Common Fixed Point Theorems for Six Mappings in Symmetric Spaces

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Abstract: In this paper, we establish a common fixed-point theorem for six mappings in symmetric spaces with compatible mapping of type (E) and occasionally weakly compatible mappings. Our work extends the results of Rajopadhyaya et al. [23,25], Aamri and Moutawakil [2] and other similar results in semi-metric space.

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1 Introduction

Polish mathematician Banach published his contraction Principle in1922. In 1928, Menger [21] introduced semi-metric space as a generalization of metric space. In 1976, Cicchese [6] introduced the notion of a contractive mapping in semi-metric space and proved the first fixed point theorem for this class of spaces. Hicks and Rhoades [12] generalized Banach contraction principle in semi-metric space. Jungck [18] introduced the concept of compatible mappings in metric spaces. This concept has been frequently used to prove the existence of theorem in common fixed-point theory. However, the study of common fixed-point theorems for non-compatible mappings has also become an interesting concept. Pant et al. [22] initially proved some common fixed-point theorems for non-compatible mappings. Aamri et al. [3] gave a notion of E.A. property and established some common fixed-point theorems for non-compatible mappings under contractive conditions. Cho and Kim [4] generalized the paper of M. Aamri and Moutawakil by replacing W4 property by C.C property with different contractive conditions. Cho et al. [5] proved some common fixed-point theorems for weakly compatible mappings in symmetric spaces and gave some counter examples. Among various types of compatible mappings, Singh and Singh [26] introduced the concept of compatible mappings of type (E). Jungck and Rhoades [19] introduced the notion of weakly compatible mappings and showed that compatible mappings are weakly compatible but not conversely. Jungck and Rhoades [20] introduced occasionally weakly compatible mappings which is more general among the commutativity concepts. Jungck and Rhoades [20] obtained several common fixed-point theorems using the idea of occasionally weakly compatible mappings. Several interesting and elegant results have been obtained by various authorsin this direction. There have been interesting generalized and formulated results in semi- metric space initiated by Frechet [9], Menger [21] and Wilson [27]. In this paper, we prove common fixed-point theorem for three pairs of self-mappings by using compatible mapping of type (E) and occasionally weakly compatible(owc) mapping in semi-metric space that extends the results of Rajopadhyaya et al. [23,25], Aamri and Moutawakil [2] and other similar results in semi-metric space. We also prove a common fixed-point theorem for three pairs of self-mappings using occasionally weakly compatible mappings which improves and extends similar known results in the literature.

2 Basic Definitions

Definition 2.1. Let X be a non-empty set. A symmetric (semi-metric) on a set X is a non-negative real valued function d on $X \times X$ such that

(i) $d(x,y) = 0$ if and only if $x = y$ for $x, y \in X$.

(ii) $d(x,y) = d(y,x)$, for $x, y \in X$.

Example 2.1. Consider $X = R$ be the set of all real numbers. Let a function d be defined as follows.

$$d(x,y) = \begin{cases} |x-y|, & x \text{ and } y \text{ both are rational or irrational} \\ |x-y|^{-1} & \text{Otherwise} \end{cases}$$

Then, (X, d) is a symmetric (semi-metric) space but not a metric space because the property of triangle inequality is not satisfied by d.

Example 2.2. Consider $X = [0,1]$. Let a function d be defined by $d(x,y) = (x-y)^2$. Then, (X, d) is a symmetric (semi-metric) space but not a metric space because the property of triangle inequality is not satisfied by d.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x,r) = \{y \in X: d(x,y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$if and only if for each $x \in U$ $B(x,r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x,r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n,x) = 0$if and only if $x_n \to x$ in the topology $t(d)$.

The difference between symmetric and metric space occurring due to triangle inequality. To obtain fixed point theorems in a symmetric (semi-metric) space, we require some additional axioms.

The following two axioms were given by Wilson [27].

Let (X,d) be a symmetric space.

(W3) Given a sequence $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y) = 0$ implies $x = y$.

(W4) Given sequences $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y_n) = 0$ implies that $\lim_{n\to\infty} d(y_n,x) = 0$
.

It is easy to see that for a semi-metric d, if $t(d)$ is a Hausdorff, then (W3) holds.

The following axiom was given by Aamri et al. [2].

Let (X,d) be a symmetric space.

