Common Fixed Point Theorems in Dislocated Cone Metric Spaces

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Abstract
Jungck and Rhoades [18] introduced the notion of weakly compatible mappings, which is weaker than compatibility. Many interesting fixed point theorems for weakly compatible maps satisfying contractive type conditions have been obtained by various authors. In this paper, notion of dislocated cone is introduced and a common fixed point theorem for three pairs of weakly compatible mappings satisfying a rational inequality without any continuity requirement which generalize several previously known results due to Imdad and Ali [13], Goyal ([4], [5]), Goyal and Gupta ([6], [7]), Imdad-Khan [14], Jeong-Rhoades [15] and others.

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1. Introduction And Preliminaries

The notion of a metric space, introduced by Maurice Fréchet [3] in 1906, is a foundational concept not only in mathematics but also in various quantitative sciences. Due to its significance and potential applications, this notion has been extended, refined, and generalized in numerous ways. The concept of a dislocated metric (d-metric) was introduced by Hitzler and Seda [11], particularly valuable in logic programming semantics. Over time, numerous papers have been published on fixed point and common fixed point theorems that satisfy certain contractive conditions in dislocated metric spaces.

Huang and Zhang [9] generalized the concept of a cone metric space, re-placing the set of real numbers by an ordered Banach space and obtained some common fixed point theorems for mappings satisfying different contractive conditions over cone metric space. Subsequently, Abbas and Jungck[1] and Abbas and Rhoades[2] studied common fixed point theorems in cone metric spaces. Moreover, Huang and Zhang [9], Abbas and Jungck[1], IIlic and Rakocevic [12] proved their results for normal cones. Jungck [17] generalized the concept of weak commuting by defining the term compatible mappings and proved that the weakly commuting mappings are compatible but the converse is not true. Jungck and Rhoades [18] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space utilizing these concepts. In this paper, we prove a common fixed point theorem for three pairs of weakly compatible mappings satisfying a rational inequality without any continuity requirement in complete dislocated cone metric spaces. Our work generalizes some earlier results of Imdad and Ali [13], Jeong and Rhoades [15], Goyal ([4], [5]),Goyal and Gupta ([6], [7]). Some examples are also furnished to demonstrate the validity of the hypothesis.

We give the definition of dislocated cone metric space and some of their properties. The following notions will be used to prove the main result.

Definition 1.1. Let E be a real Banach space and P be a subset of E. The subset P is called a cone if and only if
(a) P is closed, non-empty and $P \neq \{0\}$
(b) $a, b \in R,\ a, b \geq 0,\ x, y \in P$ implies $ax + by \in P$
(c) $x \in P$ and $-x \in P \Rightarrow x = 0$ i.e $P \cap (-P) = \{0\}$

Definition 1.2. Let P be a cone in a Banach space E i.e. given a cone $P \subset E$, define partial ordering '$\leq$' with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$while $x \ll y$ will stand for $y - x \in Int\ P$, where $Int\ P$ denote the interior of the set P. This cone P is called an order cone.

Definition 1.3. Let E be a real Banach space and $P \subset E$ be an order cone. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4. Let X be a non-empty and E be a real Banach space. Suppose that the mapping $d: X \times X \to E$ satisfies

(i) $d(x,x) = 0 \; \forall \, x \in X$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) = d(y,x) = 0$ implies $x = y$

(iv) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space. If d satisfies the conditions from (ii)- (iv), then d is said to be dislocated cone metric on X and the pair (X, d) is called dislocated cone metric space.

The above definition change to usual definition of cone metric space if $\emptyset(t) = I$.

It is obvious that cone metric spaces generalize metric spaces because each metric space is a cone metric space with $E = R$ and $P = [0, +\infty[$

Example 1.1. (a) Let $E = R^2$, $P = \{(x,y) \in E \,|x,y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$ such that

$d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant.

Then (X, d) is a cone metric space.

(b) Let $E = R^n$ with $P = \{(x_i, \ldots, x_n): x_i \geq 0, \forall i = 1,2,\ldots,n\} X = R$ and $d: X \times X \to E$ such that

$d(x,y) = (|x-y|, \alpha_i|x-y|, \ldots, \alpha_{n-1}|x-y|)$

where $\alpha_i \geq 0$ for all $1 \leq i \leq n-1$. Then (X, d) is a cone metric space.

