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On Parseval's Identity and Plancherel Relation in 2-Inner Product Spaces

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Abstract

This paper investigates the foundational concepts of linear 2-normed spaces and 2-inner product spaces, presenting essential definitions and theorems that elucidate their structure. We establish Parseval's identity for finite orthonormal sets within these spaces and examine its relationship with the Plancherel relation. Our findings include theorems related to orthonormal sets and Bessel's inequality, highlighting their implications for countably infinite orthonormal sets. By contributing to the theoretical framework of 2-inner product spaces, we enhance the understanding of orthogonality and summability in this context. Introduced by Diminnie et al. (1973), the 2-inner product extends traditional inner product notions into a two-dimensional framework, offering valuable insights into unique mathematical structures. Recent contributions, such as the formalization of orthonormal sets by Elumalai and Patricia (2000) and the rigorous proofs of the Riesz theorems by Harikrishnan et al. (2011), further deepen our understanding of these spaces. This paper aims to develop important inequalities based on orthonormal sets and present equivalent statements related to them, fostering continued research in various mathematical contexts.

Keywords: *Linear 2-norm, 2-inner product, Orthonormal set, Parseval's identity, Bessel's inequality.*

1. Introduction

The concept of a 2-inner product represents a significant advancement in functional analysis, extending the classical notion of inner products into a two-dimensional framework. Initially introduced by Diminnie et al. in 1973, this innovative approach has laid the foundation for a deeper understanding of the mathematical structures and properties unique to 2-inner product spaces. Unlike traditional inner products, which operate within the confines of a single-dimensional space, the 2-inner product facilitates a more complex interaction between vectors, enriching the analysis of geometric and algebraic relationships. As research has progressed, it has become increasingly clear that a comprehensive understanding of orthogonality in this context is crucial for deriving significant mathematical results and inequalities.

Recent work by Elumalai and Patricia (2000) formalized the notion of orthonormal sets within 2-inner product spaces, significantly advancing the field. Their contributions not only defined these sets but also established key mathematical results, such as Bessel's inequality and the Riesz representation theorem. By employing the concept of orthogonality, originally defined by Cho et al. (1996), they illustrated how these foundational elements could be utilized to deepen the understanding of the relationships between orthonormality and other critical mathematical constructs. Moreover, studies by Kafouros and Kouloumentas (2018) further explored the implications of orthonormal sets in various functional frameworks, providing new insights into their applications in approximation theory and signal processing.

Following this foundational work, Harikrishnan et al. (2011) provided rigorous proofs of the Riesz theorems specifically within the context of 2-inner product spaces. Their research solidified the theoretical underpinnings necessary for various applications of these theorems, thereby enhancing the credibility and applicability of ongoing research in this area. Concurrently, Mazaheri and Kazemi (2007) have employed the Riesz theorem to derive new results, illustrating the theorem's versatility and its pivotal role in expanding the body of knowledge regarding 2-inner product spaces. Additionally, the work of Dutta and Dutta (2020) investigated the connections between 2-inner product spaces and operator theory, revealing potential applications in quantum mechanics and other fields.

The primary objective of this paper is to advance the discourse surrounding 2-inner product spaces by developing important inequalities rooted in the concept of orthonormal sets. Additionally, we aim to present equivalent statements related to these inequalities, thereby contributing to a richer theoretical framework and fostering deeper insights into the mathematical intricacies of 2-inner product spaces. By addressing these aspects, we hope to stimulate further research and application of these concepts in various mathematical and applied contexts.

2. Preliminaries

Some related well known definitions and theorems which are used in this paper are stated below.

Definition 2.1 [2]: Let X be a linear space over reals of dimension greater than one and let $\|.,.\|$ be a real valued function on $X \times X$ satisfying

(N_1) $\|x,y\| = 0 \; iff \; x \; and \; y \; are \; dependent$,

(N_2) $\|x,y\| = \|y,x\| \; for \; every \; x,y \in X$,

(N_3) $\|\alpha x, y\| = |\alpha|\|x,y\|, where \; \alpha \; is \; real \; and$

(N_4) $\|x,y+z\| \leq \|x,y\| + \|x,z\| \; for \; every \; x,y,z \in X$.

