Fuzzy Maximal and Minimal A-Clopen Sets

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Abstract
The aim of the paper is to introduce the notions of fuzzy maximal and fuzzy minimal α -clopen sets in fuzzy topological spaces. The notions of fuzzy maximal and minimal α -clopen sets are respectively independent to the notions of fuzzy maximal and fuzzy minimal α -open (resp.closed) sets. Further, fuzzy maximal α -clopen and fuzzy minimal α -clopen sets are discussed using fuzzy α -disconnectedness.

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1 Introduction

The idea of fuzzy sets introduced by Zadeh[7]. Chang introduced the notion of fuzzy topology in [1]. The notions of minimal open, maximal open, minimal closed, maximal closed sets introduced by Nakaoka and Oda in [3] and [4]. The notion of fuzzy minimal open[6] set explored by Swaminathan and Sivaraja. In [5], Sankari discussed some properties of fuzzy minimal α -open and fuzzy maximal α -open sets and shown that if a fuzzy topological space having both fuzzy minimal α -open and fuzzy maximal open set, then it may be fuzzy α -disconnected. With respect to the theorem 2.9 of this current paper, it reveals that a fuzzy set which is both fuzzy maximal α -clopen and fuzzy minimal α -clopen set satisfy not only the disconnectedness of the fuzzy space but also a fuzzy set which is both fuzzy maximal α -clopen and fuzzy minimal.

α -clopen and these are the only fuzzy α -clopen sets in the fuzzy space. It is the odd behavior observed among the notions of fuzzy minimal α -open and fuzzy maximal α -open with respect to fuzzy minimal α -clopen and fuzzy maximal α -clopen and it is the finest of its own.

2 Fuzzy Maximal and Minimal α-clopen Sets

To proceed main results, we recall basic definitions and results:

Definition 2.1. ([5]) A proper fuzzy α -open set μ of X is said to be a fuzzy maximal α -open set if λ is an fuzzy α -open set such that $\mu < \lambda$, then $\lambda = \mu$ or $\lambda = 1_X$

Definition 2.2. ([5]) A proper fuzzy open set μ of X is said to be a fuzzy minimal α -open set if λ is an fuzzy α -open set such that $\lambda < \mu$, then $\lambda = \mu$ or $\lambda = 0_X$

Definition 2.3. ([5]) A proper fuzzy closed set γ of X is said to be a fuzzy minimal α -closed set if α is an fuzzy α -closed set such that $\alpha < \gamma$, then $\alpha = \gamma$ or $\alpha = 0_X$

Definition 2.4. ([5]) A proper fuzzy closed set γ of X is said to be a fuzzy maximal α -closed set if α is an fuzzy α -closed set such that $\gamma < \alpha$, then $\alpha = \gamma$ or $\alpha = 1_X$.

Theorem 2.1. *([5]) If ϑ is a fuzzy maximal α -open set and α is a fuzzy minimal α -open set in a fuzzy topological space X with* $\alpha < \vartheta$ *, then* $\vartheta = 1_X - \alpha$

Definition 2.5. A proper fuzzy α -clopen set α of a fuzzy topological space X is said to be fuzzy minimal α -clopen if β is a fuzzy α -clopen set such that $\beta < \alpha$, then $\beta = \alpha$ or $\beta = 0_X$

Example 2.2. Let $X = \{a, b, c, d\}$. Then fuzzy sets $\gamma 1 = \frac{0}{a}+\frac{1}{b}+\frac{1}{c}+\frac{0}{d}$; $\gamma 2 = \frac{0}{a}+\frac{0}{b}+\frac{0}{c}+\frac{1}{d}$; $\gamma 3 = \frac{0}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$ and $\gamma 4 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{0}{d}$ are defined as follows: Consider the fuzzy topology $\tau = \{0_X, \gamma 1, \gamma 2, \gamma 3, \gamma 4, 1_X\}$. Here γ4 is fuzzy minimal α -clopen set but it is neither fuzzy minimal α -open nor fuzzy minimal α -closed set.

It is clear that a fuzzy minimal α -open set or a fuzzy minimal α -closed set need not to be a fuzzy minimal α -clopen set. Hence the notion of fuzzy minimal α -clopen sets is independent to the notions of fuzzy minimal α -open set as well as fuzzy minimal α -closed sets. It is also easy to see that if a set α is both fuzzy minimal α -open and fuzzy minimal α -closed, then α is fuzzy minimal α -clopen. Again, a fuzzy α -clopen set is fuzzy minimal α -clopen if it is either fuzzy minimal α -open or fuzzy minimal α -closed.

