Research Article

RUNGE–KUTTA LIKE METHOD FOR THE SOLUTION OF OPTIMAL CONTROL MODEL OF REAL INVESTMENT AND FISH MANAGEMENT

Adamu, S.[a], Alkali, A. M.[b], Odekunle, M. R.[c]

[a] Department of Mathematics, Nigerian Army University Biu, Borno State, Nigeria

[b,c] Department of Mathematics, Modibbo Adama University Yola, Adamawa State, Nigeria

*Corresponding Author: malgwisa@gmail.com

TO CITE THIS ARTICLE:

Adamu, S., Alkali, A. M., & Odekunle, M. R. (2024). RUNGE–KUTTA LIKE METHOD FOR THE SOLUTION OF OPTIMAL CONTROL MODEL OF REAL INVESTMENT AND FISH MANAGEMENT. Turkish Journal of Computer and Mathematics Education (TURCOMAT), 15(2), 155–169. https://doi.org/10.61841/turcomat.v15i2.14646

Abstract: This study develops the Runge-Kutta Like Method (RKLM), which uses Pontryagin's principle to solve optimal control problems numerically using forward-backward sweep methods. It is based on the Patade and Bhalekar methodology. The RKLM's stability properties and its convergence are examined. The Forward-backward sweep algorithm and the RKLM algorithm are implemented using MATLAB code. Physical optimum control problems are solved with the RKLM. The first problem's conclusion demonstrates that, when investment declines, the capital first grow to boost production before it depreciates. The outcome of the second problem demonstrates that a larger weight parameter causes the harvesting rate to reach zero more quickly and the total fish mass to reach its maximum level more quickly. The findings obtained demonstrate the effectiveness of using RKLM in conjunction with forward-backward sweep methods to solve optimal control problems.

Keywords: Fish, First Boubaker polynomials, Investment, Model, Optimal control problem

1. Introduction

First order ordinary differential equations of initial value problems arise from a variety of physical problems that exist in our surroundings nowadays (Alkali et al., 2023). Traditionally, analytical approaches have been used to find solutions to these differential equations; however, certain differential equations have extremely difficult solutions, if any exist at all, with the exception of an approximate solution found through the use of numerical methods (Areo & Adeniyi, 2013). Numerous numerical techniques have been developed in response to the need to acquire more precise approximate solutions to mathematical models that arise in the fields of engineering, sciences, medical sciences, economics, and social sciences.

Finding the best control trajectory and related state trajectory for an Optimal Control (OC) problem aims to optimize a user-specified performance measure while taking into account process restrictions and system dynamics (Garrete, 2015; Andres-Martinez, 2022; Aduroja et al., 2024). A country's economy could be the dynamical system, with the goal of minimizing unemployment; in this scenario, fiscal and monetary policy could be the control. Finding the best answer might help you spot possible constraints that could be broken and cause harm or other unfavorable outcomes (Lenhart & Workman, 2007; Adamu, 2023). OC has been used to tackle physical system problems in a variety of sectors, including aerospace, science, engineering, economics, management, biology, and medicine (Rodrigues et al., 2014; Adamu, 2023). The numerical solution to the optimal control model of actual investment and fish management was presented in this research. Naeval (2002) explained how to solve continuous time optimal control models using Microsoft Excel. Real investment and fishery management models are solved to show how academics can use Excel's Solver Tool to tackle optimal control problems.

The Runge-Kutta (R-K) formulas represent a significant family of iterative techniques for the approximation of ODE solutions (Odekunle, 2000; Odekunle et al., 2004) and are one of the most ancient schemes in numerical analysis (Musa et al., 2010). Runge-Kutta techniques, which were initially examined by Carle Runge and Martin Kutta (Hildebrand, 1974), are often single-step techniques with several phases in each step. The introduction of Butcher tableau and John Butcher's improvements are largely responsible for modern advances. Other researchers, such as (Odekunle et al., 2004), (Musa et al., 2010) explored Runge-Kutta methods, and (Usman et al., 2013), also attempted to simplify the Runge-Kutta technique's derivation process or create an implicit Runge-Kutta approach for solving initial value problems involving ordinary differential equations.

Forward Backward Sweep (FBS) is an iterative technique named based on the way the algorithm solves the problem state and adjoint ODEs (Garret, 2015; Adamu, 2023). In order to solve optimal control problems using the Classical Runge Kutta Method (CRKM) using Pontryagin's principle, (Garrete, 2015) and (Rodrigues et al., 2014) employed Forward Backward Sweep approaches. Runge-Kutta methods struggle to solve issues when they grow stiff and the development of CRKM is time-consuming and difficult. However, the classical Runge Kutta method has become a prominent instrument for solving optimum control problems utilizing the Forward Backward Sweep approach.

In order to convert a single step trapezoidal approach into a three stage Runge Kutta type method with constant step length for the solution of ordinary differential equations, (Patade & Bhalekar, 2015) applies the Daftardar-Gejji and Jafari technique. The second-order techniques that have been developed have shown to be effective in solving ordinary differential equations. In order to build a Runge-Kutta-like method that follows the Patade and Bhalekar approach for the solution of optimal control problems utilizing the Forward Backward Sweep method via Pontryagin's principle. Adamu et al. (2024) consider a single step method with hybrid point not taking at the middle and developed four stage numerical method. This work considers a single step trapezoidal method with hybrid point taking at the middle for the development of the method for solving some application problems.

The other sections are arranged as follows: Section 2 presents preliminary studies. Methodology is covered in Section 3, while RKLM analysis is covered in Section 4. Section 5 presents numerical experiments, while Section 6 concludes the paper.

2. Preliminary Studies

Definition 2.1 A general single-step method can be written in the form

$$y_{j+1} - y_j = h\phi(x_j, y_j, h). \tag{2.1}$$

The function ϕ is called the increment function (Patade & Bhalekar, 2015).

Definition 2.2 (Patade & Bhalekar, 2015**)** A single step method defined in (2.1) is said to be consistent if

$$\phi(x, y, 0) = f(x, y). \tag{2.2}$$

Definition 2.3 A single step method defined in (2.1) is said to be regular if the function $\phi(x, y, h)$ is defined and continuous in the domain $a \le x \le b, -\infty < y^j < \infty, i = 1, 2, .., n, 0 \le h \le h_0$ and if there exist a constant L such that

$$\|\phi(x, y, h) - \phi(x, y^*, h)\| \le L\|y - y^*\|, \tag{2.3}$$

for every $x \in [x_0, b], y, y^* \in (-\infty, \infty)$, $h \in (0, h_0)$ (Patade & Bhalekar, 2015).