$\|.,.\|$ is called a **2-norm** and the linear space X equipped with the 2-norm is called the **linear 2-normed space.**

Definition 2.2 [2]**:** Let X be a linear space of dimension greater than one and $(.,.|.)$ be a real valued function on X × X × X which satisfies the following conditions

(I_1) $(x,x|z) \geq 0, (x,x|z) = 0$ *if and only if x and z are linearly dependent,*

(I_2) $(x,x|z) = (z,z|x)$,

(I_3) $(x,y|z) = (y,x|z)$,

(I_4) $(\alpha x,y|z) = \alpha(x,y|z)$ *where* α *is a real,*

(I_5) $(x+x',y|z) = (x,y|z) + (x',y|z)$, *for every* $x,x',y,z \in X$.

$(.,.|.)$ is called a **2-inner product** and (X, $(.,.|.)$) is called a **2-inner product space**(or a 2-Pre- Hilbert space).

Definition 2.3 [4]: Let $(.,.|.)$ be a 2-inner product on X. An element $a \in X$ is said to be ***orthogonal*** to a non-empty subset S of X (relative to $(.,.|.)$) *if* $(a,b|c) = 0$ for arbitrary $b,c \in S$.

Definition 2.4 [4]: A non-empty subset $\{e_i\}$ of a 2-inner product space $(X,(.,.|.))$ is said to be ***orthonormal*** set if

(2.1) $(e_i,e_j|z) = 1$ *for* $i = j$ *and* $z \in X\setminus\{e_i\}$.

(2.2) $(e_i,e_j|z) = 0$ *for every* $i \neq j$ *and for every* $z \in X\setminus\{e_i\}$.

S. Elumalai and A. Patricia Jennifer Asha [4] have obtained the following results on orthonormal set and developed Bessel's inequality in 2-inner product space.

Theorem 2.1: Let $(X,(.,.|.))$ be a 2-inner product space and let $\{e_1,e_2,e_3,\dots,e_n\}$ be a finite orthonormal set in X. If x is vector in X, then

$$\sum_{i=1}^{n} |(x,e_i|z)|^2 \leq \|x,z\|^2 \quad \text{for every} \quad z \in X\setminus V(x),$$

where $V(x)$ is the space generated by $x \in X$. Further

$$x - \sum_{i=1}^{n} (x, e_i | z) e_i \perp_z e_j, \quad j = 1,2, \dots n.$$

$(x \perp_z y)$ *means x is orthogonal to y with respect to z*

Theorem 2.2[4]: Let $(X, (.,.|.))$ be a 2-inner product space and $\{e_i\}$ is an orthonormal set in X. If x is any vector in X then the set $S = \{e_i : (x, e_i | z) \neq 0\}$ is either empty or countable.

Theorem 2.3 (Bessel's inequality in linear 2-normed spaces)[4]: Let $(X, (.,.|.))$ be a 2-inner product space and $\{e_i\}$ is an orthonormal set in X then

$$\sum_i |(x, e_i | z)|^2 \leq \|x, z\|^2$$

for every $x \in X$ *and* $z \in X \backslash V(x)$, *where* $V(x)$ is the space generated by $x \in X$.

Our main results give Parseval's identity for finite orthonormal set and its relationship with Plancherel relation in 2-inner product spaces.

3. Main Results:

Theorem 3.1(Parseval's identity in linear 2-normed spaces): Let $(E\ (.,.|.))$ be a 2-inner product space and $\{e_i\}$ is a finite orthonormal set in X then, the following relation holds

$$\sum_{i=1}^{n} |(u, e_i | w)|^2 = \|u, w\|^2$$

iff $u = \sum_{i=1}^{n} (u, e_i | w) e_i$ for every $u \in E \backslash V(u)$, where $V(u)$ is the space generated by $u \in E$.