Definition 1.5. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$is a

(a) convergent sequence or $\{x_n\}$ converges to x if for every c in E with $c \gg 0$, there is an $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \ll c$ for some fixed point x in X where x is that limit of $\{x_n\}$. This is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x, n \to \infty$. Completeness is defined in the standard way.

It was proved in [9] if (X, d) be a cone metric space, P be a normal cone with normal constant K and $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(b) Cauchy sequence if for c in E with $c \gg 0$, there is an $n_0 \in N$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$.

It was proved in [9] if (X, d) be a cone metric space, P be a normal cone with normal constant K and $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 1.6. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence in unique provided P is a normal cone with normal constant K.

In recent years several definitions of conditions weaker than commutativity have appeared which facilitated significantly to extend the Jungck's [16] theorem and several others. Foremost among of them is perhaps the weak commutativity condition introduced by Sessa [22] which can be described as follows:

Definition 1.7. Let S and T be mappings from a cone metric space (X, d) into itself. Then S and T are said to be weakly commuting mappings on X if

$d(STx, TSx) \leq d(Sx, Tx)$, for all $x \in X$.

Obviously, a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

Example 1.2. Consider the set $X = [0,1]$ with the usual metric defined by

$d(x,y) = |x-y| = \|x-y\|$

Define S and $T: X \to X$ by

$Sx = \frac{x}{3-2x}$ and $Tx = \frac{x}{3}$ for all $x \in X$.

Then, we have to any x in X

$STx = \frac{x}{9-2x}$ and $TSx = \frac{x}{9-6x}$

Hence $ST \neq TS$. Thus, S and T do not commute.

Again, $d(STx, TSx) = \left\|\frac{x}{9-2x} - \frac{x}{9-6x}\right\|$

$$= \frac{4x^2}{(9-2x)(9-6x)}$$

$$\le \frac{2x^2}{3(3-2x)} = \frac{x}{3-2x} - \frac{x}{3} = d(Sx, Tx)$$

and thus S and T commute weakly.

Example 1.3. Consider the set $X = [0,1]$ with the usual metric $d(x, y) = \|x - y\|$. Let $Sx = \frac{x}{2}$ and $Tx = \frac{x}{2+x}$, for every $x \in X$. Then, for all $x \in X$

$STx = \frac{x}{4+2x}$ and $TSx = \frac{x}{4+x}$

Hence, $ST \ne TS$. Thus, S and T do not commute.

Again, $d(STx, TSx) = \left\|\frac{x}{4+2x} - \frac{x}{4+x}\right\|$

$$= \frac{x^2}{(4+x)(4+2x)}$$

$$\le \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx)$$

and thus, S and T commute weakly.

Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.

Jungck [17] has observed that for $X = R$ if $Sx = x^3$ and $Tx = 2x^3$ then S and T are not weakly commuting. Thus it is desirable to a less restrictive concept which he termed as 'compatibility' the class of compatible mappings is still wider and includes weakly commuting mappings as subclass as is evident from the following definition of Jungck [17].

Definition 1.8. Let S and T be self mappings on a cone metric space (X, d). Then S and T are said to be compatible mappings on X if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some point } t \in X.$$

Obviously, any weakly commuting pair $\{S, T\}$ is compatible, but the converse is not necessarily true, as in the following example.

Example 1.4. Let $Sx = x^3$ and $Tx = 2x^3$ with $X = R$ with the usual metric. Then S and T are compatible, since $|Tx - Sx| = |x^3| \to 0$ if and only if $|STx - TSx| = 6|x^9| \to 0$ But $|STx - TSx| \le |Tx - Sx|$ is not true for all $x \in X$, say for example at $x = 1$.

Definition 1.9. Let S and T be self maps of a set X. If $w = Sx = Tx$ for some x in X, then x is called a coincidence point of S and T and w is called a point of coincidence of S and T.

Jungck-Rhoades [18] obtained the concept of weakly compatible as follows:

Definition 1.10. A pair of self mappings (S, T) on a cone metric space (X, d) is said to be weakly compatible if the mappings commute at their coincidence points i.e $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$.

Lemma 2.1([19]) Let A and Bbe weakly compatible self maps of a set X. IfA and B have a unique point of coincidence $w = Ax = Bx$, then w is the unique common fixed point of A and B.

Example 1.5.