Theorem 2.1 Suppose the single step method defined in (2.1) is regular. Then the relation (2.2) is a necessary and sufficient condition for the convergence of the method.

Definition 2.4 A method is stable if the cumulative effect of all errors, including round-off error is bounded, independent of the number of mesh points.

Definition 2.5 A Runge-Kutta method is said to be of order p if

$$N(y(x); h = O(h^p). \tag{2.4}$$

And the local truncation error T_{n+1} at x_{n+1} is given by N(y(x); h).

Definition 2.6 (Butcher, 1996) For a method given by the tableau

c	A
	b

the stability for a $y' = qy$ is the set of points in the complex plane satisfying $|R(z)| \le 1.$

3. Methods

3.1 Development of Linear Multistep Method

Approximating the exact solution $x(t)$ of a differential equation $x'(t) = f(t, x(t))$ on the partition $\pi[a, b] = [a = t_0 < t_1 < \ldots < t_n < t_{n+1} < \ldots < t_N = b]$ of the integration interval $[a, b]$ using first Boubaker polynomials of the form

$$x(t) = \sum_{n=0}^{3} a_n B_n(t), \tag{3.1}$$

where $t \in [a, b]$, $a_n \in \mathrm{R}$ are unknown parameters to be determined. To generate the collocation points, closed

Research Article

Newton Cote's collocation point is considered as

$$t_i = a + hi, \quad h = \frac{(b-a)}{N}, \quad i = 0,1,2. \tag{3.2}$$

First Boubaker polynomials satisfy the following recursive formula

$$B_0(t) = 1, \; B_1(t) = t, \; B_2(t) = t^2 + 2, \; B_3(t) = t^3 + t.$$

Interpolating and collocating (3.1) using the points

$$x(t_{n+j}) = x_{n+j}, \quad j = 0; \; x'(t_{n+j}) = f_{n+j}, \quad j = 0, \frac{1}{2}, 1$$

gives a system of equations

$$XA = U, \tag{3.3}$$

where

$$X = \begin{bmatrix} 3 & 2t_n & t_n^2 & t_n^3 \\ 0 & 2 & 2t_n & 3t_n^2 \\ 0 & 2 & 2t_{n+\frac{1}{2}} & 3t_{n+\frac{1}{2}}^2 \\ 0 & 2 & 2t_{n+1} & 3t_{n+1}^2 \end{bmatrix},$$

$$A = [a_0, a_1, a_2, a_3]^T, \; U = [x_n, \; f_n, \; f_{n+\frac{1}{2}}, \; f_{n+1}]^T.$$

Solving the system (3.3) for the unknown parameters and substitute the results into the approximate solution (3.1) to get the continuous scheme

$$x_{n+t} = \alpha_0(t)x_n + \beta_0(t)hf_n + \beta_{\frac{1}{2}}(t)hf_{n+\frac{1}{2}} + \beta_1(t)hf_{n+1}, \tag{3.4}$$

Where the coefficients $\alpha_0(t), \beta_0(t), \beta_{\frac{1}{2}}(t), \beta_1(t)$ are:

$$\alpha_0(t) = 1,$$
$$\beta_0(t) = t\left(\tfrac{2}{3}t^2 - \tfrac{3}{2}t + 1\right),$$
$$\beta_{\frac{1}{2}}(t) = t^2\left(2 - \tfrac{4}{3}t\right),$$
$$\beta_1(t) = t^2\left(\tfrac{2}{3}t - \tfrac{1}{2}\right).$$

Evaluating equation (3.4) at point x_{n+1} gives the discrete scheme

$$x_{n+1} = x_n + \frac{h}{6}\left(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}\right). \tag{3.5}$$

3.2 Development of the Iterative Method

3.2.1 Runge-Kutta like method

Equation (3.5) can be written as

$$x_{n+1} = x_n + \frac{h}{6}f_n + \frac{4h}{6}f_{n+\frac{1}{2}} + \frac{h}{6}f_{n+1}. \tag{3.6}$$

Let

$$x = x_{n+1}, \tag{3.7}$$

$$x_0 = f = x_n + \frac{h}{6}f(t_n, x_n) + \frac{4h}{6}f\left(t_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}\right), \tag{3.8}$$

and

$$N(x_{n+1}) = N(x) = \frac{h}{6}f\left(t_{n+1}, x_{n+1}\right). \tag{3.9}$$

Using 3-terms solution of DJM series then,

$$x = x_0 + x_1 + x_2, \tag{3.10}$$

$$x = x_0 + N(x_0 + N(x_0)). \tag{3.11}$$

Substituting (3.7) and (3.8) into (3.11) to get

$$x_{n+1} = x_n + \frac{h}{6} f(t_n, x_n) + \frac{4h}{6} f\left(t_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}\right) + N\left(\begin{array}{c} x_n + \frac{h}{6} f(t_n, x_n) + \frac{4h}{6} f\left(t_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}\right) \\ + N\left(x_n + \frac{h}{6} f(t_n, x_n) + \frac{4h}{6} f\left(t_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}\right)\right) \end{array} \right). \tag{3.12}$$

This can be simplified (by using equation 3.9) as

$$x_{n+1} = x_n + \frac{h}{6} f(t_n, x_n) + \frac{4h}{6} f\left(t_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}\right) + \frac{h}{6} f\left(\begin{array}{c} t_{n+1}, x_n + \frac{h}{6} f(t_n, x_n) + \frac{4h}{6} f\left(t_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}\right) \\ + \frac{h}{6} f\left(t_{n+1}, x_n + \frac{h}{6} f(t_n, x_n) + \frac{4h}{6} f\left(t_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}\right)\right) \end{array} \right). \tag{3.13}$$

Equation (3.13) reduces to

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 4k_2 + k_4), \tag{3.14}$$

where

$$k_1 = f(t_n, x_n),$$
$$k_2 = f\left(t_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}\right),$$
$$k_3 = f\left(t_{n+1}, x_n + \frac{h}{6}k_1 + \frac{4h}{6}k_2\right),$$
$$k_4 = f\left(t_{n+1}, x_n + \frac{h}{6}k_1 + \frac{4h}{6}k_2 + \frac{h}{6}k_3\right).$$

This is the required RKLM

Lemma 3.1 The RKLM (3.14) is not a Runge-Kutta method.