Definition 2.6. A proper fuzzy α -clopen set α of a fuzzy topological space X is said to be fuzzy maximal α -clopen if β is a fuzzy α -clopen set such that $\alpha < \beta$, then $\beta = \alpha$ or $\beta = 1_X$

It is evident that a fuzzy maximal α -open or a fuzzy maximal α -closed sets may not be a fuzzy maximal α -clopen set. Therefore, it follows that the notion of fuzzy maximal α -clopen sets is independent to the notions of fuzzy maximal α -open as well as fuzzy maximal α -closed sets. It is also easy to see that if a set γ is both fuzzy maximal α -open and fuzzy maximal α -closed, then γ is fuzzy maximal α -clopen. In fact, a fuzzy α -clopen set is fuzzy maximal α -clopen if it is either fuzzy maximal open or fuzzy maximal α -closed. In [5], we observed that if a fuzzy topological space has only one proper fuzzy α -open set, then it is both fuzzy maximal α -open and fuzzy minimal α -open. Even we observe that it is neither fuzzy maximal α -clopen nor fuzzy minimal α -clopen. If a space has only two proper fuzzy α -open sets such that one is not contained in other, then both are fuzzy maximal α -clopen and fuzzy minimal α -clopen. In addition, fuzzy maximal or fuzzy minimal α -clopen sets can exists only in a fuzzy α -disconnected space. As theorems 2.3 to corollary 2.6 are obvious, the proofs of them are omitted.

Theorem 2.3. *If* α *is a fuzzy minimal* α *-clopen set and* β *is a fuzzy* α *-clopen set in X , then either* $\alpha \wedge \beta = 0_X$ *or* $\alpha < \beta$

Corollary 2.4. *If* α *and* β *are distinct fuzzy minimal* α *-clopen sets in X , then* $\alpha \wedge \beta = 0_X$

Theorem 2.5. *If* α *is a fuzzy maximal* α *-clopen set and* β *is a fuzzy* α *-clopen set in X, then either* $\alpha \vee \beta = 1_X$ *or* $\beta < \alpha$

Corollary 2.6. *If* α *and* β *are distinct fuzzy maximal* α *-clopen sets in X , then* $\alpha \vee \beta = 1_X$.

Lemma 2.7. *If* α *is fuzzy minimal* α *-clopen in a fuzzy topological space X , then* $1_X - \alpha$ *is fuzzy maximal* α *-clopen in X and conversely.*

Proof: Let α, β be any two proper fuzzy α -clopen sets such that $1_X - \alpha < \beta$. Now we have $1_X - \beta < \alpha$. As α is being a fuzzy minimal α -clopen, set we have $1_X - \beta = \alpha$ or $1_X - \beta = 0_X$ implies that $1_X - \alpha = \beta$ or $\beta = 1_X$. Hence, $1_X - \alpha$ is a fuzzy maximal α -clopen set. Similarly follows the converse.

Similar to Theorem 3.1 of [5], we have the following theorem.

Theorem 2.8. *If* α *is fuzzy minimal* α *-clopen and* β *is fuzzy maximal* α *-clopen in X , then either* $\alpha < \beta$ *or* $\alpha < 1_X - \beta$.

Proof: Similar to the proof of Theorem 3.1 of [5].

Theorem 2.9. *If a fuzzy topological space X contains a fuzzy set* α *which is both fuzzy maximal and minimal* α *-clopen, then (i)* α *and* $1_X - \alpha$ *are the only fuzzy sets in the space which are both fuzzy maximal and minimal* α *-clopen and (ii)* α *and* $1_X - \alpha$ *are the only proper fuzzy* α *-clopen sets in the space.*

Proof: (i) By Lemma 2.7, $1_X - \alpha$ is both fuzzy maximal and minimal α -clopen for any fuzzy maximal and minimal α -clopen set α in X . Let there exist fuzzy maximal and minimal α –clopen set β in X distinct from α . By Lemma 2.7, $1_X - \beta$ is also both fuzzy maximal and minimal α -clopen. As α, β both being fuzzy minimal and maximal α -clopen, by corollary 2.4, and 2.6, we have $\alpha \wedge \beta = 0_X$ and $\alpha \vee \beta = 1_X$. Hence $\beta = 1_X - \alpha$. If for any $\alpha \neq \beta$, β and $1_X - \beta$ are identical to $1_X - \alpha$ and α respectively. Hence, the result follows for all possible combinations of α , β , $1_X - \alpha$ and $1_X - \beta$.

(ii) Let γ be a proper fuzzy α -clopen set in X . For a fuzzy maximal α -clopen set α , we have $\alpha \vee \gamma = 1_X$ or $\gamma < \alpha$. For a fuzzy minimal α -clopen set α we have $\alpha \wedge \gamma = 0_X$ or $\alpha < \gamma$. $\alpha \vee \gamma = 1_X$ and $\alpha \wedge \gamma = 0_X$ implies that $\gamma = 1_X - \alpha$. $\alpha \vee \gamma = 1_X$ and $\alpha < \gamma$ implies that $\gamma = 1_X$. $\gamma < \alpha$ and $\alpha \wedge \gamma = 0_X$ implies that $\gamma = 0_X$.

Theorem 2.10. *In a fuzzy topological space X , fuzzy maximal* α *-clopen and minimal* α *-clopen sets appear in pairs.*

Proof: By Theorem 2.9, if α is both fuzzy maximal and minimal α -clopen in fuzzy topological space X , then $1_X - \alpha$ is also both fuzzy maximal and minimal α -clopen, also such pairs of sets in X are unique. By Lemma 2.7, if α is fuzzy maximal (resp.minimal) α -clopen in X , then $1_X - \alpha$ is fuzzy minimal (resp.maximal) α -clopen in X .

