamma s \subseteq \tau$ and $s \Gamma \tau \subseteq \tau$. (J) *Idempotent* if $\tau^2 = \tau \Gamma \tau = \tau$.

3. FUZZY SIMPLE PARTIALLY ORDERED Γ -SEMIGROUPS

DEFINITION 3.1: If S itself is the only left ideal of S then S is LSPOFS.

DEFINITION 3.2: PO- Γ - semigroup S is FLSPoFS if each fuzzy left ideal of S is a constant function.

DEFINITION 3.3: 'f' be a fuzzy subset of a po- Γ - semigroup S .

$f_{(S\Gamma a]}(x)$ is defined as

$$f_{(S\Gamma a]}(x) = \begin{cases} 1 & \text{if } x \in (S\Gamma a] \\ 0 & \text{otherwise} \end{cases}$$

THEOREM 3.4: $f_{(S\Gamma a]}$ is a fuzzy left ideal of a PO- Γ - Semigroup $S, \forall a \in S$.

Proof: (i) Let $x, y \in S$ and $x \leq y$

If $y \in (S\Gamma a]$ since $x \leq y \Rightarrow x \in (S\Gamma a]$ then $f_{(S\Gamma a]}(x) = 1 = f_{(S\Gamma a]}(y)$

If $y \notin (S\Gamma a]$ then $f_{(S\Gamma a]}(y) = 0 \leq f_{(S\Gamma a]}(x)$.

By summarizing the above $f_{(S\Gamma a]}(x) \geq f_{(S\Gamma a]}(y)$

(ii) If $y \notin (S\Gamma a]$ then $f_{(S\Gamma a]}(y) = 0 \leq f_{(S\Gamma a]}(x\gamma y)$.

If $y \in (S\Gamma a]$ then $f_{(S\Gamma a]}(y) = 1$

Since $y \in (S\Gamma a]$ and $(S\Gamma a]$ is a left ideal of S

we have $x\gamma y \in (S\Gamma a], \forall x \in S, \gamma \in \Gamma \Rightarrow f_{(S\Gamma a]}(x\gamma y) = 1 = f_{(S\Gamma a]}(y)$

Therefore $f_{(S\Gamma a]}(x\gamma y) \geq f_{(S\Gamma a]}(y)$.

Hence $f_{(S\Gamma a]}$ is a fuzzy left ideal of S .

THEOREM 3.5: In a po- Γ - semigroup S , "TFAE".

(p) S is a LSPOFS

(q) S is a FLSPoFS.

Proof: (p) \Rightarrow (q)

Assume (P) holds.

suppose T is any fuzzy left ideal of S which implies that \exists elements $x, y \in S$ and $\alpha, \gamma \in \Gamma \ni b = x\alpha a$ and $a = y\gamma b$.

By definition of T we have $T(a) = T(y\gamma b) \geq T(b) = T(x\alpha a) \geq T(a)$

$\Rightarrow T(a) = T(b) \forall a, b \in S$ and $\alpha, \gamma \in \Gamma \Rightarrow T$ is a constant fuzzy ideal.

Hence (q).

(q) \Rightarrow (p):

Assume (q) holds.

Suppose A is po left ideal of S . $\therefore C_A$ is a fuzzy left ideal of S (from [9]).

$\Rightarrow C_A$ is a constant function.

let $x, y \in S, \gamma \in \Gamma$ since $A \neq \emptyset, C_A(x\gamma y) = 1 \Rightarrow x\gamma y \in A \Rightarrow S \subseteq A$

Therefore $A=S$. Hence (p).

THEOREM 3.6: S is FLSPoFS $\Leftrightarrow f_{(S\Gamma a]} = f_S = S \forall a \in S$, where S is a po- Γ -semigroup.

Proof: Suppose S is a FLSPoFS. From Th.3.5 S is a LSPOFS. Then from [5], $(S\Gamma a]=S$. Therefore $f_{(S\Gamma a]} = f_S = S$.

Conversely assume that $f_{(S\Gamma a]} = f_S = S \Rightarrow f_{(S\Gamma a]}(x) = f_S(x)$

$\Rightarrow (S\Gamma a]=S$. Then from [5] S is a LSPOFS. \therefore From Th.3.5 S is a FLSPoFS.

DEFINITION 3.7: APO- Γ -semigroup S is known as a RSPOFS if S itself is the only po right ideal of S .

DEFINITION 3.8: If each fuzzy right ideal of 'S' is a constant function then S is termed as FRSPoFS.

DEFINITION 3.9: f be a fuzzy subset of a po- Γ -semigroup S .

$f_{(a\Gamma S]}(x)$ is defined as

$$f_{(a\Gamma S]}(x) = \begin{cases} 1 & \text{if } x \in (a\Gamma S] \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION 3.10: If every fuzzy ideal of 'S' is a constant function then S is fuzzy simple po- Γ -semigroup.