Let X = [0, 3] be equipped with the usual metric space $d(x, y) = |x - y|$.

Define S, T : [0, 3] → [0, 3] by

$$Sx = \begin{cases} x, & x \in [0,1) \\ 3, & x \in [1,3] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 3 - x, & x \in [0,1) \\ 3, & x \in [1,3] \end{cases}$$

Then for any $x \in [1, 3]$, $STx = TSx$, showing that S and T are weakly compatible maps on [0, 3]

Example 1.6.

Let X = R and define S, T : R→ R by $Sx = x/3,\ x \in R$ and $Tx = x^2$, $x \in R$. Here 0 and 1/3 are two coincidence points for the maps S and T. Note that S and T commute at 0, i.e. $ST(0) = TS(0) = 1/27$, but $ST(1/3) = S(1/9) = 1/27$ and $TS(1/3) = T(1/9) = 1/81$ and so S and T are not weakly compatible maps on R.

Example 1.7. Let X = [2, 20] with usual metric. Define

$$Tx = \begin{cases} 2, & if\ x = 2 \\ 12 + x, & if\ 2 < x \le 5 \\ x - 3, & if\ 5 < x \le 20 \end{cases} \quad \text{and } Sx = \begin{cases} 2 & if\ x \in \{2\} \cup (5,20] \\ 8 & if\ 2 < 5 \le 5, \end{cases}$$

S and *T* are weakly compatible mappings which is not compatible.

Theorem 1.1 Let (X, d) be a complete d-cone metric space and let $T: X \to X$ be a contraction mapping, then T has a unique fixed point.

Remark 1.1. Let (X, d) be a cone metric space with a cone P. If $d(x, y) \le hd(x, y)$ for all $x, y \in X, h \in (0,1)$, then $d(x, y) = 0$, which implies that $x = y$.

2. MAIN RESULTS

The following Lemma is the key in proving our result. Its proof is like that of Jungck [16].

Lemma 2.1: Let $\{y_n\}$ be a sequence in a complete metric space (X, d). If there exists a $k \in (0,1)$ such that $d(y_{n+1}, y_n) \le\ d(y_{n+1}, y_n)$,for all n, then $\{y_n\}$ converges to a point in X.

Motivated by the contractive condition given by, Jeong Rhoades [15], we prove the following theorem.

Theorem 2.1. Let (X, d) be a complete dislocated cone metric space and P be a normal cone with normal constant K. Let A, B, S, T, I and J be self-mappings of a cone metric space (X, d) satisfying

(i) $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ such that for each $x, y \in X$

(ii) $d(ABx, STy) \le \beta_1 \left[\frac{\{d(ABx,Ix)\}^2 + \{d(STy,Jy)\}^2}{d(ABx,Ix) + d(STy,Jy)}\right] + \beta_2 d(Ix, Jy) + \beta_3[d(ABx, Jy) + d(STy, Ix)]$, ……(2.1)

for all $x, y \in X$ and $\beta_i \ge 0$ $(i = 1,2,3)$ with at least one β_i non zero and $2\beta_1 + \beta_2 + 2\beta_3 < 1$.

If one of the $AB(X)$, $ST(X)$, $J(X)$ and $I(X)$ is a complete subspace of X, then

(a) (AB, I) has a coincidence point (b) (ST, J) has a coincidence point.

Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point. Moreover, if the pairs (A, B), (A, I), (B, I), (S, T), (S, J) and (T, J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $AB(X) \subset J(X)$, we can choose a point x_1 in X such that $ABx_0 = Jx_1$. Again, since $ST(X) \subset I(X)$, we can choose a point x_2 in $X$with $STx_1 = Ix_2$. Using this process repeatedly, we can construct a sequence $\{z_n\}$ such that

$z_{2n} = ABx_{2n} = Jx_{2n+1}$ and $z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$for $n = 0, 1, 2, ...$