Proof To show that, the RKLM is not a Runge-Kutta method, Runge-Kutta properties are tested. The RKLM is written as

$$x_{n+1} = x_n + h(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4),$$

while k_i is written as

$$k_i = f\left(t_j + c_1 h, x_j + h(a_{i1}k_1 + a_{i2}k_2 + a_{i3}k_3 + a_{i4}k_4)\right), i = 1,2,3,4$$

The following parameters are extracted from the RKLM

$$b_1 = \frac{1}{6},\ b_2 = \frac{2}{3},\ b_3 = 0,\ b_4 = \frac{1}{6};\ c_1 = 0,\ c_2 = \frac{1}{2}, c_3 = c_4 = 1;$$
$$a_{11} = a_{12} = a_{13} = a_{14} = 0,\ a_{21} = a_{22} = a_{23} = a_{24} = 0,$$
$$a_{31} = \frac{1}{6},\ a_{32} = \frac{4}{6},\ a_{33} = a_{34} = 0, a_{41} = a_{43} = \frac{1}{6},\ a_{42} = \frac{2}{3}, a_{44} = 0,$$

Thus the Butcher table for RKLM is

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & a_{13} & a_{14} \\
c_2 & a_{21} & a_{22} & a_{23} & a_{24} \\
c_3 & a_{31} & a_{32} & a_{33} & a_{34} \\
c_4 & a_{41} & a_{42} & a_{43} & a_{44} \\
\hline
 & b_1 & b_2 & b_3 & b_4
\end{array}
=
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 \\
1 & \frac{1}{6} & \frac{4}{6} & 0 & 0 \\
1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\
\hline
 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6}
\end{array}
$$

For Runge-Kutta method, it is necessary that $\sum_{j=1}^{4} a_{ij} = c_i$ (Butcher,1996). From the parameters, $a_{31} + a_{32} + a_{33} + a_{34} = \frac{5}{6} \neq c_3$. This sufficiently shows that RKLM is not a Runge-Kutta method.

4 Analysis of the New Method

4.1 Order of the method

Theorem 4.1 The Runge-Kutta like method in (3.14) is of order four.

Proof We have

$$k_1 = f_n. \tag{4.1}$$

Using second order Taylor's series, we can write k_2, k_3 and k_4 as

$$k_2 = f_n + \frac{h}{2} f_{n,t} + \frac{h}{4} f_{n,x} + \frac{h^2}{4} f_{n,tt} + \frac{h^2}{4} f_{n,tx} + \frac{h^2}{8} f_{n,xx} + \dots \tag{4.2}$$

$$k_3 = f_n + hf_{n,t} + \frac{h^2}{2} f_{n,tt} + \frac{h^2}{6} f_n f_{n,tx} + \frac{h}{6} f_n f_{n,x} + \frac{h^2}{24} f_n^2 f_{n,xx} + \dots \tag{4.3}$$

$$k_4 = f_n + hf_{n,t} + \frac{h^2}{2} f_{n,tt} + W_3 \left[hf_{n,tx} + f_{n,x} + \frac{h}{4} f_n f_{n,xx} \right] + \dots \tag{4.4}$$

where

$$W_3 = \begin{pmatrix} \frac{h}{6} f_n + \frac{4h}{6} \left(f_n + \frac{h}{2} f_{n,t} + \frac{h^2}{4} f_{n,tt} + \frac{h^2}{4} f_{n,tx} + \frac{h}{4} f_{n,x} + \frac{h^2}{8} f_{n,xx} \right) \\ + \frac{h}{6} \left(f_n + hf_{n,t} + \frac{h^2}{2} f_{n,tt} + \frac{h^2}{6} f_n f_{n,tx} + hf_{n,x}^2 + \frac{h^2}{4} f_n^2 f_{n,xx} \right) \end{pmatrix},$$

$$f_n = f(t_n, x_n), f_{n,t} = \left(\frac{\partial f(t,x)}{\partial t} \right)_{(t_n,x_n)}, f_{n,x} = \left(\frac{\partial f(t,x)}{\partial x} \right)$$

$$f_{n,tt} = \left(\frac{\partial^2 f(t,x)}{\partial t^2} \right)_{(t_n,x_n)}, \quad f_{n,xx} = \left(\frac{\partial^2 f(t,x)}{\partial x^2} \right)_{(t_n,x_n)}, \quad f_{n,tx} = \left(\frac{\partial^2 f(t,x)}{\partial t \partial x} \right)_{(t_n,x_n)}$$

Using (4.1), (4.2), (4.3) and (4.4) in (3.14), we get

$$x_{n+1} = x_n + \frac{h}{6} \begin{pmatrix} f_n + 4\left(f_n + \frac{h}{2} f_{n,t} + \frac{h^2}{4} f_{n,tt} + \frac{h^2}{4} f_{n,tx} + \frac{h}{4} f_{n,x} + \frac{h^2}{8} f_{n,xx} \right) \\ + f_n + hf_{n,t} + \frac{h^2}{2} f_{n,tt} + W_3 \left[hf_{n,tx} + f_{n,x} + \frac{h}{4} f_n f_{n,xx} \right] \end{pmatrix} \tag{4.5}$$