Proof:

'If' part: Suppose $\{e_i\}$ is a closed orthonormal set in E. And $u = \sum_{i=1}^{n} (u, e_i | w) e_{i.}$

Then,

$$\|u, w\|^2 = \|\sum_{i=1}^{n} (u, e_i | w) e_i, \ w\|^2$$

$$= \sum_{i=1}^{n} |(u, e_i | w)|^2 \|e_i, w\|^2$$

$$= \sum_{i=1}^{n} |(u, e_i | w)|^2. \qquad [\because \ \|e_i, w\|^2 = (e_i, e_i | w) = 1]$$

'Only if' part:

Let $\|u,w\|^2 = \sum_{i=1}^{n} |(u, e_i|w)|^2$.

Consider the vector $v = u - \sum_{i=1}^{n} (u, e_i|w)\, e_i$

Now, we have

$$\|v,w\|^2 = (v, v|w)$$

$$= ((u - \sum_{i=1}^{n} (u, e_i|w)e_i), (u - \sum_{i=1}^{n} (u, e_i|w)e_i) \mid w)$$

$$=(u,u|w) - \sum_{i=1}^{n} (u,e_i|w)(e_i,u|w) - \sum_{i=1}^{n} (u,e_i|w)(e_i,u|w)$$

$$+ \sum_{i=1}^{n} \sum_{i=1}^{n} (u, e_i|w)(u, e_i|w)(e_i, e_i|w)$$

$$= (u,u|w) - 2\sum_{i=1}^{n} (u, e_i|w)(u, e_i|w) + \sum_{i=1}^{n} |(u,e_i|w)|^2$$

$$= \|u,w\|^2 - 2\sum_{i=1}^{n} |(u, e_i|w)|^2 - \sum_{i=1}^{n} |(u, e_i|w)|^2$$

$$= \|u,w\|^2 - \sum_{i=1}^{n} |(u, e_i|w)|^2$$

$$= \|u,w\|^2 - \|u,w\|^2$$

$$\Rightarrow v = 0$$

$$\Rightarrow u = \sum_{i=1}^{n} (u, e_i|w)e_i$$

Remark (1): The relation $\|u,w\|^2 = \sum_{i=1}^{\infty} |(u, e_i|w)|^2$ is the Parseval's identity for an infinite orthonormal set in linear 2-normed spaces.

Theorem 3.2: Let $\left(E, (.,.|.)\right)$ be a 2-inner product space, and let $\{e_i, i \in \Lambda\}$ be a countably infinite orthonormal set in X. Then the following statements are equivalent.

For each $u, v \in \left(E, (.,.|.)\right),\ w \in E\backslash V(u), where\ V(u)\ is\ space\ generated\ by\ u \in E.$

(3.1) $\|u,w\|^2 = \sum_{i\in\Lambda} |(u, e_i|w)|^2, \quad \boldsymbol{Parseval's\ identity}.$

(3.2) $u = \sum_{i\in\Lambda} (u, e_i|w)e_i \quad as\ a\ norm\ convergent\ series.$

(3.3) $(u, v|w) = \sum_{i\in\Lambda} (u, e_i|w)\overline{(v, e_i|w)}\ ,\ (\boldsymbol{Plancherel\ relation})\ holds.$

Proof: For an arbitrary finite subset S of Λ, the set

$$u_s = \sum_{i\in S}(u, e_i|w)e_i \text{ and } K = span\{e_i,\ i \in S\}$$

Then, for each given $u \in (E, (.,.)), we\ have$

$$(u - u_s, v|w) = (u - \sum_{i\in S}((u, e_i|w)e_i, v|w) = 0, \text{for all } v \in K,$$

which shows that $v \perp (u - u_s)$.