Then on using inequality (1), we have

$$\begin{aligned} d(z_{2n+1}, z_{2n+2}) &= d(STx_{2n+1}, ABx_{2n+2}) \\ &\le \beta_1 \left[\frac{\{d(ABx_{2n+2}, Ix_{2n+2})\}^2 + \{d(STx_{2n+1}, Jx_{2n+1})\}^2}{d(ABx_{2n+2}, Ix_{2n+2}) + (STx_{2n+1}, Jx_{2n+1})}\right] + \beta_2 d(Ix_{2n+2}, Jx_{2n+1}) \\ &\quad + \beta_3[(d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})] \\ &= \beta_1[d(ABx_{2n+2}, Ix_{2n+2}) + d(STx_{2n+1}, Jx_{2n+1})] + \beta_2 d(Ix_{2n+2}, Jx_{2n+1}) \\ &\quad + \beta_3[d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})] \\ &= \beta_1[d(z_{2n+2}, z_{2n+1}) + d(z_{2n+1}, z_{2n})] + \beta_2 d(z_{2n+1}, z_{2n}) + \beta_3[d(z_{2n+2}, z_{2n+1}) + d(z_{2n+1}, z_{2n})] \\ &= (\beta_1 + \beta_3)\, d(z_{2n+2}, z_{2n+1}) + (\beta_1 + \beta_2 + \beta_3) d(z_{2n}, z_{2n+1}), \end{aligned}$$

which implies that

$$d(z_{2n+2}, z_{2n+1}) \le \left(\frac{\beta_1 + \beta_2 + \beta_3}{1 - \beta_1 - \beta_3}\right) d(z_{2n+1}, z_{2n})$$

Following the same process, we can show that

$d(z_{2n},z_{2n+1}) \le \left(\frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3}\right) d(z_{2n-1},z_{2n})$

Thus, for every n, we can show that

$d(z_n,z_{n+1}) \le \alpha d(z_{n-1},z_n),$ … (2.2)

where $\alpha = \frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3} < 1$

Now, by induction

$$\begin{aligned} d(z_n,z_{n+1}) &\le \alpha d(z_{n-1},z_n) \\ &\le \alpha^2 d(z_{n-2},z_{n-1}) \\ &\vdots \\ &\le \alpha^n d(z_0,z_1) \end{aligned}$$

For any $m > n$, we get,

$$\begin{aligned} d(z_n,z_m) &\le d(z_n,z_{n+1}) + d(z_{n+1},z_{n+2}) + \ldots + d(z_{m-1},z_m) \\ &\le [\alpha^n+\alpha^{n+1}+\ldots\ldots+\alpha^{m-1}] d(z_0,z_1) \\ &\le \frac{\alpha^n}{1-\alpha} d(z_0,z_1) \end{aligned}$$

Now, using normality of cone, we get

$||d(z_n,z_m)|| \le \frac{\alpha^n}{1-\alpha}.K||d(z_0,z_1)||$

This implies that $d(z_n,z_m) \to 0$ as $n,m \to \infty$

Hence, sequence $\{z_n\}$ described by

$\{ABx_0, STx_1, ABx_2, \ldots STx_{2n-1}, ABx_{2n}, STx_{2n+1}, \ldots\}$

is a Cauchy sequence in a cone metric space (X,d). Now, let $ST(X)$ is a complete subspace of X, then the subsequence $\{z_{2n+1}\}$ which is contained in $ST(X)$ also get a limit z in $ST(X)$ i.e.

$\lim_{n\to\infty} STx_{2n+1} = z$

Since, $ST(X) \subset I(X)$, there exists a point $z' \in X$ such that $Iz' = z$.

Again, as $\{z_n\}$ is a Cauchy sequence containing a convergent subsequence $\{z_{2n+1}\}$, therefore the sequence $\{z_n\}$ also converges which implies the convergence of $\{z_{2n}\}$ being a subsequence of the convergent sequence $\{z_n\}$ i.e.

$\lim_{n\to\infty} Jx_{2n+1} = z.$

To prove that $ABz' = z$ put $x = z'$ and $y = x_{2n-1}$ in (1), we get

$$d(ABz',STx_{2n-1}) \le \beta_1\left[\frac{\{d(ABz',Iz')\}^2+\{d(STx_{2n-1},Jx_{2n-1})\}^2}{d(ABz',Iz)+d(STx_{2n-1},Jx_{2n-1})}\right] + \beta_2 d(Iz',Jx_{2n-1}) + \beta_3[d(ABz',Jx_{2n-1}) + d(STx_{2n-1},Iz')]$$