$$x_{n+1} = x_n + hf_n + h^2\left(\frac{1}{2} f_{n,t} + \frac{1}{6} f_{n,x} + \frac{1}{6} f_{n,x}^3 + \frac{1}{6} f_{n,x} f_n\right)$$
$$+h^3\left(\begin{array}{c}\frac{1}{12} f_{n,xx} + \frac{1}{4} f_{n,tt} + \frac{1}{6} f_{n,tx} + \frac{1}{24} f_{n,xx} f_n^2 + \frac{1}{6} f_{n,x}^2 f_{n,tx} + \frac{1}{36} f_{n,x}^2 \\ + \frac{1}{12} f_{n,t} f_{n,x} + \frac{1}{6} f_{n,tx} f_n + \frac{1}{144} f_{n,xx} f_{n,x} f_n^2 + \frac{1}{24} f_{n,xx} f_{n,x}^2 f_n\end{array}\right)$$
$$+h^4\left(\begin{array}{c}\frac{1}{576} f_{n,xx}^2 f_n^3 + \frac{1}{72} f_{n,xx} f_{n,x} + \frac{1}{12} f_{n,t} f_{n,tx} + \frac{1}{24} f_{n,x} f_{n,tt} \\ + \frac{1}{18} f_{n,x} f_{n,tx} + \frac{1}{144} f_{n,xx} f_{n,tx} f_n^2 + \frac{1}{48} f_{n,xx} f_{n,t} f_n + \frac{1}{144} f_{n,xx} f_{n,x} f_n + \frac{1}{216} f_{n,x} f_{n,tx} f_n\end{array}\right)$$
$$+h^5\left(\begin{array}{c}\frac{1}{288} f_{n,xx}^2 f_n + \frac{1}{864} f_{n,xx} f_{n,tx} f_n^2 + \frac{1}{144} f_{n,xx} f_{n,tx} f_n \\ + \frac{1}{72} f_{n,xx} f_{n,tx} + \frac{1}{96} f_{n,tt} f_{n,xx} f_n + \frac{1}{216} f_{n,tx}^2 f_n + \frac{1}{36} f_{n,tx}^2 + \frac{1}{24} f_{n,tt} f_{n,tx}\end{array}\right) + \ldots$$

The Taylor's series expansion of exact value $x(t)$ about t_n is

$$x(t_{n+1}) = x_n + hf_n + h^2\left(\frac{1}{2} f_{n,t} + \frac{1}{6} f_{n,x} + \frac{1}{6} f_{n,x}^3 + \frac{1}{6} f_{n,x} f_n\right)$$
$$+h^3\left(\begin{array}{c}\frac{1}{12} f_{n,xx} + \frac{1}{4} f_{n,tt} + \frac{1}{6} f_{n,tx} + \frac{1}{24} f_{n,xx} f_n^2 + \frac{1}{6} f_{n,x}^2 f_{n,tx} + \frac{1}{36} f_{n,x}^2 \\ + \frac{1}{12} f_{n,t} f_{n,x} + \frac{1}{6} f_{n,tx} f_n + \frac{1}{144} f_{n,xx} f_{n,x} f_n^2 + \frac{1}{24} f_{n,xx} f_{n,x}^2 f_n\end{array}\right) + O(h^4) \tag{4.6}$$

The truncation error gives

$$T_{n+1} = x_{i+1} - x(t_{i+1})$$

$$T_{n+1} = x_n + hf_n + h^2\left(\frac{1}{2} f_{n,t} + \frac{1}{6} f_{n,x} + \frac{1}{6} f_{n,x}^3 + \frac{1}{6} f_{n,x} f_n\right)$$
$$+h^3\left(\begin{array}{c}\frac{1}{12} f_{n,xx} + \frac{1}{4} f_{n,tt} + \frac{1}{6} f_{n,tx} + \frac{1}{24} f_{n,xx} f_n^2 + \frac{1}{6} f_{n,x}^2 f_{n,tx} + \frac{1}{36} f_{n,x}^2 \\ + \frac{1}{12} f_{n,t} f_{n,x} + \frac{1}{6} f_{n,tx} f_n + \frac{1}{144} f_{n,xx} f_{n,x} f_n^2 + \frac{1}{24} f_{n,xx} f_{n,x}^2 f_n\end{array}\right)$$
$$+h^4\left(\begin{array}{c}\frac{1}{576} f_{n,xx}^2 f_n^3 + \frac{1}{72} f_{n,xx} f_{n,x} + \frac{1}{12} f_{n,t} f_{n,tx} + \frac{1}{24} f_{n,x} f_{n,tt} \\ + \frac{1}{18} f_{n,x} f_{n,tx} + \frac{1}{144} f_{n,xx} f_{n,tx} f_n^2 + \frac{1}{48} f_{n,xx} f_{n,t} f_n + \frac{1}{144} f_{n,xx} f_{n,x} f_n + \frac{1}{216} f_{n,x} f_{n,tx} f_n\end{array}\right) \tag{4.7}$$
$$+h^5\left(\begin{array}{c}\frac{1}{288} f_{n,xx}^2 f_n + \frac{1}{864} f_{n,xx} f_{n,tx} f_n^2 + \frac{1}{144} f_{n,xx} f_{n,tx} f_n \\ + \frac{1}{72} f_{n,xx} f_{n,tx} + \frac{1}{96} f_{n,tt} f_{n,xx} f_n + \frac{1}{216} f_{n,tx}^2 f_n + \frac{1}{36} f_{n,tx}^2 + \frac{1}{24} f_{n,tt} f_{n,tx}\end{array}\right)$$

$$T_{n+1} = h^5\left(\begin{array}{c}\frac{1}{288} f_{n,xx}^2 f_n + \frac{1}{864} f_{n,xx} f_{n,tx} f_n^2 + \frac{1}{144} f_{n,xx} f_{n,tx} f_n \\ + \frac{1}{72} f_{n,xx} f_{n,tx} + \frac{1}{96} f_{n,tt} f_{n,xx} f_n + \frac{1}{216} f_{n,tx}^2 f_n + \frac{1}{36} f_{n,tx}^2 + \frac{1}{24} f_{n,tt} f_{n,tx}\end{array}\right) + O(h^6). \tag{4.8}$$

Hence, the new iterative method is of fourth order.

4.1.1 Convergence of the iterative methods

Theorem 4.2 An RKLM (3.14) is said to be convergent by Theorem 2.1 if the following results hold:

(i) Consistency holds

(ii) Regularity holds

Proof Consider the increment function in (3.14),

$$\phi(x_n, u_n, h) = h^{-1}\left(\frac{h}{6} k_1 + \frac{4h}{6} k_2 + \frac{h}{6} k_4\right),$$

$$\phi(x_n, u_n, h) = \frac{1}{6}(k_1 + 4k_2 + k_4). \tag{4.9}$$

If $h = 0$, then

$$\phi(x_n, u_n, 0) = f(t, x). \tag{4.10}$$

Therefore, by Definition 2.2, the method (3.14) is consistent.

