Research Article

PRIME SERIAL RINGS WITH KRULL DIMENSION

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Abstract: A Characterization of Prime Serial rings with Krull dimention as rings with a unique maximum ideal. A description of the laltice of deals and laltice submodules of a module over ring with Krull dimension is a direct sum of indecomposable modules.

Keywords: Prime Serial Ring, Krull Dimension, Laltice Of Ideal, Laltice Of Submodules, Indecomposable Modules.

Introduction :

A module is said to be serial if it has a unique chain of sub-modules, and a ring is serial if it is a direct sum of serial right ideals and a direct sum of serial left ideals. The serial rings of Krull dimension 0 are the Artinian serial rings. Noetherian serial ring is a direct sum of Artinian serial rings and prime Noetherian serial rings, and he gave a structure theorem in the prime Noetherian case. A Noetherian non-Artinian serial ring has Krull dimension.

In this paper, we extend some of these results to serial rings of arbitrary Krull dimension. The methods used rely heavily on making use of weak chain conditions. That certain indecomposable serial rings with Krull dimension are prime or can be written as triangular matrices.

1. Background Material And Conventions :

All rings considered here are associative with identity element. We shall use N or N(R) to denote the nil radical of a ring R, and J or J(R) to denote the Jacobson radical of R.

Let R be a ring. that is an R-module M is said to be serial if for any sub-modules A and B of M we have $A \subseteq B$ or $B \subseteq A$. The ring R is said to be serial if it is the direct sum of serial right ideals and is also the direct sum of serial left ideals. Serial rings with Krull dimension have left and right Krull dimensions of a serial ring are equal. A serial ring A is Noetherian if and only if $\cap J^n = 0,$ and under these conditions

Kdim(R) $\leq$ 1.

Let R be a serial ring with Krull dimension. Also R/N is a serial semi-prime Goldie ring every finitely-generated one-sided ideal is projective. Therefore R/N is a direct sum of prime rings. Also R has an Artinian quotient ring if and only if R satisfies the ascending chain condition for right annihilators.

Let R be an indecomposable non-singular serial ring. Then A has a two-sided quotient ring which is a blocked triangular matrix ring over a division ring D. The corresponding full matrix ring over D is both the maximal left and right quotient ring of R. Hence every complement one-sided ideal of R is generated by an idempotent.

2. Certain Serial Rings Are Prime:

We shall establish some sufficient conditions for a serial ring with Krull dimension to be prime. The first result of this type was Warfield's theorem that an indecomposable Noetherian non-Artinian serial ring R is prime.

LEMMA 2.1. Let R be a serial ring with Krull dimension and let K be a non-zero serial right R-module with KN = 0. Then there is a unique minimal prime ideal P of R such that KP = 0. Also KQ = K for every minimal prime ideal Q of R with $Q \neq P$.

Proof. We know that KN=0 and that N contains a product of minimal primes (because R/N is a Goldie ring). Also K $\neq$ 0. Therefore there is a minimal prime ideal P with KP $\neq$ K. Let Q be a minimal prime ideal of R with Q $\neq$ P. Because R/N is a direct sum of prime rings, we have P + Q = R. Hence K = KP + KQ. But KP≠K and K is serial. Therefore K = KQ. Let Q_1 . ., Q_n, P be the distinct minimal primes of R. Then KQ_i = K for all i and $Q_1 Q_2 \ldots Q_nP \subseteq N$. Therefore 0 = KN $=KQ_1 \ldots Q_nP$ = KP.

LEMMA 2.2. Let R be a serial ring and let K be a semi-prime ideal of R such that R/K is a Goldie ring and K contains no non-zero idempotent elements of R. Then K = Kc = cK for all c e C(K).

Proof. There are non-zero orthogonal idempotents $e_1, \ldots, e_n$ of R adding to 1 such that each e_iR is a serial right R-module and $cR = \sum_i (e_i R \cap cR)$, We fix an integer; j with $1 \leq j \leq n.$. We have $e_j \notin$ K. Also (cR + K)/K is an essential right ideal of R/K. Hence $e_jR \cap$ (cR + K) is not contained in K. Because e_jR is serial, it follows that $e_j R \cap (cR + K) \supseteq e_j K.$ But

$$e_jR \cap (cR + K) = e_jR \cap \left(\sum_i (e_i R \cap cR) + \sum_i e_i K \right) = e_j R \cap cR + e_j K.$$

Hence $e_j R \cap cR + e_j K \supseteq e_j K.$. Because $e_j R$ is serial, it follows that $e_j K \subseteq e_j R \cap cR.$ Hence $e_j K \subseteq cR$ for all j and $e_1 + \ldots\ldots + e_n = 1$. Therefore $K \subseteq cR.$ Because $c \in C(K)$, we have $cR \cap K = cK.$ Therefore $K = cK.$

THEOREM 2.3. Let R be an indecomposable serial ring with Krull dimension and suppose that R has a non-nilpotent ideal X such that $\bigcap_{n=1}^{\infty} X^b = 0$. Then R is a prime ring.