By orthogonality

$$\left\|\sum_{i\in S}(u, e_i|w)e_i, w\right\|^2 = \sum_{i\in S}|(u, e_i|w)|^2$$

In the particular, the map

$$P_K: x \to \sum_{i\in S}(u, e_i|w)e_i$$

is the orthogonal projection of $(E, (.,.|.))$ onto K, and

(3.i) $\qquad \|u, w\|^2 = \sum_{i\in S}|(u, e_i|w)|^2 + \|u - P_K u, w\|^2$ holds for each $u \in (E, (.,.|.))$.

Now we are to prove $(3.1) \Rightarrow (3.2) \Rightarrow (3.3) \Rightarrow (3.1)$

$\mathbf{(3.1) \Rightarrow (3.2)}$**:** Using $(3.i)$ for every $u \in X$, and for each finite subset Λ_1 of Λ, we have

$$\left\|u - \sum_{i\in\Lambda_1}(u, e_i|w)e_i, w\right\|^2 = \|u, w\|^2 - \sum_{i\in\Lambda_1}|(u, e_i|w)|^2$$

$$= \sum_{i\in\Lambda}|(u, e_i|w)|^2 - \sum_{i\in\Lambda_1}|(u, e_i|w)|^2 \qquad \text{[by (3.1)]}$$

$$= \sum_{i\in\Lambda\setminus\Lambda_1}|(x, e_i|z)|^2$$

By theorem (2.2) $\{e_i: (e_i, u|w) \neq 0\}$ is countable. So that $\{(u, e_i|w)e_i\}$ is summable to x, and hence the above equation can be written as

$$\left\|u - \sum_{i=1}^{n}(u, e_i|w)e_i, w\right\|^2 = \sum_{i=n+1}^{\infty}|(u, e_i|w)|^2$$

i.e. $u = \sum_{i\in\Lambda}(u, e_i|w)e_i$, which is (3.2)

$\mathbf{(3.2) \Rightarrow (3.3)}$**:** Let $u = \sum_{i\in\Lambda}(u, e_i|w)e_i\ and\ v = \sum_{j\in\Lambda}(v, e_j|w)e_j$

Then, the continuity of the inner product gives

$$(u, v|w) = \left(\sum_{i\in\Lambda}(u, e_i|w)e_i, \sum_{j\in\Lambda}(v, e_i|w)e_j|w\right)$$

$$= \sum_{i\in\Lambda}\sum_{j\in\Lambda}(u, e_i|w), \overline{(v, e_j|w)}\,(e_i, e_j|w)$$

$$= \sum_{i\in\Lambda}(u, e_i|w)\,\overline{(v, e_i|w)}$$

So, (3.3) holds.

$\mathbf{(3.3) \Rightarrow (3.1)}$**:**

This part follows if we take $u = v$ in (3.3).

This completes the proof.

4. Conclusion

The investigation of 2-inner product spaces marks a significant advancement in functional analysis, broadening the traditional inner product theories into a more complex two-dimensional framework. This study highlights the necessity of understanding the unique structural properties of these spaces, especially in relation to orthonormal sets and their connections to key mathematical results. Important findings, such as Bessel's inequality and Parseval's identity, illustrate the intricate relationship between orthogonality and summability within these frameworks. The results not only reinforce the theoretical foundations of 2-inner product spaces but also enhance their relevance across a variety of mathematical and applied fields. The formulation of Parseval's identity for finite orthonormal sets, along with its implications for countably infinite orthonormal sets, provides critical insights that can further stimulate research in areas such as approximation theory and signal processing.

Furthermore, by developing important inequalities based on the concept of orthonormality and offering equivalent statements regarding these inequalities, this paper contributes to a richer understanding of the complexities inherent in 2-inner product spaces. As the discourse on these subjects continues to progress, it is anticipated that this work will encourage additional exploration and application of 2-inner product spaces in various mathematical contexts, ultimately enriching the field and paving the way for new insights in functional analysis and its related disciplines. The findings underscore the potential of 2-inner product spaces to enhance both theoretical knowledge and practical applications in contemporary mathematics.

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