THEOREM 3.11: $f_{(a\Gamma S]}$ is a fuzzy right ideal of a po- Γ -semigroup S for every $a \in S$.

Proof: (p) Let $x, y \in S, \alpha, \beta \in \Gamma$ and $x \leq y$

If $y \in (a\Gamma S]$ since $x \leq y \Rightarrow x \in (a\Gamma S]$ then $f_{(a\Gamma S]}(x) = 1 = f_{(a\Gamma S]}(y)$

If $y \notin (a\Gamma S]$ then $f_{(a\Gamma S]}(y) = 0 \leq f_{(a\Gamma S]}(x)$.

By summarizing the above $f_{(a\Gamma S]}(x) \geq f_{(a\Gamma S]}(y)$

(q) If $x \notin (a\Gamma S]$ then $f_{(a\Gamma S]}(x) = 0 \leq f_{(a\Gamma S]}(x\gamma y)$.

If $x \in (a\Gamma S]$ then $f_{(a\Gamma S]}(x) = 1$

Since $x \in (a\Gamma S]$ and $(a\Gamma S]$ is a **poright ideal** of S , then $x\gamma y \in (a\Gamma S], \forall y \in S, \forall \gamma \in \Gamma$ (from theorem [5]).

$$\Rightarrow f_{(a\Gamma S]}(x\gamma y) = 1 = f_{(a\Gamma S]}(x)$$

$$\therefore f_{(a\Gamma S]}(x\gamma y) \geq f_{(a\Gamma S]}(x).$$

From (p) and (q) $f_{(a\Gamma S]}$ is a fuzzy right ideal of S .

THEOREM 3.12: In a **po- Γ -semigroup S** , “**TFAE**”.

l) S is a **RSPOFS.**

m) S is a **FRSPOFS.**

Proof : (l) \Rightarrow (m):

Assume S is a **RSPOFS**.

suppose f is fuzzy right ideal of S . Then \exists elements $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $\alpha x = b$ and $a = b\beta y$ (from [7]).

$\therefore f$ is a fuzzy right ideal of $S, f(a) = f(b\beta y) \geq f(b) = f(\alpha x) \geq f(a) \Rightarrow f(a) = f(b) \forall a, b \in S \Rightarrow f$ is a constant fuzzy ideal.

$\therefore S$ is **FRSPOFS**.

(m) \Rightarrow (l):

Suppose S is a **FRSPOFS**.

Assume A as any **po right ideal** of $S \Rightarrow C_A$ is a fuzzy right ideal of S (from [9]).

$\Rightarrow C_A$ is a constant function.

Let $x, y \in S, \gamma \in \Gamma$ Since $A \neq \emptyset, C_A(x\gamma y) = 1 \Rightarrow x\gamma y \in A \Rightarrow S \subseteq A$

Therefore $A = S$.

Hence S is **RSPOFS**.

THEOREM 3.13: S is **FRSPOFS** $\Leftrightarrow f_{(a\Gamma S]} = f_S = S \forall a \in S$.

Proof: Suppose that S is a **FRSPOFS**. S is a **RSPOFS** (From theorem 3.12). Then from [5], $(a\Gamma S] = S$.

Therefore $f_{(a\Gamma S]} = f_S = S$.

Conversly assume that $f_{(a\Gamma S]} = f_S = S \Rightarrow f_{(a\Gamma S]}(x) = f_S(x)$

$\Rightarrow (a\Gamma S] = S$. Then S is a **RSPOFS** (from [5]). Then S is a **FRSPOFS** (from Theorem 3.12).

DEFINITION 3.14: f and g are any two fuzzy subsets of S , define **(fog)** as

$$(fog)(x) = \bigvee_{x \leq y\gamma z} (f \circ g)(y\gamma z), \forall x, y, z \in S, \gamma \in \Gamma.$$

DEFINITION 3.15 : A fuzzy ideal f of a **po - Γ - semigroup ‘ S ’** is known as **globally idempotent** if $(f^n) = (f), \forall n$.

DEFINITION 3.16 : A **po - Γ - semigroup ‘ S ’** is known as **fuzzy globally idempotent** if $(S^n) = S, \forall n$.

THEOREM 3.17 : A **po - Γ - semigroup ‘ S ’ with ‘ e ’, unity and f , a fuzzy ideal of S with $f(e) = 1$ then $f = S = f_S$.**

Proof: Let $x \in S, \gamma \in \Gamma$. Consider $f(x) = f(x\gamma e) \geq f(e) = 1$

$\Rightarrow f(x) \geq 1 \Rightarrow f(x) = 1, \forall x \in S$. Therefore $f = f_S = S$.

DEFINITION 3.18: A fuzzy ideal f which is non-zero of S is known as a **proper fuzzy ideal** if $f \neq C_S = S$.

DEFINITION 3.19: A fuzzy ideal ‘ f ’ of ‘ S ’ is **maximal** if no proper fuzzy ideal ‘ g ’ of ‘ S ’ existed $\exists f \subset g$.