(H.E) for given sequences $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(y_n,x) = 0$ implies $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Preposition 2.1. For axioms in symmetric space (X,d),

(a) $(W4) \Rightarrow (W3)$

Proof: Let $\{x_n\}$ be a sequence in X and $x, y \in X$ such that $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y) = 0$

By substituting $y_n = y$ for each $n \in N$, we get $\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x_n,y) = 0$. By using (W4) we get, $0 = \lim_{n\to\infty} d(y_n,x) = d(y,x)$.

In the sequel $\emptyset: R^+ \to R^+$ be a function satisfying $0 < \emptyset(t) < t, t > 0$.

Example 2.3. Let $X = [0,\infty)$. Consider

$$d(x,y) = \begin{cases} |x-y|, & (x \neq 0, y \neq 0) \\ \frac{1}{x}, & (x \neq 0) \end{cases}$$

Thus, (X,d) is a symmetric space which satisfies (W4) but does not satisfy (H.E) for $x_n = n,\ y_n = n+1$.

Example 2.4. Let $X = [0,1] \cup \{2\}$. Consider

$$d(x,y) = \begin{cases} |x|, & (0 < x \leq 1, y = 2) \\ |x-y| & (0 \leq x \leq 1, 0 \leq y \leq 1) \end{cases}$$

and $d(0,2) = 1$.

Thus, (X,d) is a symmetric space which satisfies (H.E). Let $x_n = \frac{1}{n}$

Then, $\lim_{n\to\infty} d(x_n,0) = \lim_{n\to\infty} d(x_n,2) = 0$ but $d(0,2) \neq 0$. Therefore, symmetric space (X,d) does not satisfy (W3).

Example 2.5.

(i) Every metric space (X,d) satisfies property (H.E).

(ii) Let $X = [0,\infty)$ with the symmetric function d defined by $d(x,y) = e^{|y-x|} - 1$.Clearly, symmetric space (X,d) satisfies property (H.E).

Definition 2.2. Let X be a non-empty set and $S,T: X \to X$ be an arbitrary mapping. A point $x \in X$ is called a coincidence point of S and T if and only if $Sx = Tx$.

Example 2.6. Let S and T be two self-maps on $X = R$ defined by $S(x) = x^2 + 1$ and $T(x) = e^x$. Here $S(0) = T(0) = 1$ this implies, $S(0) = T(0)$. Hence $0 \in X$ is a coincidence point of S and T.

Definition 2.3. Let S and T be two self-mappings of a symmetric space(X,d). S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$,

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$, for some $t \in X$

Definition 2.4. Two self-mappings S and T of a symmetric space (X,d) will be non-compatible if there exist at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$, for some $t \in X$. but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non-zero or does not exist. Therefore, two non-compatible self-mapping of a symmetric space (X,d) satisfy the property (E.A).

Definition 2.5. Two self-mappings S and T of a symmetric space (X,d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.6. Let S and T be two self-mappings of a symmetric space (X,d). We say that S and T satisfy the property (E.A) if there exist a sequence $\{x_n\}$ such that.

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0, \text{for some } t \in X.$$

Example 2.7. Let $X = [0,\infty)$. Let d be a symmetric on X defined by $d(x,y) = e^{|y-x|} - 1$ for all x, y in X. Define $S,T: X \to X$as follows:

$$Sx = 2x + 1 \text{ and } Tx = x + 2 \quad \text{for all } x \in X.$$

Here function d is not a metric. Consider the sequence $x_n = 1 + \frac{1}{n}, n = 1,2,3 - -$

Clearly $\lim_{n\to\infty} d(Sx_n, 3) = \lim_{n\to\infty} d(Tx_n, 3) = 0.$

Then S and T satisfy property (E.A).

Definition 2.8. A subset A of a symmetric space (X,d) is said to be d –closed if for a sequence $\{x_n\}$ in A and a point $x \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ implies $x \in A$. For a symmetric space (X,d), d –closedness implies $t(d)$ –closedness.

Definition 2.9. Let S and T be two self-mappings of a semi-metric space (X,d). Then S and T are said to be compatible mapping of type (E) if

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = T(t) \text{ and}$$

$$\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} TSx_n = S(t),$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0 \text{ for some } t \in X$$

Example 2.8. Let $X = [0,1]$ with usual-metric space $d(x,y) = |(x-y)|$. We define a self-maps A and S as $Ax = 1, Sx = 0$, for

$x \in \left[0,\frac{1}{2}\right] - \left\{\frac{1}{4}\right\}, Ax = 0, Sx = 1$ for $x = \frac{1}{4}$ and

$Ax = \frac{1-x}{2}, Sx = \frac{x}{2}$ for $x \in \left(\frac{1}{2}, 1\right]$. Clearly A and S are not continuous at $x = \frac{1}{2}, \frac{1}{4}$

Suppose that $x_n \to \frac{1}{2}, x_n > \frac{1}{2}$ for all n.