Proof Denote $k_1^* = f(t_n, x_n^*)$, $k_2^* = f(t_{n+\frac{1}{2}}, x^*_{n+\frac{1}{2}})$, $k_3^* = f(t_{n+1}, x_n^* + \frac{h}{6}k_1^*)$ and $k_4^* = f(t_{n+1}, x_n^* + \frac{h}{6}k_1^* + \frac{4h}{6}k_2^* + \frac{h}{6}k_3^*)$ for every (t,x), $(t,x^*) \in S$ and k_n $(n = 1,2,3,4)$ are defined in (3.14). Since $f(t,x)$ is Lipschitz, we have

$$\|k_1 - k_1^*\| = \|f(t_n, x_n) - f(t_n, x_n^*)\|,$$
$$\leq L\|x_n - x_n^*\|,$$

(where L is the Lipschitz constant)

$$\|k_2 - k_2^*\| = \left\| f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}\right) - f\left(t_n + \frac{h}{2}, x_n^* + \frac{h}{2}\right)\right\|,$$
$$\leq L\|x_n - x_n^*\|,$$

$$\|k_3 - k_3^*\| = \left\| f\left(t_{n+1}, x_n + \frac{h}{6}k_1 + \frac{4h}{6}k_2\right) - f\left(t_{n+1}, x_n^* + \frac{h}{6}k_1^* + \frac{4h}{6}k_2^*\right)\right\|,$$
$$\leq L\left(1 + \frac{5h}{6}L\right)\|x_n - x_n^*\|,$$

$$\|k_4 - k_4^*\| = \left\| f\left(t_n + h, x_n + \frac{h}{6}k_1 + \frac{4h}{6}k_2 + \frac{h}{6}k_3\right) - f\left(t_n + h, x_n^* + \frac{h}{6}k_1^* + \frac{4h}{6}k_2^* + \frac{h}{6}k_3^*\right)\right\|,$$
$$\leq L\left(1 + 3Lh + \frac{5L^2h^2}{6}\right)\|x_n - x_n^*\|.$$

Now using equation (4.9)

$$\phi(x_n, u_n, h) - \phi(x_n, u_n^*, h) = \frac{1}{6}\left(\|k_1 - k_1^*\| + 4\|k_2 - k_2^*\| + \|k_4 - k_4^*\|\right),$$
$$= L\left(1 + \frac{18Lh}{36} + \frac{5L^2h^2}{36}\right)\|x_n - x_n^*\|,$$
$$= L'\|x_n - x_n^*\|,$$

where $L' = L\left(1 + \frac{18Lh}{36} + \frac{5L^2h^2}{36}\right)$.

Therefore, the increment ϕ satisfies a Lipschitz condition in x and hence, by Definition 2.3, the method (3.14) is regular.

Lemma 4.1 Suppose the function $f(t,x)$ is defined and continuous in the strip $S(|t - t_0| \leq a,\ \|x\| < \infty, a > 0)$ and satisfy Lipschitz condition

$$\|f(t,x) - f(t,x^*)\| \leq L\|x - x^*\|,$$

for every $(t,x),(t,x^*) \in S$, where L is Lipschitz constant; then, method (3.14) is said to be convergent

Proof Since consistency and regularity holds for method (3.14), by Theorem 2.1, it is convergent.

4.1.2 Stability of the iterative methods

Theorem 4.3 For the iterative method (3.14) to be stable, we must have

$$\left|\frac{\left(1+\frac{2h\lambda}{3}+\frac{h^2\lambda^2}{6}+\frac{h^3\lambda^3}{216}\right)}{\left(1-\frac{4h\lambda}{12}+\frac{4h^2\lambda^2}{36}\right)}\right| \le 1.$$

Proof Using Definition 2.4. Applying the iterative method (3.14) to the test equation $x' = \lambda x,$ we obtain

$$k_1 = \lambda x_n,$$

$$k_2 = \lambda x_{n+\frac{1}{2}},$$

$$k_3 = \lambda\left(1+\frac{h\lambda}{6}\right)x_n + \frac{4h\lambda}{6}x_{n+\frac{1}{2}},$$

$$k_4 = \lambda\left[\left(1+\frac{h\lambda}{3}+\frac{1}{36}h^2\lambda^2\right)x_n + \frac{4h\lambda}{3}x_{n+\frac{1}{2}}\right].$$

From (3.14), we can write

$$x_{n+1} = x_n + \frac{h}{6}\lambda x_n + \frac{4h}{6}\lambda x_{n+\frac{1}{2}} + \frac{h}{6}\lambda\left[\left(1+\frac{h\lambda}{3}+\frac{1}{36}h^2\lambda^2\right)x_n + \frac{4h\lambda}{3}x_{n+\frac{1}{2}}\right],$$

$$x_{n+1} = \left(1+\frac{h\lambda}{3}+\frac{h^2\lambda^2}{18}+\frac{h^3\lambda^3}{216}\right)x_n + \left(\frac{4h\lambda}{6}+\frac{4h^2\lambda^2}{18}\right)x_{n+\frac{1}{2}}.$$

Writing $x_{n+\frac{1}{2}}$ as $\frac{1}{2}(x_{n+1}+x_n),$ then

$$x_{n+1} = \left(1+\frac{h\lambda}{3}+\frac{h^2\lambda^2}{18}+\frac{h^3\lambda^3}{216}\right)x_n + \frac{1}{2}\left(\frac{4h\lambda}{6}+\frac{4h^2\lambda^2}{18}\right)(x_{n+1}+x_n).$$

So that,

$$x_{n+1} = \frac{\left(1+\frac{2h\lambda}{3}+\frac{h^2\lambda^2}{6}+\frac{h^3\lambda^3}{216}\right)}{\left(1-\frac{4h\lambda}{12}+\frac{4h^2\lambda^2}{36}\right)}x_n.$$

Now, for stability of RKLM and by Definition (2.6), we must have

$$\left|\frac{\left(1+\frac{2h\lambda}{3}+\frac{h^2\lambda^2}{6}+\frac{h^3\lambda^3}{216}\right)}{\left(1-\frac{4h\lambda}{12}+\frac{4h^2\lambda^2}{36}\right)}\right| \le 1.$$