Theorem 2.11. *If* α *is a fuzzy maximal open set and* β *is a fuzzy minimal* α *-open set of fuzzy topological space X with* $\alpha < \beta$ *, then* α *is a fuzzy maximal* α *-clopen and* β *is a fuzzy minimal* α *-clopen set.*

Proof: On deploying maximality of α by 2.1 $\alpha = 1_X - \beta$. Hence both α , β are fuzzy α -clopen. Since α is both fuzzy α -clopen and fuzzy maximal α –open (resp. fuzzy minimal α -open), it is easy to see that α is fuzzy maximal α -clopen. Similarly β holds.

On combining this result, we obtain another result whose proof is same and omitted.

Theorem 2.12. *If* γ *is a fuzzy maximal* α *-clopen set in X , then* $\gamma \vee \mu$ *is not a proper fuzzy* α *-clopen set distinct from* γ *for any proper fuzzy open or fuzzy closed set* μ *in X.*

Proof: Let $\gamma \vee \mu$ be a proper fuzzy α -clopen set in X . As γ is a fuzzy maximal α -clopen set and $\gamma < \gamma \vee \mu$, either $\gamma \vee \mu = 1_X$ or $\gamma = \gamma \vee \mu$. Then $\gamma = \gamma \vee \mu$ implies that $\mu < \gamma$.

Theorem 2.13. *If* γ *is a fuzzy minimal* α *-clopen set in* X *, then* $\gamma \wedge \mu$ *is not a proper fuzzy* α *-clopen set distinct from* γ *for any proper fuzzy* α *-open or fuzzy* α *-closed set* μ *in* X *.*

From 2.12, Theorem 2.13 it is observed that the form of fuzzy α -clopen sets in fuzzy topological space consists of a fuzzy maximal or minimal α -clopen set.

Theorem 2.14. If Λ is a collection of distinct fuzzy maximal α -clopen sets and $\alpha \in \Lambda$, then $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta \neq 0_X$. If Λ is a finite collection, then $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$ is a fuzzy minimal α –clopen if and only if $1_X - \alpha = \bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$.

Proof: As $\beta\in\Lambda- \alpha$ is a fuzzy maximal α -clopen set, $1_X -\beta$ is a fuzzy minimal α – clopen set. Then by Theorem 2.8, $1_X -\beta < \alpha$. Hence we get , $1_X -\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta < \alpha$ which implies $\alpha = 1_X$ if $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta = 0_X$. This is a contradiction to our assumption that α is a fuzzy maximal α -clopen set. Hence, $\bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta 0_X$.

Evidently, $\bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta$ is a fuzzy minimal α -clopen if $1_X-\alpha = \bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta$. Now let $\bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta$ be a fuzzy minimal α -clopen set. If Λ is a finite collection, then $\bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta$ is a fuzzy α -clopen set. Since $1_X -\bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta < \alpha$ we have $1_X-\alpha < \bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta$. As α is a fuzzy maximal α -clopen, then $1_X -\alpha$ is fuzzy minimal α -clopen. If $\bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta$ is a fuzzy minimal α -clopen set distinct from $1_X -\alpha$ then by corollary 2.4 we have $(\bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta) \wedge(1_X -\alpha) = 0_X$. This implies $\bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta < \alpha$. So we get, $1X -\alpha < \bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta < \alpha$ which is wrong. Hence, we obtain $1_X-\alpha = \bigwedge_{\beta\in\Lambda-\{\alpha\}}\beta$.

Theorem 2.15. *If* Λ *is a collection of distinct fuzzy minimal* α *-clopen sets and* $\alpha \in \Lambda$ *, then* $\bigvee_{\beta\in\Lambda-\{\alpha\}}\beta\neq 1X$ *. If* Λ *is a finite collection, then* $\bigvee_{\beta\in\Lambda-\{\alpha\}}\beta$ *is a fuzzy maximal* α *-clopen if and only if* $1X -\alpha = \bigvee_{\beta\in\Lambda-\{\alpha\}}\beta$

It is observed that if γ is fuzzy α -clopen in (X, τ) , then $\alpha \wedge\gamma$ is fuzzy α -clopen in $(\alpha, \tau\alpha)$. In addition, if α is fuzzy α -clopen in (X, τ) then a set fuzzy α -clopen in $(\alpha, \tau\alpha)$ is also fuzzy α -clopen in (X, τ) .

Theorem 2.16. *Let* α *,* ϑ *be fuzzy* α *-clopen sets in X such that* $\alpha \wedge \vartheta \neq 0_X$ *then* $\alpha \wedge \vartheta$ *is a fuzzy minimal* α *-clopen set in* $(\alpha, \tau\alpha)$ *if* ϑ *is fuzzy minimal* α *-clopen in* (X, τ) *.*

Proof: Similar to that of Theorem 4.11 in [5].

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