Then, we have $Ax_n = \frac{1-x_n}{2} \to \frac{1}{4} = t$ and $Sx_n = \frac{x_n}{2} \to \frac{1}{4} = t$. Also, we have

$$Ax_n = A\left(\frac{1-x_n}{2}\right) = 1 \to 1, \quad Sx_n = A\left(\frac{x_n}{2}\right) = 1 \to 1$$

$$S(t) = 1 \text{ and } SSx_n = S\left(\frac{x_n}{2}\right) = 0 \to 0,$$

$$SAx_n = S\left\{\frac{(1-x_n)}{2}\right\} = 0 \to 0, \quad A(t) = 0$$

Therefore, (A,S) is compatible of type E.

Example 2.9. Let $X = [0,1)$ and $d(x,y) = |x-y|$.

$$Ax = Bx = \begin{cases} \frac{1+x}{3}, if\ x \in \left[1,\frac{1}{3}\right) \\ \frac{1}{3}, if\ x \in \left[\frac{1}{3}, 1\right) \end{cases}$$

$$Tx = Sx = \begin{cases} \frac{1}{3} + x, if\ x \in \left[0, \frac{1}{3}\right) \\ \frac{1}{3}, if\ x = \frac{1}{3} \\ \frac{2}{5}, if\ x \in \left(\frac{1}{3}, 1\right) \end{cases}$$

$$Ax = Bx = \left[\frac{1}{3}, \frac{4}{9}\right] \cup \left\{\frac{1}{3}\right\},\ Sx = Tx = \left[\frac{1}{3}, 1\right) \cup \left\{\frac{1}{3}\right\} \cup \left\{\frac{2}{5}\right\}$$

Take a sequence $\{x_n\}$ such that $x_n \to 0, x_n > 0$ for all n. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{3} = t \text{ (say)}$$

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ASx_n = S(t) = \frac{1}{3}$$

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = \frac{2}{5} \neq A(t) = \frac{1}{3}$$

Pair (A, S) is not compatible mapping of type E.

Preposition 2.2. Let S and T be two compatible mappings of type (E). If one of the mappings is continuous, then

(a) $S(t) = T(t)$ and $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} TTx_n = \lim_{n\to\infty} STx_n = \lim_{n\to\infty} TSx_n$
where, $\lim_{n\to\infty} Sx_n = t$ and $\lim_{n\to\infty} Tx_n = t$

If there exists $w \in X$ such that $Sw = Tw = t$, then $STw = TSw$.

Definition2.10. Let A and B be two self-mappings of a semi-metric space (X, d). Then, A and B are said to be **occasionally weakly compatible (owc)** if there is a point $x \in X$ which is coincidence point of A and B at which A and B commute.

Example2.10. Let us consider $X = [2.20]$ with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define a self-map A and B by
$A(2) = 2\ at\ x = 2$ and $A(x) = 6$ for $x > 2$
$B(2) = 2\ at\ x = 2$, $B(x) = 12$ for $2 < x \leq 5$ and $B(x) = x - 3$ for $x > 5$.
Now, $A(9) = B(9) = 6$, besides $x = 2$, $x = 9$ is another coincidence point of A and B.
$AB(2) = BA(2)$ but $AB(9) = 6, BA(9) = 3$, $AB(9) \neq BA(9)$ Therefore A and B are owc but not weakly compatible. Hence weakly compatible mappings are owc but not conversely.

Lemma 2.1. Let (X, d) be a semi-metric space. If the self-mappings A and B on X have a unique point of coincidence $w = Ax = Bx$, then w is the unique common fixed point of A and B.

3. Main Results

We establish a common fixed point theorem in symmetric space with compatible mapping of type (E) which improves and extends similar known results in the literature.