4.2 Algorithm for Forward Backward Sweep Methods

4.2.1 Algorithm for FBS implementation for RKLM

Applying (3.14), the algorithms are given as

Algorithm 4.1 Forward algorithm

for $n = 1 : N$

$$k_1 = f(t_n, x_n, u_n)$$

$$k_2 = f(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(x_n + x_{n+1}), \tfrac{1}{2}(u_n + u_{n+1}))$$

$$k_3 = f\left(t_n + h,\ x_n + \tfrac{h}{6}k_1 + \tfrac{4h}{6}k_2,\ u_{n+1}\right)$$

$$k_4 = f\left(t_n + h,\ x_n + \tfrac{h}{6}k_1 + \tfrac{4h}{6}k_2 + \tfrac{h}{6}k_3,\ u_{n+1}\right)$$

$$x_{n+1} = x_n + \tfrac{h}{6}(k_1 + 4k_2 + k_4)$$

Algorithm 4.2 Backward algorithm

for $j = 1 : N$

$$n = N + 2 - j$$
$$k_1 = f\left(t_n, x_n, \lambda_n, u_n\right)$$
$$k_2 = f\left(t_n - \tfrac{1}{2}h, \tfrac{1}{2}(x_n + x_{n-1}), \tfrac{1}{2}(\lambda_n + \lambda_{n-1}), \tfrac{1}{2}(u_n + u_{n-1})\right)$$
$$k_3 = f\left(t_n - h,\ x_{n-1}, \lambda_n - \tfrac{h}{6}k_1 - \tfrac{4h}{6}k_2, u_{n-1}\right)$$
$$k_4 = f\left(t_n - h, x_{n-1}, \lambda_n - \tfrac{h}{6}k_1 - \tfrac{4h}{6}k_2 - \tfrac{h}{6}k_3, u_{n-1}\right)$$
$$\lambda_{n-1} = \lambda_n - \tfrac{h}{6}\left(k_1 + 4k_2 + k_4\right)$$

5. Numerical Experiment

This section provide solution to optimal control model for real investment and fish management using RKLM. Let $x_N(t)$ and $x(t)$ be the approximate and numerical solutions for the state respectively, then the absolute error of the state is given by $\left|x_N(t) - x(t)\right|$. Let $u_N(t)$ and $u(t)$ be the approximate and numerical solutions for the control respectively; then, the absolute error of the control is given by $\left|u_N(t) - u(t)\right|$. All numerical solutions are given in figure and tabular forms. All computations in this section are done with the aid of a written MATLAB codes, which were run on a Window 8.1 computer.

Table 1: Notations

Abbreviations	Meaning
NM2	Naevdal (2002) Method
Err	Absolute Error

Example 1 (Real Investment) A firm has a production $y = x^a$. Here y is output and x is capital. The stock of capital is assumed to be driven by the differential equation $x' = u - \delta x$. Here u is investment and δ is the rate of capital depreciation. Assume that the cost of investment is given by $\frac{c}{2}u^2$ and the market price of y is equal to one. This leads to the following maximization problem

$$\max_u J(x,u) = \int_0^T \left(x^a - \frac{c}{2}u^2 \right) e^{-rt} dt,$$

subject to

$$x' = u - \delta x,$$
$$x(0) = 0.$$

Solving with the following weight parameters (the choice of the parameters implies that, the investment has a constant rate of depreciation)

$$c = a = \delta = T = 1,\ r = 0,$$

lead to optimal control with optimality solution

$$x^*(t) = 1 - \frac{1}{2}e^{t-1} + \left(\frac{1}{2}e^{-1} - 1\right)e^{-t},$$

$$u^*(t) = 1 - e^{t-1}.$$

Source: (Naevdal, 2002).

Solution The optimality system of the problem is developed by first constructing the Hamiltonian

$$H = \left(x^a - \frac{c}{2}u^2 \right) e^{-rt} + \lambda(u - \delta x).$$

The problem is a maximization problem as

$$\frac{\partial^2 H}{\partial u^2} = -ce^{-rt} < 0.$$

The optimality condition

$$0 = \frac{\partial H}{\partial u} = \lambda - cue^{-rt} \Rightarrow u = -\frac{\lambda}{ce^{-rt}},$$

and the adjoint equation is

$$\lambda' = -\frac{\partial H}{\partial x} = \lambda\delta - ax^{a-1}e^{-rt}.$$

Using the optimality system, the numerical code is generated, written in MATLAB R2018a. This problem is solved with $N = 10$: The results are shown in Figure 1, Figure 2, and Table 2.

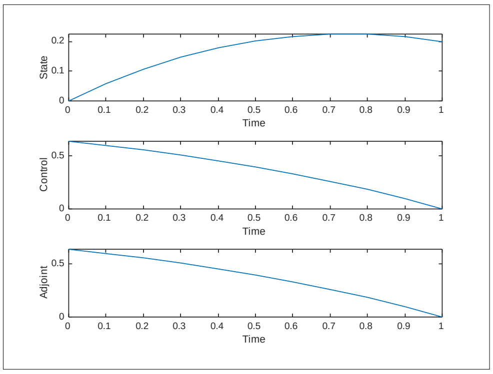

Figure 1: The optimal state, adjoint and control of RKLM for Example 1

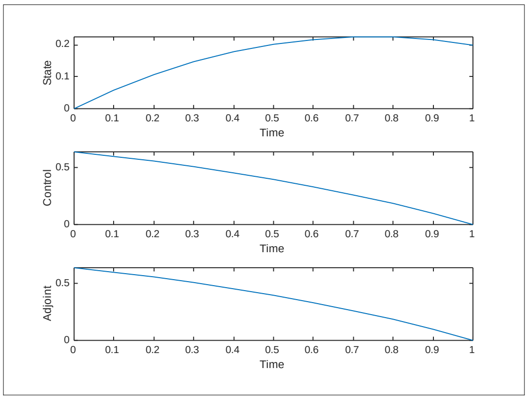

Figure 2: The optimal state, adjoint and control of NM2 for Example 1

Table 2: State and Control for Example 1

State				Control		
t	Exact	$RKLM_{Err}$	$NM2_{Err}$	Exact	$RKLM_{Err}$	$NM2_{Err}$
0	0	0	0	0.632	4.666e-05	7.666e-05
0.2	0.107	1.038e-05	2.353e-05	0.550	2.212e-05	7.491e-5
0.4	0.178	1.350e-05	5.422e-05	0.451	1.777e-06	6.862e-05
0.6	0.216	1.32e-05	6.291e-05	0.329	1.150e-06	5.587e-05
0.8	0.223	4.802e-06	6.097e-05	0.181	1.382e-05	3.412e-05
1.0	0.199	6.007e-6	4.846e-6	0	0	0

To maximize productivity is the goal. As the investment progressively decreases in Figures 1 and 2, the capital x first increases to boost production. Between $t = 0.7$ and $t = 0.8$, capital accumulates. Then allowing capital to degrade for the remaining time horizon. It is evident from a comparison of Figures 1 and 2 that RLKLM and NM2 are identical. Table 2 demonstrates how capital increases as investment declines until it reaches its optimal level, at which point it starts to decline. Additionally, Table 2 makes it quite evident that RKLM is more accurate than NM2.