Proof. We fix a minimal prime ideal P of R such that P does not contain X. We wish to show that N = NP. Let e be an idempotent element of R such that eR is serial. It is enough to show that eN = eNP, and this is trivial if eN = 0. Suppose that eN≠O and that eN≠NP. By taking K= eN/eN^2. We see that eNP=eN^2. Because X is not contained in P, there is no minimal prime of R which contains X + P. Therefore $c \in$ X + P for some $c \notin$ C(N). But Nc = N by Lemma 2.2

Hence N = eNc = eN(X + P) = eNX + eNP = eNX + eN^2.

Therefore eN=eNX=eNX^2 =........ This is a contradiction because eN≠0 and $\bigcap X^n = 0.$.

Thus N = NP. Because R/N is a direct sum of prime rings, we know that P/N is generated by a central idempotent of R/N. Hence there is an idempotent element f of R such that P=fR + N = Rf + N. Thus,

$$N = NP = Nf + N^2 = Nf + N(Nf + N^2) = Nf + N^3 = \ldots$$

Hence N = Nf and P = Rf + N = Rf + Nf = Rf. By symmetry, we have P = Rf =fR. It follows that f is a central idempotent element of R. Therefore P = 0 because R is indecomposable.

COROLLARY 2.4. Let R be a serial ring. Then R/ $\bigcap_{n=1}^{\infty} J^n$ is a direct sum of Artinian rings and prime rings.

Proof. Without loss of generality, we may suppose that $\bigcap J^n = 0$ and that R is indecomposable. Either J is nilpotent and R is Artinian, or J is not nilpotent and R is prime by the Theorem 2.3.

COROLLARY 2.5. Let R be an indecomposable serial ring with Krull dimension. Suppose that R has an Artinian quotient ring and that Kdim(N) < Kdim(R) as both left R-modules and right R-modules. Then R is a prime ring.

3. Prime serial rings of finite Krull dimension. We shall determine the structure of prime serial rings with finite Krull dimension in terms of blocked matrices over integral domains. The proof will be module-theoretic and will require that the domain of definition of certain homomorphisms can be extended.

LEMMA 3.1. Let R be a seriol ring with primitive idempotents e and f. Let x $\in$ eR, and let a : xR $\to$ fR be o homomorphism of right R-modules. Then one at least of the following statements is true.

1. There is a homomorphism $b : eR \to fR$ with $b(x) = a(x)$, or
2. There is a homomorphism $b : fR \to eR$ with $ba(x) = x$.

Proof. There are orthogonal primitive idempotents $g_1, \ldots, g_n$ of R adding 1 such that each Rg_i is a serial left R-module. We have

$xR = xg_1R + \ldots\ldots + xg_nR.$. Because eR is serial, we have $xR = xg_iR$ for some i. Thus xR = xgR for some primitive idempotent g. Set $w = a(xg) = a(x)g.$ Because Rg is serial, we have $Rxg \subseteq Rw$ or $Rw \subseteq Rxg$.

Case (a). Suppose that $Rxg \subseteq Rw$. Then xg=rw for some $r \in R$. But $x = ex$ and $w = fw$. Hence, without loss of generality, we may suppose that r =erf. Define $b : fR \to eR$ by b(fs) = rs for all $s \in R$. Then ba(xg)=b(w) =rw=xg. But xR= xgR. Hence x=xgt for some $t \in R$. Therefore ba(x) = ba(xgt) = (ba(xg))t= xgt= x.

Cese (b). Suppose that $Rw \subseteq Rxg$. Then w= rxg for some $r \in R$ with r= fre. Define $b : eR \to fR$ by b(es) = rs too all $s \in R$.
Then b(xg)= rxg = w = a(xg). As in (a), it follows that b(x) = a(x).