Theorem 3.1. Let (X, d) be a symmetric (semi-metric) space that satisfy (W4) and (H.E.). Let A, B, S, T, I and J be self-mappings of X such that

(i) $AB(X) \subset I(X)$ and $ST(X) \subset J(X)$ … (3.1)

(ii) the pair (ST, I) satisfies properties E.A. (resp. (AB,J) satisfy property E.A) … (3.2)

(iii) $d(ABx, STy) \leq \emptyset(max\{d(Jx, Iy), d(Jx, STy), d(Iy, STy)\})$ for all $(x, y) \in X \times X$ … (3.3)
where $\emptyset: R^+ \to R^+$ be a function satisfying $0 < \emptyset(t) < t, t > 0$

(iv) J(X) is a d-closed ($\tau(d)$-closed) subset of X (resp. I(X) is a d-closed ($\tau(d)$ –closed) subset of X). … (3.4)

Then, the pair (AB, J)as well as (ST, I)have a coincidence point.

Moreover, if the pairs $(ST, I)$and $(AB, J)$are compatible mapping of type (E) and one of the mappings AB, ST, I and J is continuous then AB, ST, I and J have a unique common fixed point.

Further, if $AB = BA, AJ = JA, BJ = JB, ST = TS, SI = IS$ and $TI = IT$, then the mappings A, B, S, T, I and J have a unique common fixed point.

Proof: In view of (3.2), the pair (ST, I)satisfies the property E.A therefore, there exists a sequence $\{x_n\}$in X and a point $u \in X$ such that

$$\lim_{n\to\infty} d(STx_n, u) = \lim_{n\to\infty} d(Ix_n, u) = 0$$

From (3.1), since $ST(X) \subset J(X)$ there exists a sequence $\{y_n\}$ in X such that

$$STx_n = Jy_n.$$

Hence, $\lim_{n\to\infty} d(Jy_n, u) = 0$

By using property (H.E), we get

$$\lim_{n\to\infty} d(STx_n, Ix_n) = 0 \text{ and } \lim_{n\to\infty} d(Jy_n, Ix_n) = 0$$

From (3.4), since $J(X)$ is a d-closed subset of X, hence

$\lim_{n\to\infty} Jy_n = u \in J(X)$. Therefore, there exists a point $v \in X$ such that $Jv = u$.

Also, we have

$$\lim_{n\to\infty} d(STx_n, Jv) = \lim_{n\to\infty} d(Ix_n, Jv) = \lim_{n\to\infty} d(Jy_n, Jv) = 0$$

Now, we assert that $ABv = Jv$.

From condition (3.3), we have

$$d(ABv, STx_n) \le \emptyset(\max\{d(Jv, Ix_n), d(Jv, STx_n), d(Ix_n, STx_n)\})$$

By taking $n \to \infty$, we have

$$\lim_{n\to\infty} d(ABv, STx_n) = 0$$

By using property (W4),

$$\lim_{n\to\infty} d(Ix_n, ABv) = 0$$

$$\left[\lim_{n\to\infty} d(ABv, STx_n) = 0, \lim_{n\to\infty} d(STx_n, Ix_n) = 0 \Rightarrow \lim_{n\to\infty} d(Ix_n, ABv) = 0\right]$$

This implies that $ABv = Jv$

[By using (W3),

$$\lim_{n\to\infty} d(Ix_n, Jv) = 0, \lim_{n\to\infty} d(Ix_n, ABv) = 0 \Rightarrow ABv = Jv]$$

This shows that v is a coincidence point of pair (AB, J).

Again, $AB(X) \subset I(X)$, there exists a point $w \in X$ such that $ABv = Iw$. This implies that $ABv = Jv = Iw$.

We assert that $STw = Iw$.

On using condition (3.3), we get

$$\begin{aligned} d(ABv, STw) &\le \emptyset(\max\{d(Jv, Iw), d(Jv, STw), d(Iw, STw)\}) \\ &\le \emptyset(\max\{d(ABv, Iw), d(ABv, STw), d(STw, ABv)\}) \\ &\le \emptyset(\max\{0, d(ABv, STw), d(STw, ABv)\}) \\ &\le \emptyset\, d(ABv, STw) \end{aligned}$$

or, $d(ABv, STw) < d(ABv, STw)$, which is a contradiction.

Hence, we get $ABv = STw = Iw$.

This shows that w is a coincidence point of pair (ST, I). Consequently,

$$ABv = Jv = STw = Iw$$

which shows that the pairs (AB, J) and (ST, I) have a coincidence point *v* and *w* respectively.

For the existence of a common fixed point for four mappings of a semi-metric space, we apply compatible mapping of type (E).

Let us suppose that J is continuous and the pair (AB, J) is compatible mapping of type (E), then by proposition (2.2), we have

$ABv = Jv$ then $ABJv = JABv$.

Now, we have $ABABv = ABJv = JABv = JJv$.

We show that ABv is a common fixed point of AB and J.