Example 2 (Fish Harvesting) Suppose at some point, designated as $t = 0$, a fish population is introduced into a fishery of some kind (for example, an artificial tank or a netted area in a body of water). Let $x(t)$ be the population level (scaled) at time t, where $x(0) = x_0 > 0$ is the initial concentration, as determined by the introduction. Suppose that, when introduced, the fish are very small and that the average mass of the fish at time $t = 0$ is essentially 0. Further, the average mass of the fish as a function of time is given by

$$f_{mass}(t) = \frac{kt}{t+1},$$

where k is the maximum mass of this specie. We will assume the time interval $[0,T]$, over which we are to consider harvesting, is small enough that no reproduction will occur. Specifically, the population will have no natural growth. Let $u(t)$ be the harvest rate at time t and m be the natural death rate of the fish. We wish to maximize the total mass harvested over the interval taking into account the cost of harvesting. So, the optimal control problem can be stated as

$$\max_u J(x,u) = \int_0^T A\frac{kt}{t+1}x(t)u(t) - u(t)^2\,dt$$

subject to

$$x'(t) = -(m+u(t))x(t),\ x(0) = x_0$$
$$0 \le u(t) \le M.$$

The upper bound M is added to take physical limitations of harvesting into account, and A is a nonnegative weight parameter. Note, if u is set to 0, then $x(t) = x_0 e^{-mt}$ which naturally decreases. Any positive control will cause the state to decrease even more.
Source: (Lenhart & Workman, 2007)

Solution The optimality system of the problem is developed by first constructing the Hamiltonian

$$H = A\frac{kt}{t+1}x(t)u(t) - u^2 + \lambda(-(m+u(t))x(t)).$$

The optimality condition is

$$0 = \frac{\partial H}{\partial u} = -\frac{1}{t+1}(2u + x\lambda + 2tu + tx\lambda - Aktx),$$

$$\Rightarrow u^* = \frac{1}{2t+2}(x\lambda + tx\lambda - Aktx).$$

The adjoint equation is

$$\lambda'(t) = -\frac{\partial H}{\partial x} = \frac{1}{t+1}(m\lambda + u\lambda + mt\lambda + tu\lambda - Aktu)$$

Using the optimality system, the numerical code is generated, written in MATLAB R2018a. This problem is solved with $N = 1000$: The results are shown in Figure 3 to Figure 5, and Table 3 to Table 5. First considering the parameters

$$A = 5, k = 10, m = 0.2, x_0 = 0.4, M = 1, \text{ and } T = 10$$

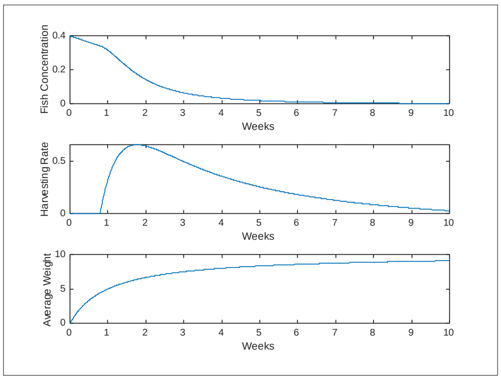

Figure 3: The fish concentration and harvesting rate of RKLM for Example 2

Table 3: Sate and Control for Example 2

t	Fish concentration	Harvesting rate	Average weight
0	0.4000000000000000	0	0
0.2	0.141961458996662	0.641543817352862	0.666666666666667
0.4	0.032681251561386	0.353633034550845	8.000000000000000
0.6	0.011065553849465	0.179886503543253	8.571428571428571
0.8	0.004136441229281	0.084028905440520	8.888888888888889
1.0	0.001167497704697	0.026617744299209	9.090909090909092

Here, maximizing the overall mass of fish harvested while accounting for harvesting time is the goal. With the species' maximum mass, $k = 10$, and the weight parameter, $A = 5$, Figure 3 illustrates how, in the absence of early harvesting, the fish concentration first decreases extremely slowly. The total mass of fish harvested to this point is zero. Fish concentration started to decline after harvesting got underway, and as week one, when harvesting is happening at a fast pace, dwindled dramatically and average weight increased from zero to roughly five. The average weight peaked at week 10 at about nine, and the harvesting rate decreased to nearly zero when the concentration of fish approached zero.

Varying the parameters,

$$A = 5, k = 15, m = 0.2, x_0 = 0.4, M = 1, \text{ and } T = 10$$

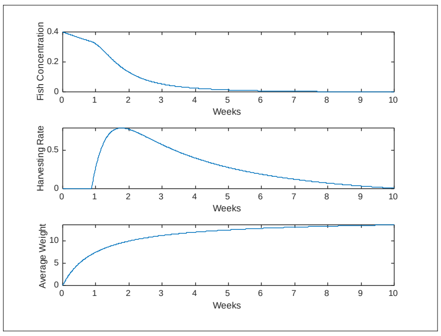

Figure 4: The fish concentration and harvesting rate of RKLM for Example 2

Table 4: Sate and Control for Example 2

t	Fish concentration	Harvesting rate	Average weight
0	0.400000000000000	0	0
0.2	0.131130289853804	0.767736885244573	10.000000000000000
0.4	0.025066899041166	0.396627546025515	12.000000000000000

0.6	0.007521947106378	0.187081795142835	12.857142857142858
0.8	0.002334938847361	0.072837344661134	13.333333333333334
1.0	0.000196827505200	0.006602086784370	13.636363636363637

Figure 4 shows that the weight parameter stays at $A = 5$, but the maximum mass of the species is raised to $k = 15$. The main distinction from Figure 3 is that at week 9.5, the average weight reached its maximum of roughly 13, and the harvesting rate decreased to zero as the fish concentration approached zero.