on letting $n \to \infty$, above reduces to

$$\begin{aligned} d(ABz',z) &\le \beta_1\left[\frac{\{d(ABz',z)\}^2+\{d(z,z)\}^2}{d(ABz',z)+d(z,z)}\right] + \beta_2 d(z,z) + \beta_3[d(ABz',z) + d(z,z)] \\ &\le \beta_1[d(ABz',z)+d(z,z)] + \beta_2 d(z,z) + \beta_3[d(ABz',z)+d(z,z)] \\ &\le (\beta_1+\beta_3)[d(ABz',z)+d(z,z)] + \beta_2 d(z,z) \\ &\le (\beta_1+\beta_3)[d(ABz',z)+\{d(z,ABz')+d(ABz',z)\}] + \beta_2[d(z,ABz')+d(ABz',z)] \\ &\le (3\beta_1+2\beta_2+3\beta_3)\, d(ABz',z), \end{aligned}$$

which is a contradiction, since $3\beta_1+2\beta_2+3\beta_3 \neq 1$.

Implying thereby $ABz' = z$ [by using Remark (1.1)].

Thus, we get $ABz' = Iz' = z$ and result (a) is established i.e the pair (AB, I) has a coincidence point.

Since z is in the range of AB i.e. $ABz' = z$ and $AB(X) \subset J(X)$ there always exists a point z" such that $Jz" = z$

Now, $d(z,STz") = d(ABz',STz")$

$$\begin{aligned} &\le \beta_1\left[\frac{\{d(ABz',Iz')\}^2+\{d(STz",Jz")\}^2}{d(ABz',Iz')+d(STz",Jz")}\right] + \beta_2 d(Iz',Jz") + \beta_3[d(ABz',Jz") + d(STz",Iz')] \\ &\le \beta_1\left[\frac{\{d(z,z)\}^2+\{d(STz",z)\}^2}{d(z,z)+d(STz",z)}\right] + \beta_2 d(z,z) + \beta_3[d(z,z) + d(STz",z)] \\ &\le \beta_1[d(z,z)+d(STz",z)] + \beta_2 d(z,z) + \beta_3[d(z,z)+d(STz",z)] \\ &\le (\beta_1+\beta_2+\beta_3)[d(z,STz")+d(STz",z)] + (\beta_1+\beta_3) d(z,STz") \\ &\le (3\beta_1+2\beta_2+3\beta_3)\, d(z,STz"), \end{aligned}$$

which is a contradiction, since $3\beta_1+2\beta_2+3\beta_3 \neq 1$.

Implying thereby that $STz" = z = Jz"$ [by using Remark (1.1)], i.e. the pair (ST, J) has a coincidence point. This establishes the result (b).

If we assume that $I(X)$ is a complete subspace of X, then similar arguments establish results (a) and (b). The remaining two cases pertain essentially to the previous cases.

Infect, if $ST(X)$ is complete then $z \in ST(X) \subset I(X)$ and if $AB(X)$ is complete, then, $z \in AB(X) \subset J(X)$. Thus, the results (a) and (b) are completely established.

Furthermore, if the pairs (AB, I) and (ST, J) are coincidentally commuting at z' and $z"$ respectively then

(i) $z = ABz' = Iz' = STz" = Jz"$

(ii) $ABz = AB(Iz') = I(ABz') = Iz$

(iii) $STz = ST(Jz") = J(STz") = Jz$

$$d(z, STz) = d(ABz', STz)$$
$$\leq \beta_1\left[\frac{\{d(ABz', Iz')\}^2 + \{d(STz, Jz)\}^2}{d(ABz', Iz') + d(STz, Jz)}\right] + \beta_2 d(Iz', Jz) + \beta_3[d(ABz', Jz) + d(STz, Iz')]$$
$$\leq \beta_1\left[\frac{\{d(z,z)\}^2 + \{d(STz,z)\}^2}{d(z,z) + d(STz,z)}\right] + \beta_2 d(z,z) + \beta_3[d(z,z) + d(STz,z)]$$
$$\leq \beta_1[d(z,z) + d(STz,z)] + \beta_2 d(z,z) + \beta_3[d(z,z) + d(STz,z)]$$
$$\leq (\beta_1 + \beta_2 + \beta_3)[d(z,STz) + d(STz,z)] + (\beta_1 + \beta_3)d(z,STz)$$
$$\leq (3\beta_1 + 2\beta_2 + 3\beta_3)\, d(z, STz),$$

, which is a contradiction, since $3\beta_1 + 2\beta_2 + 3\beta_3 \neq 1$.