If possible, $ABABv \ne ABv$.

On using condition (3.3), we get,

$$\begin{aligned} d(ABv, ABABv) &= d(ABABv, STw) \\ &\le \emptyset(\max\{d(JABv, Iw), d(JABv, STw), d(Iw, STw)\}) \\ &\le \emptyset(\max\{d(ABABv, ABv), d(ABABv, ABv), 0\}) \end{aligned}$$

or, $d(ABv, ABABv) < d(ABv, ABABv)$, which is a contradiction.

Hence, we get $ABABv = ABv$, which implies that

$$ABABv = JABv = ABv.$$

Therefore, ABv is a common fixed point of *AB* and *J*.

Let us assume that *I* is continuous and (ST, I) is compatible mapping of type (E), then we have

$STw = Iw$ then $STIw = ISTw$

Now, we have $IIw = ISTw = STIw = STSTw$

We show that *STw* is a common fixed point of ST and I, if possible $STSTw \ne STw$.

Using condition (3.3), we have

$$\begin{aligned} d(STw, STSTw) &= d(ABv, STSTw) \\ &\le \emptyset(\max\{d(Jv, ISTw), d(Jv, STSTw), d(ISTw, STSTw)\}) \\ &\le \emptyset(\max\{d(ABv, STSTw), d(ABv, STSTw), d(STSTw, STSTw)\}) \\ &\le \emptyset(\max\{d(STw, STSTw), d(STw, STSTw), 0\}) \\ &\le \emptyset\left(d(STw, STSTw)\right) \\ &< d(STw, STSTw), \text{ which is a contradiction.} \end{aligned}$$

Hence we have $STSTw = STw$, which implies that

$$STSTw = ISTw = STw.$$

Therefore, *STw* is a common fixed point of *ST* and *I*.
Since, $ABv = STw$, ABv is a common fixed point of *AB*, *ST*, *I* and *J*.

To prove that ABv is unique, let *z* be another common fixed point of *AB*, *ST*, *I* and *J*, $z \neq ABv$ such that $ABz = STz = Jz = Iz = z$.
By using condition (2.7), we get

$$\begin{aligned} d(ABv,z) &= d(ABABv,STz) \\ &\leq \emptyset(\max\{d(JABv,Iz),d(JABv,STz),d(Iz,STz)\}) \\ &\leq \emptyset(\max\{d(ABv,z),d(ABv,z),d(z,z)\}) \\ &\leq \emptyset(\max\{d(ABv,z),d(ABv,z),0\}) \end{aligned}$$

or, $d(ABv,z) < d(ABv,z)$, which is a contradiction.

Hence, we get $ABv = z$. Therefore, *AB*, *ST*, *I* and *J* have a unique common fixed point.

Now, we claim that *z* is also a common fixed point of mappings *A*, *B*, *S*, *T*, *I* and *J* and let both the pairs (AB, J) and (ST, I) have a unique common fixed point *z*. Then,

$$Az = A(ABz) = A(BAz) = AB(Az)$$
$$Az = A(Jz) = J(Az)$$
$$Bz = B(ABz) = B\big(A(Bz)\big) = BA(Bz) = AB(Bz)$$
$$Bz = B(Jz) = J(Bz)$$

which implies that (AB, J) has a common fixed point which are *Az* and *Bz*.

We get thereby $Az = z = Bz = Jz = ABz$, by virtue of uniqueness of common fixed point of pair (AB, J).

Similarly, on using $ST = TS, SI = IS$ and $TI = IT$, $Sz = z = Tz = Iz = STz$ can be shown.

Now, we require to show that $Az = Sz(Bz = Tz)$, a common fixed point of both the pairs (AB, J) and (ST, I).
We have,

$$\begin{aligned} d(Az,Sz) &= d(A(ABz),S(TSz)) \\ &= d(AB(Az),ST(Sz)) \\ &\leq \emptyset(\max\{d(J(Az),I(Sz)),d(J(Az),ST(Sz)), \\ &\quad d(I(Sz),ST(Sz))\}) \\ &\leq \emptyset(\max\{d(Az,Sz),d(Az,Sz),d(Sz,Sz)\}) \\ &\leq \emptyset(\max\{d(Az,Sz),d(Az,Sz),0\}) \\ &< d(Az,Sz), \text{ which is a contradiction.} \end{aligned}$$

Hence, $Az = Sz$. Similarly, $Bz = Tz$ can be shown. Thus, z is a unique common fixed point of *A*, *B*, *S*, *T*, *I* and *J*.