Now varying the parameter A to 10,

$$A = 10, k = 10, m = 0.2, x_0 = 0.4, M = 1 \text{ and } T = 10.$$

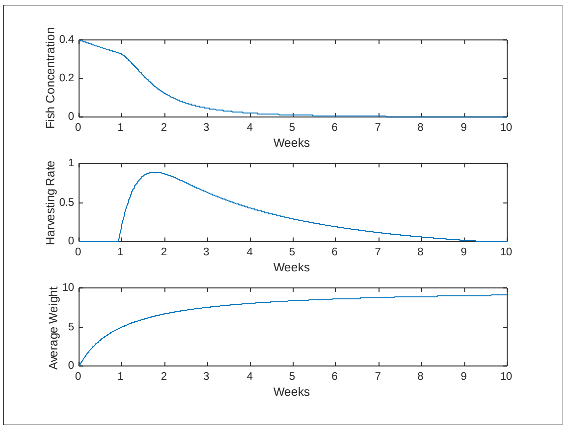

Figure 5: The fish concentration and harvesting rate of RKLM for Example 2

Table 5: Sate and Control for Example 2

t	Fish concentration	Harvesting rate	Average weight
0	0.4000000000000000	0	0
0.2	0.123418956244941	0.869330790901199	6.666666666666667
0.4	0.020321085948559	0.427140131370255	8.000000000000000
0.6	0.005466509254597	0.187138329274792	8.571428571428571
0.8	0.001342538425360	0.057379813095424	8.888888888888888
1.0	0.000320569138487	1.69406589450e-21	9.090909090909092

Figure 5 shows that the weight parameter is changed to $A = 10$ but the species' maximal mass, $k = 10$, remains constant. At week nine, the average weight reaches its peak of approximately nine, and the harvesting rate drops to zero as the fish concentration approaches zero.

Tables 3 and 5 show that the fish concentration (state) was high at first ($t = 0$), but that it started to decline as soon as heavy harvesting (control) was implemented. The state drops even more at $t = 0.2$ when the control is higher. The state and the control both steadily decrease until they reach their lowest points at $t = 1$. The control starts to do so at $t = 1$.

As a result, examining Figures 3 through 5 and Tables 3 through 5 demonstrates the best harvesting approach. A greater weight parameter causes the harvesting rate to reach zero more quickly and the total fish mass to reach its maximum level more quickly. However, a change in the species' maximum mass has no appreciable impact on the overall mass of fish.

6. Conclusion

This study develops a Runge-Kutta-like method for solving optimum control problems, based on the Patade and Bhalekar methodology. The developed RKLM is of order four, and it has been confirmed to be stable and convergent, much like the conventional CRKM. When it comes to method development, the Runge Kutta method is more complex to create than the RKLM, particularly when higher order methods are needed. Once more, it is established that RKLM outperforms CRKM in terms of accuracy. For this reason, it is recommended to utilize RKLM instead than CRKM. This research also demonstrates how well RKLM works with the forward-backward

sweep method to address optimum control problems modeled in ordinary differential equations. MATLAB R2018a is used to write codes for the RKLMs' implementation.
This study will help fish farmers who have several fish pond and want to engage the costumers throughout the year. It will guide the farmers to balance the rate of harvesting so that there will be sale continuously until the next pond is ready. It will also help a real investor to know the right time to increase or decrease investment.

Declaration of competing interest:
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding:
This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

References

[1] Adamu, S. (2023). Numerical solution of optimal control problems using block method. Electronic Journal of Mathematical Analysis and Applications, 11(2), 1-12. http://ejmaa.journals.ekb.eg/

[2] Adamu, S., Aduroja, O. O., Onanaye, A. S. & Odekunle, M. R. (2024). Iterative method for the numerical solution of optimal control model for mosquito and insecticide. J. Nig. Soc. Phys. Sci. 6(2024) 1965. DOI: https://doi.org/10.46481/jnsps.2024.1965.

[3] Aduroja O. O., Adamu S., Kida M. & Buhari H. L. (2024). Bi-Basis Function Second Derivative Block Method for the Solution of Optimal Control Problems. International Journal of Development Mathematics 1(1) 16 - 24. https://doi.org/10.62054/ijdm/0101.02.

[4] Alkali, A. M., Ishiyaku, M., Adamu, S., & Umar, D. (2023). Third derivative integrator for the solution of first order initial value problems. Savannah Journal of Science and Engineering Technology, 1(5), 300-306.

[5] Andres-Martinez, O. (2022). Optimal control and Pontryagin's principle in chemical engineering: History, theory, and challenges. AIChE Journal, 1-76.

[6] Areo, E. A., & Adeniyi, R. B. (2013). A self-starting linear multistep method for direct solution of initial value problems of second order ordinary differential equations. International Journal of Pure and Applied Mathematics, 82(3), 345-364.

[7] Butcher, J. C. (1996). A history of Runge-Kutta methods. Applied Numerical Mathematics, 20, 247-260.

[8] Garret, R. R. (2015). Numerical methods for solving optimal control problems. Tennessee Research and Creative Exchange, University of Tennessee, Knoxville.

[9] Hildebrand, F. B. (1974). Introduction to numerical analysis (2nd ed.). Tata McGraw-Hill Publishing Co. Limited.

[10] Lenhart, S., & Workman, J. T. (2007). Optimal control applied to biological models. Chapman & Hall/CRC.

[11] Musa, H., Saidu, I., & Waziri, M. Y. (2010). A simplified derivation and analysis of fourth order Runge-Kutta method. International Journal of Computer Applications, 9(8), 0975-8887.

[12] Naevdal, E. (2002). Numerical optimal control in continuous time made easy. Computer in Higher Education Economics Review, 15(1), 1-14.

[13] Odekunle, M. R. (2000). Some semi-implicit rational R-K schemes. Bagale Journal of Pure and Applied Sciences, 1(1), 11-14.

[14] Odekunle, M. R., Oye, N. D., & Adee, S. O. (2004). A class of inverse Runge-Kutta schemes for the numerical integration of singular problems. Applied Mathematics and Computation, 158, 149-158.

[15] Patade, J., & Bhalekar, S. (2015). A new numerical method based on Daftardar-Gejji and Jafari technique for solving differential equations. World Journal of Modeling and Simulation, 11(4), 256-271.

[16] Rodrigues, H. S., Monteiro, M. T. T. D., & Torres, F. M. (2014). Systems theory: Perspectives, applications and developments. Nova Science Publishers.

[17] Usman, A. S., Odekunle, M. R., & Ahmad, M. M. (2013). A class of three-stage implicit rational Runge-Kutta schemes for approximation of second order ordinary differential equations. Mathematical Theory and Modeling, 3(11), 121-130.