Implying thereby that $z = STz$ [by using Remark (1.1)].

$$d(ABz, z) = d(ABz, STz")$$
$$\leq \beta_1\left[\frac{\{d(ABz, Iz)\}^2 + \{d(STz", Jz")\}^2}{d(ABz, Iz) + d(STz", Jz")}\right] + \beta_2 d(Iz, Jz") + \beta_3[d(ABz, Jz") + d(STz", Iz)]$$
$$\leq \beta_1\left[\frac{\{d(ABz, ABz)\}^2 + \{d(z,z)\}^2}{d(ABz, ABz) + d(z,z)}\right] + \beta_2 d(ABz, z) + \beta_3[d(ABz, z) + d(z, ABz)]$$
$$\leq \beta_1[d(ABz, ABz) + d(z,z)] + \beta_2 d(ABz, z) + \beta_3[d(ABz, z) + d(z, ABz)]$$
$$\leq \beta_1[\{d(ABz,z) + d(z,ABz)\} + \{d(z,ABz) + d(ABz,z)\}] + \beta_2 d(ABz,z) + \beta_3[d(ABz,z) + d(z,ABz)]$$
$$\leq (4\beta_1 + \beta_2 + 2\beta_3) d(ABz, z)$$

which is a contradiction, since $4\beta_1 + \beta_2 + 2\beta_3 \neq 1$

yielding, thereby $z = ABz$[by using Remark (1.1)].

Thus, $ABz = Iz = STz = Jz = z$, which shows that z is a common fixed point of AB, ST, I and J.

To show that z is unique, let u be another fixed point of I, J, AB and ST. Then,

$$d(z, u) = d(ABz, STu)$$
$$\leq \beta_1\left[\frac{\{d(ABz, Iz)\}^2 + \{d(STu, Ju)\}^2}{d(ABz, Iz) + d(STu, Ju)}\right] + \beta_2 d(Iz, Ju) + \beta_3[d(ABz, Ju) + d(STu, Iz)]$$
$$\leq \beta_1\left[\frac{\{d(ABz, Iz) + d(STu, Ju)\}^2}{d(ABz, Iz) + d(STu, Ju)}\right] + \beta_2 d(Iz, Ju) + \beta_3[d(ABz, Ju) + d(STu, Iz)]$$
$$\leq \beta_1[d(ABz, Iz) + d(STu, Ju)] + \beta_2 d(Iz, Ju) + \beta_3[d(ABz, Ju) + d(STu, Iz)]$$
$$\leq \beta_1[d(z,z) + d(u,u)] + \beta_2 d(z,u) + \beta_3[d(z,u) + d(u,z)]$$
$$\leq \beta_1[\{d(z,u) + d(u,z)\} + \{d(u,z) + d(z,u)\}] + \beta_2 d(z,u) + \beta_3[d(z,u) + d(u,z)]$$
$$\leq (4\beta_1 + \beta_2 + 2\beta_3) d(z,u)$$

which is a contradiction, since $4\beta_1 + \beta_2 + 2\beta_3 \neq 1$,

yielding, thereby $z = u$.

Thus, z is a unique common fixed point of AB, ST, I and J.

Finally, we prove that z is also a common fixed point A,B, S,T, I and J. For this, let both the pairs (AB, I) and (ST, J) have a unique common fixed point z.

Then $Az = A(ABz) = A(BAz) = AB(Az)$, $Az = A(Iz) = I(Az)$

and $Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz)$, $Bz = B(Iz) = I(Bz)$,

which shows that (AB, I) has common fixed points, which are Az and Bz.

We get thereby, $Az = z = Bz = Iz = ABz$, by virtue of uniqueness of common fixed point of pair (AB, I).

Similarly, using the commutativity of (S,T), (S,J) and (T,J), $Sz = z = Tz = Jz = STz$ can be shown.

