Research Article

Some Fixed Point Theorems For Mappings Satisfying Contractive Conditions Of Integral Type

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Abstract: In this paper, we establish some common fixed-point theorems for two pairs of weakly compatible mappings satisfying integral type contractive conditions integral type in dislocated metric spaces by using E.A. which improves and extends similar known results in the literature.

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1.Introduction

In 1922, S. Banach proved a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by different authors, and many generalizations of this theorem have been established. Jungck [7] generalized the Banach contraction principle by introducing a contractive condition for a pair of commuting self-mappings on metric space and pointed out the potential of commuting mappings for generalizing fixed point theorems in metric spaces. Jungck's [7] results have been further generalized.

Sessa [24], initiated the tradition of improving commutativity conditions in metrical common fixed point theorems. While doing so Sessa [24] introduced the notion of weak commutativity. Motivated by Sessa [24], Jungck [8] defined the concept of compatibility of two mappings, which includes weakly commuting mappings as a proper subclass. Jungck and Rhoades [10] introduced the notion of weakly compatible (coincidentally commuting) mappings and showed that compatible mappings are weakly compatible but not conversely. Many interesting fixed point theorems for weakly compatible maps satisfying contractive type conditions have been obtained by various authors. The concept of compatible mappings was frequently used to show the existence of common fixed points. However, the study of the existence of common fixed points for noncompatible mappings is also very interesting. Aamri and Moutawakil [4] gave a notion (E.A) which generalizes the concept of noncompatible mappings in metric spaces.

Branciari [2] introduced the notion of contraction of integral type and proved first fixed point theorem for this class of mapping. Further results on this class of mappings were obtained by Rhoades [22], Aliouche [3], Djoudi and Merghadi [6] and many others.

. Matthews [11] introduced some concepts of metric domains in the context of domain theory The notion of a dislocated metric (*d-metric*) space was introduced by Pascal Hitzler in [12] as a part of the study of logic programming semantics. The study of common fixed point mappings in dislocated metric space satisfying certain contractive conditions has been at the center of vigorous research activity, see for example in [13-21].

In this article, we have established some common fixed point results of integral type contractive conditions using the concept of weakly compatible mappings with (E. A.) property in dislocated metric (*d-metric*) space. Our obtained results generalizes some well known results of the literature.

2. Preliminary Notes

We begin by recalling some basic concepts of the theory of dislocated metric (*d-metric*) spaces.

Definition 2.1 Let X be a non empty set and let $d: X \times X \to [0, \infty)$ be a function satisfying the following conditions:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = d(y, x) = 0$ implies $x = y$

(iii) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$

Then d is called dislocated metric (or simply d-metric) on X.

Definition 2.2 A sequence $\{x_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m. n \ge n_0$, we have $d(x_m, x_n) < \epsilon$.

Definition 2.3 A sequence in d-metric space converges if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

Definition 2.4 A d-metric space (X, d) is called complete if every Cauchy sequence is convergent.

Definition 2.5 Let (X, d) be a d-metric space. A map $T: X \to X$ is called contraction if there exists a number λ with $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)$.

Definition 2.6 Let A and S be two self mappings on a set X. Mappings A and S are said to be commuting if $ASx = SAx \quad \forall x \in X$.

Definition 2.7. Let S and T be mappings of a metric space (X, d) into itself. Then (S, T) is said to be **weakly commuting** pair if

$$d(STx, TSx) \leq d(Tx, Sx) \text{ for all } x \in X.$$

Obviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

Example 2.1. Consider the set X = [0, 1] with the usual metric. Let $Sx = \frac{x}{2}$ and $Tx = \frac{x}{2+x}$ for every $x \in X$. Then for all $x \in X$

$$STx = \frac{x}{4+2x}, TSx = \frac{x}{4+x} .$$

Hence $ST \neq TS$. Thus S and T do not commute.

Again

$$d(STx, TSx) = \left|\frac{x}{4+2x} - \frac{x}{4+x}\right| = \frac{x^2}{(4+x)(4+2x)}$$

$$\leq \frac{x^2}{(4+2x)} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx),$$

and so, S and T commute weakly.

Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.

Definition 2.8. Two self mappings S and T from a *d*-metric space (X, d) into itself are called compatible if and only if

$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X.$$

Very recently concept of **weakly compatible** obtained by Jungck-Rhoades [10] stated as the pair of mappings is said to be weakly compatible if they commute at their coincidence point.