THEOREM 3.20: If $\{f_i\}$ is a sequence fuzzy ideals of a **po- Γ -semigroup S** then $\cup f_i(x)$ is fuzzy ideal of S .

Proof: let $\{f_i\}$ be a sequence of fuzzy ideals of S .

Let $x, y \in S$ such that $x \leq y$.

Consider $\cup f_i(x) = \max\{f_1(x), f_2(x), f_3(x), \dots \dots \dots\}$

$$= f_1(x) \vee f_2(x) \vee f_3(x) \vee \dots \dots \dots$$

$$\geq f_1(y) \vee f_2(y) \vee f_3(y) \vee \dots \dots \dots \text{ since each } f_i \text{ is a fuzzy ideal.}$$

$$= \max\{f_1(y), f_2(y) \dots \dots\} = \cup f_i(y)$$

Therefore $\cup f_i(x) \geq \cup f_i(y)$ if $x \leq y$

Let $x, y \in S$ and $\gamma \in \Gamma$

Consider $\cup f_i(x\gamma y) = f_1(x\gamma y) \vee f_2(x\gamma y) \vee f_3(x\gamma y) \vee \dots \dots \dots$

$$\geq f_1(y) \vee f_2(y) \vee f_3(y) \vee \dots \dots \dots \text{ since each } f_i \text{ is a fuzzy left ideal.}$$

$$= \cup f_i(y)$$

Therefore $\cup f_i(x\gamma y) \geq \cup f_i(y)$, Similarly $\cup f_i(x\gamma y) \geq \cup f_i(x)$. Hence the theorem.

THEOREM 3.21: The union of all proper fuzzy ideals ($\cup f_i$) of a **PO-Semigroup S** with ‘ e ’ unity is the unique fuzzy maximal ideal of S .

Proof: let's take $f_M = \cup f_i$.

$\Rightarrow f_M$ would be a fuzzy ideal of S (from Th 3.20).

Suppose f_M is not proper then $f_M = C_S \Rightarrow f_M(x\gamma y) = 1, \forall x \in S, \gamma \in \Gamma$

$\Rightarrow \square_{\square}(\square) = 1$ for some fuzzy ideal \square_{\square} since $\cup \square_{\square} = \square_{\square}$
 $\Rightarrow \square_{\square} = \square_{\square}$ but \square_{\square} is proper. $\therefore \square_{\square}$ is a fuzzy ideal of S which is proper.
 Since \square_{\square} contains all proper fuzzy ideals of S therefore \square_{\square} is maximal fuzzy ideal of S . suppose \square_{\square} is any other maximal fuzzy ideal of S then $\square_{\square} \subseteq \square_{\square} \subseteq \square_{\square}$.

$\therefore \square_{\square} = \square_{\square}$. Hence \square_{\square} is fuzzy maximal ideal of S , which is Unique.

THEOREM 3.22: If S is FLSPOTS(FRSPOTS) then S is fuzzy simple po- Γ -semigroup.

Proof: Assume S is a FLSPOTS(FRSPOTS).
 consider any fuzzy ideal 'f' of $S \Rightarrow f$ is a fuzzy left (right) ideal of S .

$\Rightarrow f$ is a constant function $\Rightarrow S$ is fuzzy simple po- Γ -semigroup.

THEOREM 3.23: If fuzzy ordered element \square_{\square} of S is semisimple and idempotent then $\square_{\square} \subseteq \langle \square_{\square} \rangle^{\square}, \forall \square$.

Proof: Suppose \square_{\square} is semisimple and idempotent, $\square \in \square$ and $n \in \square$.

$\square_{\square} \subseteq \langle \square_{\square} \rangle^{\square}$ which holds for $n=2$ since \square_{\square} is fuzzy semisimple.

Assume the result holds for $n=n-1$. i.e. $\square_{\square} \subseteq \langle \square_{\square} \rangle^{\square-1}$

Consider $\langle \square_{\square} \rangle^{\square} = \langle \square_{\square} \rangle^{\square-1} \Gamma \langle \square_{\square} \rangle$

$$\supseteq \square_{\square} \Gamma \square_{\square} = \square_{\square}^2 = \square_{\square}, \text{ since } \square_{\square} \text{ is idempotent.}$$

Therefore $\langle \square_{\square} \rangle^{\square} \supseteq \square_{\square}, \forall \square$.

4. CONCLUSION:

The objective of this paper is describing fuzzy left ($\square_{(\square\square\square)}$) and fuzzy right ($\square_{(\square\square\square)}$) ideals of S . We established the relationship between fuzzy left simple po- Γ - semigroup and fuzzy simple Γ - semi group. The union of all proper fuzzy ideals ($\cup \square_{\square}$) of po- Γ - semigroup S with unity 'e' is the unique fuzzy maximal ideal of 'S'.

ACKNOWLEDGEMENTS

We are immensely grateful to authors for their valuable effort in the area of research.

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