Our first corollary is obtained by putting $AB = A$, $ST = B$, $I = T$ and $J = S$ in main Theorem (3.1).

Corollary 3.1.

Let (X, d) be a symmetric (semi-metric) space that satisfies (W4) and (H.E). Let *A*, *B*, *T* and *S* be self-mappings of *X* such that

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$
(ii) the pair (*B*, *T*) satisfies properties E.A (resp. (*A*, *S*) satisfy property E.A)
(iii) $d(Ax, By) \leq \emptyset(\max\{d(Sx,Ty), d(Sx,By), d(Ty,By)\})$ for all $(x,y) \in X \times X$
(iv) S(X) is d-closed ($\tau(d)$-closed) subset of X (resp. T(X) is a d-closed ($\tau(d)$-closed) subset of *X*.

Then, the pair (*A*, *S*) and (*B*, *T*) have a coincidence point.

Moreover, if the pairs (*B*, *T*) and (*A*, *S*) are compatible mapping of type (E) and one of the mappings *A*, *B*, *T* and *S* is continuous then *A*, *B*, *T* and *S* have a unique common fixed point.
The above Corollary (3.1) shows the result of Rajopadhyaya et al. [23].

Our next corollary is obtained by putting $AB = ST = A$ and $I = J = S$ in Theorem (3.1) which generalizes the result of Aamri et al. [2] in semi-metric space with compatible mapping of type E.

Corollary 3.2. Let (X, d) be a symmetric (semi-metric) space that satisfy (W4) and (H.E) and let A and T be self-mappings of X such that

(i) $A(X) \subset T(X)$
(ii) The pair (A, T) satisfies E.A property.
(iii) $d(Ax, Ay) \leq \emptyset(\max\{d(Tx,Ty), d(Tx,Ay), d(Ty,Ay)\})$ for all $(x,y) \in X \times X$
(iv) *T(X)* is a d-closed ($\tau(d)$-closed) subset of *X*
(v) The pair (*A*, *T*) are compatible mapping of type (E).

If one of the mappings *A* and *T* is continuous then *A* and *T* have a unique common fixed point.

Example 3.1. Let $X = [0,1]$ with semi-metric space (X, d) defined by $d(x,y) = (x-y)^2$. Define a self-maps *A*, *B*, *S*, *T*, *I* and *J* as

$$Ax = \frac{3x}{8}, Bx = \frac{4x}{10}, Sx = \frac{x}{5}, Tx = \frac{5x}{12}, Ix = \frac{9x}{20}, Jx = \frac{x}{3}$$

Then A, B, S, T, I and J satisfy all the conditions of the theorem (31) and have a unique common fixed point at $x = 0$.

Now we establish a common fixed point theorem in symmetric space with occasionally weakly compatible(owc) mapping which improves and extends similar known results in the literature.

Theorem 3.2: Let (X, d) be a symmetric (semi-metric) space that satisfy (W4) and (H.E.). Let *A*, *B*, *S*, *T*, *P* and *Q* be self-mappings of X such that

(i) $\{P, AB\}$ and $\{Q, ST\}$ are occasionally weakly compatible (owc), … (3.5)

(ii) $d(Px, Qy) \le \emptyset(max\left\{d(ABx, STy), \frac{1}{2}[d(Px, ABx) + d(Qy, STy)], \frac{1}{2}[d(Qy, ABx) + d(Px, STy)]\right\})$
for all $(x, y) \in X \times X$, where $\emptyset: R^+ \to R^+$ be a function satisfying $0 < \emptyset(t) < t, t > 0$ … (3.6)

Then *AB*, *ST*, *P* and *Q* have a unique common fixed point. Furthermore, if *AB*= BA and *ST*= TS,then *A*, *B*, *S*, *T*, *P* and *Q* have a unique common fixed point.