THEOREM 3.2. Let R be a prime serial ring with finite Krull dimension. Then there is o serial integral domain T such that either

(1) $R \cong M_n(T)$ for some positive integer n, or

(2) there is a positive integer $k \neq 1$ and sets H_{ij} for $1 \leq i, j \leq k$ such that R is isomorphic to the ring of k by k matrices with (i, j)-entries in H_{ij}, where

(i) if $i < j$ then H_{ij} is the set of all n_i by n_j matrices with entries in T for some positive integers n_i and n_j,

(ii) if $i > j$ then H_{ij} is the set of all n_i by n_j matrices over J(T), and

(iii) each H_{ij} is a prime serial ring of finite Krull dimension and with smaller Goldie rank than that of R.

Thus each H_{ij} can in mm be written as a matrix ring in the same way that R is, and so on.

Proof. We shall frequently use the well-known fact that a nonzero homomorphism between uniform right ideals of R is injective. Set J = J(R). Suppose firstly that J is a prime ideal of R. Then all simple right R-modules are isomorphic. There are orthogonal idempotents $e_1, \ldots\ldots, e_n$ of R adding to 1 such that each e_iR is a serial right R-module. Each e_iR / e_iJ is a simple module. Therefore $e_iR / e_iJ \cong e_iR / e_iJ$ for all i and j. It is now routine to show that $e_iR \cong e_jR$ and that (1) holds with $T = End_R(e_iR)$.

For the remainder of the proof, we suppose that J is not a prime ideal of R. Set $J_0 = J$ and $J_n = \bigcap_{i=1}^{\infty} J_{n-1}^i$, for every positive integer n. Because R has finite Krull dimension, we have $J_h = 0$ for some h. For each i, let K_i be the ideal of R containing J_i such that $K_i / J_i = N(R / J_i)$. Each K_i contains no non-zero idempotent element of R. We have $K_0 = J$ and $K_h = 0.$ Thus K_0 is not prime and K_h is prime. Let p be the largest integer such that K_p is not a prime ideal. Set $K = K_p$ and $L = K_{p+1}$.

Set $I = \bigcap_{i=1}^{\infty} K^i$ and let e be a primitive idempotent of R. We shall show that $I = L$. Because K is nilpotent modulo J_p we have

$I = \bigcap_{i=1}^{\infty} K_p^i = \bigcap J_p^i = J_{p+1} \subseteq L.$ Also $L^r \subseteq J_{p+1}$ for some non-negative integer r. Hence $L^r \subseteq J_p \subseteq K.$ Therefore $L \subseteq K$. But L is prime and K is not. Therefore $L \underset{\neq}{\subset} K$. Let i be a positive integer.

Then eK^i is not contained in the **prime** ideal L. Because eR is serial it follows that $eK^i \supsetneq eL$. Because 1 is the sum of **such** idempotents e, we have $L \subsetneq K^i$. Hence $L \subseteq Z$. Therefore $L = I = J_{p+1}$. From $eK^i \supseteq eL$, it also follows that $eK^i \neq eK^i$ if $i \neq j$.

The reason for choosing K to be semi-prime but not prime is that we take k in (2) to be the number of prime ideals of R minimal over K.

Set C = C(K). We shall show that C is an Ore set in R. Let $a \in R$ and $c \in C$. Because R/K is a semi-prime Goldie ring we have ad = cb + u for some $d \in C$, $b \in R$, $u \in K$. By Lemma 2.2, we have u = cv for some $u \in K$. Thus ad =c(b + v). Therefore C is an Ore set in R, and the elements of C are regular in R. Let S be the partial quotient ring of R with respect to C. Then K =J(S) and L is a prime ideal of S. Because $\bigcap K^i = L$ it follows that S/L is a prime Noetherian serial ring.

Let e be a primitive idempotent of R. Then eR is a serial R-module. Hence eS is serial as a right S-module. For each i, set $M_i = ann_s\left(eK^i / eK^{i+1}\right)$. Because S/L is a prime Noetherian serial ring and $eK^i \supseteq eL$ for all i, it follows, as in the proof of that the sequence $M_0, M_1, M_2, \ldots\ldots$ starts by running through the distinct maximal ideals of S and then repeats itself. The only way in which e influences this sequence is in determining its starting point. In R, this means that: the primes of R minimal over K are precisely the ideals of the form $ann_R\left(eK^i / eK^{i+1}\right)$ for some non-negative integer i; $ann_R\left(eK^{i+1}\right) = ann_R\left(eK^i / eK^{i+1}\right)$ if and only if $i \equiv j$ mod(k), where k is the number of primes of R minimal over K, $ann\left(eK^i / eK^{i+1}\right)$ is determined by $ann\left(eK^{i-1} / eK^i\right)$ and not by e.

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