Now, to show that $Az = Sz$ ($Bz = Tz$), we have

$$d(Az,Sz) = d\left(A(BAz),S(TSz)\right) = d\left(AB(Az),ST(Sz)\right)$$
$$\leq \beta_1\left[\frac{\{d(AB(Az),I(Az))\}^2+\{d(ST(Sz),J(Sz))\}^2}{d(AB(Az),I(Az))+d(ST(Sz),J(Sz))}\right]+\beta_2 d\left(I(Az),J(Sz)\right)$$
$$+\beta_3\left[d\left(AB(Az),J(Sz)\right)+d\left(I(Az),ST(Sz)\right)\right]$$
$$\leq \beta_1[d(AB(Az),\ I(Az))+d(ST(Sz),J(Sz))]+\beta_2 d\left(I(Az),J(Sz)\right)$$
$$+\beta_3\left[d\left(AB(Az),J(Sz)\right)+d\left(I(Az),ST(Sz)\right)\right]$$
$$\leq \beta_1[d(Az,Az)+d(Sz,Sz)]+\beta_2 d(Az,Sz)$$
$$+\beta_3[d(Az,Sz)+d(Az,Sz)]$$
$$\leq \beta_1[\{d(Az,Sz)+d(Sz,Az)\}+\{d(Sz,Az)+d(Az,Sz)\}]$$
$$+(\beta_2+2\beta_3)d(Az,Sz)$$
$$\leq (4\beta_1+\beta_2+2\beta_3)d(Az,Sz),$$

which is a contradiction, since $4\beta_1+\beta_2+2\beta_3 \neq 1$,
using condition (2), thereby we get $Az = Sz$.
Similarly, $Bz = Tz$ can be shown.
Hence, z is a unique common fixed point of A, B, S, T, I and J.
This completes the proof.

Putting $AB = A, ST = B$ in Theorem 2.1, we obtain the following generalization of the result of Imdad and Ali [13] in cone metric space.

Corollary 2.1. Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Let A, B, S, T, I and J be self-mappings of a cone metric space (X,d) satisfying

(i) $A(X) \subset J(X)$, $B(X) \subset I(X)$ such that for each $x,y \in X$ either

(ii) $d(Ax,By) \leq \beta_1\left[\frac{\{d(Ax,Ix)\}^2+\{d(By,Jy)\}^2}{d(Ax,Ix)+d(By,Jy)}\right]+\beta_2 d(Ix,Jy)+\beta_3[d(Ax,Jy)+d(By,Ix)]$

for each $x,y \in X$ and $\beta_i \geq 0$ $(i = 1,2,3)$ with at least one β_i non zero and $2\beta_1+\beta_2+2\beta_3 < 1$.
If one of the $A(X),\ B(X),\ I(X)$ and $J(X)$ is a complete subspace of X, then
(a) (A, I) has a coincidence point (b) (B, J) has a coincidence point.
Further, if the pairs (A,I) and (B,J) are coincidentally commuting (weakly compatible), then A, B, I and J have a unique common fixed point.

On the basis of the above Corollary (2.1), we have the following result of Singh et al. [23], whose proof is similar to that of Corollary (2.1).

Corollary 2.2. Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Let A, B, S, T, I and J be self-mappings of a cone metric space (X,d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ such that for each $x,y \in X$.

$$d(ABx,STy) \leq \beta_1[d(ABx,Ix)+d(STy,Jy)]+\beta_2 d(Ix,Jy)+\beta_3[d(ABx,Jy)+d(STy,Ix)]$$

where $\beta_i \geq 0, (i = 1,2,3)$ (with at least one β_i non zero) and $2\beta_1+\beta_2+2\beta_3 < 1$
If one of the $AB(X),\ ST(X),\ J(X)$ and $I(X)$ is a complete subspace of X, then

(a) (AB, I) has a coincidence point

(b) (ST, J) has a coincidence point

Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point.

Moreover, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.

Proof. Since

$$\frac{[d(ABx,Ix)]^2+[d(STy,Jy)]^2}{d(Ax,Fx)+d(Sy,\ Gy)} \leq \frac{[d(ABx,Ix)+d(STy,Jy)]^2}{d(Ax,Fx)+d(Sy,\ Gy)} = d(ABx,Ix)+d(STy,Jy)$$

Using above inequality in main Theorem (2.1), we get the corollary (2.4).

Taking $AB = A, ST = B, I = J = S$ in Corollary (2.2), we obtain the following result of Olaleru [20].

Corollary 2.5. Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Let A, B and S be self-mappings of a cone metric space (X, d) satisfying $A(X) \subset S(X)$, $B(X) \subset S(X)$ such that for each $x, y \in X$.

$$d(Ax, By) \le \beta_1[d(Ax, Sx) + d(By, Sy)] + \beta_2 d(Sx, Sy) + \beta_3[d(Ax, Sy) + d(By, Sx)]$$

where $\beta_i \ge 0, (i = 1,2,3)$ (with at least one β_i non zero) and $2\beta_1 + \beta_2 + 2\beta_3 < 1$

If one of the $A(X)$, $B(X)$ and $S(X)$ is a complete subspace of X, then the pair (AB, S) have unique coincidence point.