Proof: Since $\{P, AB\}$ and $\{Q, ST\}$ are occasionally weakly compatible (owc), then there exists $(x, y) \in X \times X$ such that $Px = ABx = x$,where $PABx = ABPx$ and $STy = Qy = y$, where, $STQy = QSTy$.We claim that $Px = Qy$. Using condition (ii), we get

$$\begin{aligned} d(Px, Qy) &\le \emptyset(max\left\{d(ABx, STy), \frac{1}{2}[d(Px, ABx) + d(Qy, STy)], \frac{1}{2}[d(Qy, ABx) + d(Px, STy)]\right\}) \\ &= \emptyset\left(\max\left\{d(Px, Qy), \frac{1}{2}[d(Px, Px) + d(Qy, Qy)], \frac{1}{2}[d(Qy, Px) + d(Px, Qy)]\right\}\right) \\ &= \emptyset(\max\{d(Px, Qy), 0, d(Px, Qy)\}) \\ &= \emptyset(d(Px, Qy)) \\ &< d(Px, Qy) \end{aligned}$$

which is contradiction. So, $Px = Qy$. Therefore,

$ABx = Px = STy = Qy$. … (3.7)

Moreover, if there is another point of coincidence *z* such that $z = Pz = ABz$. We claim that $Pz = Qy$. Using condition (ii), we get

$$\begin{aligned} d(Pz, Qy) &\le \emptyset(max\left\{d(ABz, STy), \frac{1}{2}[d(Pz, ABz) + d(Qy, STy)], \frac{1}{2}[d(Qy, ABz) + d(Pz, STy)]\right\}) \\ &= \emptyset\left(\max\left\{d(Pz, Qy), \frac{1}{2}[d(Pz, Pz) + d(Qy, Qy)], \frac{1}{2}[d(Qy, Pz) + d(Pz, Qy)]\right\}\right) \\ &= \emptyset(\max\{d(Pz, Qy), 0, d(Pz, Qy)\}) \\ &= \emptyset(d(Pz, Qy)) \\ &< d(Pz, Qy) \end{aligned}$$

which is contradiction. So, $Pz = Qy$. Therefore,

$Pz = ABz = STy = Qy$. … (3.8)

Similarly, if there is another point of coincidence v such that $v = STv = Qv$. It can be easily seen that $Pz = Qv$. Therefore

$Pz = ABz = STv = Qv$.

Also, from (3.7) and (3.8), it follows that $ABz = ABx$. This implies that $z = x$.Hence, $w = ABx = Px$, for $w \in X$, is the unique point of coincidence of *AB* and *P*. By Lemma 2.1, w is the unique common fixed point of *AB* and *P*. Hence $w = ABw = Pw$. Similarly, there is a unique common fixed point $u \in X$ such that $u = STu = Qu$.. Suppose that $w \ne u$.Then using condition (ii), we get.

$$\begin{aligned} d(w, u) &= d(Pw, Qu) \\ &\le \emptyset(max\left\{d(ABw, STu), \frac{1}{2}[d(Pw, ABw) + d(Qu, STu)], \frac{1}{2}[d(Qu, ABw) + d(Pw, STu)]\right\}) \\ &= \emptyset\left(\max\left\{d(w, u), \frac{1}{2}[d(w, w) + d(u, u)], \frac{1}{2}[d(u, w) + d(w, u)]\right\}\right) \\ &= \emptyset(\max\{d(w, u), 0, d(w, u)\}) \\ &= \emptyset(d(w, u)) \\ &< d(w, u) \end{aligned}$$

This is contradiction. Therefore, we have $w = u$.. Hence, w is the unique common fixed point of *AB*, *TS*, *P* and *Q*. Finally, we need to show that w is only the common fixed point of mappings *A, B, T, S, P* and *Q*.

Let both the pairs (P, AB) and (Q, ST) have a unique common fixed point w.
AB= BA ,then for this, we can write

$Aw = A(ABw) = A(BAw) = AB(Aw)$, $Aw = A(Pw) = P(Aw)$

and $Bw = B(ABw) = B(A(Bw)) = BA(Bw) = AB(Bw)$, $Bw = B(Pw) = P(Bw)$

which implies that (P, AB) has common fixed points which are Az and Bz. We get thereby $Aw = w = Bw = Pw = ABw$.

Similarly, using the commutativity of (S, T),

$Sw = w = Tw = Qw = STw$ can be shown.

Hence *A*, *B*, *T*, *S*, *P* and *Q* have a unique common fixed point.

Example 3.2. Consider $X = [0,1]$ with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define self-mappings *A*, *B*, *T*, *S*, *P* and *Q* as $Ax = \frac{x+1}{2}$, $Bx = \frac{2+3x}{5}$, $Tx = \frac{2x+1}{3}$, $S(x) = \frac{x+3}{4}$, $P(x) = \frac{3x+1}{4}$ and $Q(x) = \frac{2x+3}{5}$. Also, the mappings satisfy all the conditions of above Theorem 2.1 and hence have a unique common fixed point $x = 1$.

On the basis of above Theorem 3.2, we have the following corollary.