Further, if the pairs (A, S) and (B, S) are coincidentally commuting (weakly compatible), then A, B and S have a unique common fixed point.

Now, we furnish an example to demonstrate the validity of the hypothesis of our Corollary(2.1).

Example 2.1. Consider $X = [0,1]$ with the usual metric defined by

$d(x, y) = \|x - y\| = |x - y|$ and $F = R =$ Real Banach space.

Define self mappings A, B, S, T, I and J on X by

$$Ax = \frac{3x}{8}, Bx = \frac{4x}{10}, Sx = \frac{x}{5}, Tx = \frac{5x}{12}, \ Ix = \frac{3x}{20}, Jx = \frac{x}{3}$$

Here, $ABx = A\left(\frac{4x}{10}\right) = \frac{3}{8}\left(\frac{4x}{10}\right) = \frac{3}{20}x$

$STx = S\left(\frac{5x}{12}\right) = \frac{1}{5}\left(\frac{5x}{12}\right) = \frac{x}{12}$

$\therefore$ $AB(X) = \left[0, \frac{3}{20}\right] \subset \left[0, \frac{1}{3}\right] = J(X)$

$ST(X) = \left[0, \frac{1}{12}\right] \subset \left[0, \frac{3}{20}\right] = I(X)$

or, $AB(X) \subset J(X)$ and $ST(X) \subset I(X)$

Here all the contractive condition of the Corollary (2.2) are satisfied. Hence, mappings A, B, S, T, I and J have a unique common fixed point at $x = 0$.

Now, we furnish an example to demonstrate the validity of the hypothesis of our Corollary (2.3).

Example 2.2. Consider $X = [0,8]$ with the usual metric defined by

$d(x, y) = \|x - y\| = |x - y|$ and $F = R =$ Real Banach space.

Define self mappings A, B, S and T on X as

$A0 = 0, \ Ax = 1, \ 0 < x \le 8$

$B0 = 0, \ Bx = 1, \ 0 < x < 8, \ B8 = 0$

$S0 = 0, \ Sx = 7, \ 0 < x < 8, \ S8 = 4$

$T0 = 0, \ Tx = 8, \ 0 < x < 8, \ T8 = 1$

Here all the four maps in this example are discontinuous even at their unique common fixed point 0.

Here, $A(X) = \{0,1\} \subset T(X) = \{0,1,8\}$

And $B(X) = \{0,4\} \subset S(X) = \{0,4,7\}$

Also, the pair (A, S) and (B, T) are coincidentally commuting at $x = 0$ which is their common coincidence point.

i.e. $A0 = S0 \Rightarrow AS0 = SA0$

$B0 = T0 \Rightarrow BT0 = TB0$

By a routine calculation, we can verify that all the contractive conditions of Corollary (2.3) are satisfied for $\beta_1 = \frac{1}{20}, \ \beta_2 = \frac{1}{10}$ and $\beta_3 = \frac{3}{8}$. $(2\beta_1 + \beta_2 + 2\beta_3 = 0.95 < 1)$.

3 CONCLUSION

Fixed point theory is a rich, interesting, and exciting branch of mathematics. It is relatively young but fully developed area of research. Study of the existence of fixed points falls within several domains such as functional analysis, operator theory, general topology. Non convex analysis, especially ordered normed spaces, normal cones and topical functions have some applications in optimization theory. In these cases, an order introduced by using vector space cones. Huang and Zhang [9] used this approach and they replaced the set of real numbers by an ordered Banach space and defined cone metric space which is generalization of metric space. In this paper, we obtain some common fixed point theorems for six mappings satisfying the different contractive conditions. Common fixed point results for weakly compatible maps which are more general than compatible mappings are obtained in the setting of cone metric spaces without requirement of the notion of continuity. Our results generalize, improve and extend the results of Goyal ([4],[5]) Goyal and Gupta([6],[7]),Imdad and Ali [13], Jeong and Rhoades [15] and others. In this way we can see that our result is superior to many other results.

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