Corollary 3.3: Let (X, d) be a symmetric (semi-metric) space that satisfy (W4) and (H.E.). Let *A*, *B*, *S*, *T*, *P* and *Q* be self-mappings of X such that

(i) $\{P, AB\}$ and $\{Q, ST\}$ are occasionally weakly compatible (owc),

(ii) $d(Px, Qy) \leq \emptyset(max\left\{d(ABx, STy), d(Qy, ABx), d(Px, STy), \frac{1}{2}[d(Px, ABx) + d(Qy, STy)]\right\})$

for all $(x, y) \in X \times X$, where $\emptyset: R^+ \to R^+$ be a function satisfying $0 < \emptyset(t) < t, t > 0$

Then *AB*, *ST*, *P* and *Q* have a unique common fixed point. Furthermore, if *AB*= BA and *ST*= TS,then *A*, *B*, *S*, *T*, *P* and *Q* have a unique common fixed point.

In the above Theorem 3.2, if we take $AB = A$, $ST = S$, then we have the following corollary. This is the result of G. Jungck and B.E. Rhoades [20].

Corollary 3.4: Let (X, d) be a symmetric (semi-metric) space that satisfy (W4) and (H.E.). Let *A*, *S*, *P* and *Q* be self-mappings of X such that

(i) $\{P, A\}$ and $\{Q, S\}$ are occasionally weakly compatible (owc),

(ii) $d(Px, Qy) \leq \emptyset(max\left\{d(Ax, Sy), \frac{1}{2}[d(Px, Ax) + d(Qy, Sy)], \frac{1}{2}[d(Qy, Ax) + d(Px, Sy)]\right\})$

for all $(x, y) \in X \times X$, where $\emptyset: R^+ \to R^+$ be a function satisfying $0 < \emptyset(t) < t, t > 0$

Then *A*, *S*, *P* and *Q* have a unique common fixed point

In Corollary 3.4, if we take $P = Q$, then we have the following corollary

Corollary 3.5: Let (X, d) be a symmetric (semi-metric) space that satisfy (W4) and (H.E.). Let *A*, *S* and *P* be self-mappings of X such that

(i) $\{P, A\}$ and $\{P, S\}$ are occasionally weakly compatible (owc),

(ii) $d(Px, Py) \leq \emptyset(max\left\{d(Ax, Sy), \frac{1}{2}[d(Px, Ax) + d(Py, Sy)], \frac{1}{2}[d(Py, Ax) + d(Px, Sy)]\right\})$

for all $(x, y) \in X \times X$, where $\emptyset: R^+ \to R^+$ be a function satisfying $0 < \emptyset(t) < t, t > 0$

Then *A*, *S* and *P* have a unique common fixed point.

In Corollary 3.4, if we take $P = Q$, $A = S$ then we have the following corollary.

Corollary 3.6: Let (X, d) be a symmetric (semi-metric) space that satisfy (W4) and (H.E.). Let *A* and *P* be self-mappings of X such that

(i) *P* and A are occasionally weakly compatible (owc),

(ii) $d(Px, Py) \leq \emptyset(max\left\{d(Ax, Ay), \frac{1}{2}[d(Px, Ax) + d(Py, Ay)], \frac{1}{2}[d(Py, Ax) + d(Px, Ay)]\right\})$

for all $(x, y) \in X \times X$, where $\emptyset: R^+ \to R^+$ be a function satisfying $0 < \emptyset(t) < t, t > 0$

Then *A* and P have a unique common fixed point.

4 CONCLUSION

Menger [21] introduced the notion of symmetric space as a generalization of metric space. In metric space, if the triangle inequality property is eliminated then the metric space reduces into symmetric space. However, triangle inequality property is very important for convergence criteria to obtain a fixed point. In symmetric space, without using triangle inequality property, the establishment of fixed-point results is a challenging task. So, we will use associated useful properties in symmetric space to establish fixed point theorems as partial replacement of triangle inequality. The properties W3, W4 and W5 were introduced by Wilson [27], H.E by Aamri and Moutawakil [2] and C.C by Cho et al. [5]. In this paper, we prove a common fixed-point theorem for three pairs of self-mappings by using compatible mapping of type (E) in semi-metric space that extends the results of Rajopadhyaya et al. [23,25], Aamri and Moutawakil [2] and other similar results in semi-metric space. We also prove a common fixed-point theorem for three pairs of self-mappings using occasionally weakly compatible mappings which improves and extends similar known results in the literature.

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