Example 2.2 .

Let *X* = [0, 3] be equipped with the usual *d*- metric space $d(x, y) = |x - y|$.

Define *S*, *T* : [0, 3] → [0, 3] by

$$Sx = \begin{cases} x, & x \in [0,1) \\ 3, & x \in [1, 3] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 3 - x, & x \in [0,1) \\ 3, & x \in [1, 3] \end{cases}$$

Then for any $x \in [1, 3]$, $STx = TSx$, showing that *S* and *T* are weakly compatible maps on [0, 3].

Definition 2.9: Let *S* and *T* be two self mappings of a *d*-metric space (X, d). We say that *S* and *T* satisfy the property *(E.A)* if there exist a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$,for some u$\in X$.

Proposition 2.1 Let *S* and *T* be compatible mappings from a *d*-metric space (X, d) into itself. Suppose that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x \text{ for some } x \in X.$$

if S is continuous then $\lim_{n\to\infty} TSx_n = Sx$.

Theorem 2.1 Let (X, d) be a complete d-metric space and let $T: X \to X$ be a contraction mapping, then T has a unique fixed point.

3. Main Results

Now, we establish a common fixed point theorem for two pairs of weakly compatiblemappings using E. A. property.

Theorem 3.1 Let (X,d) be a complete dislocated metric space. Let $A,B,S,T,I,J:X\to X$ satisfying the following conditions

(i) $AB(X)\subseteq J(X)$ and $ST(X)\subseteq I(X)$ …(1)

(ii) $\int_0^{d(ABx,STy)}\emptyset(t)dt\le k\int_0^{M(x,y)}\emptyset(t)dt,\ k\in\left[0,\frac{1}{8}\right)$ … (2)

for all $(x,y)\in X\times X$ where $\emptyset:R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$\int_0^{\epsilon}\emptyset(t)dt>0$ for all $\epsilon>0$. … (3)

$M(x,y)=d(Ix,Jy)+d(Ix,ABx)+d(Jy,STy)+d(Ix,STy)+d(Jy,ABx)$ …(4)

(iii) The pairs (AB,I) or (ST,J) satisfy E.A. property.

(iv) The pairs (AB,I) and (ST,J) are weakly compatible.

If $I(X)$ is closed then the mappings AB,ST,I and J have a unique common fixed point in *X*. Furthermore, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point in X.

Proof:

Assume that the pair (AB,I) satisfy E. A. property, so there exists a sequence $\{x_n\}\in X$ Such that

$\lim_{n\to\infty}ABx_n=\lim_{n\to\infty}Ix_n=u$ …(5)

For some $u\in X$. Since $AB(X)\subseteq J(X)$, so there exists a sequence $\{y_n\}\in X$ such that $ABx_n=Jy_n$. Hence,

$\lim_{n\to\infty}ABx_n=\lim_{n\to\infty}Jy_n=u$ …(6)

From condition (2), we have

$$\int_0^{d(ABx_n,STy_n)}\emptyset(t)dt\le k\int_0^{M(x_n,y_n)}\emptyset(t)dt, \quad \text{… (7)}$$

where

$$M(x_n,y_n)=d(Ix_n,Jy_n)+d(Ix_n,ABx_n)+d(Jy_n,STy_n)+d(Ix_n,STy_n)+d(Jy_n,ABx_n)$$

Taking limit as $n\to\infty$, we get

$$\lim_{n\to\infty}\int_0^{d(ABx_n,STy_n)}\emptyset(t)dt\le k\lim_{n\to\infty}\int_0^{M(x_n,y_n)}\emptyset(t)dt \quad \text{… (8)}$$

Since

$$\operatorname{Lim}_{n\to\infty}d(Ix_n,Jy_n)=\lim_{n\to\infty}d(Ix_n,ABx_n)=\lim_{n\to\infty}d(Jy_n,ABx_n)=0$$

$$\lim_{n\to\infty}d(STy_n,Jy_n)=\lim_{n\to\infty}d(Ix_n,STy_n)=\lim_{n\to\infty}d(STy_n,u)$$

Therefore, we have

$$\lim_{n\to\infty}\int_0^{d(u,STy_n)}\emptyset(t)dt\le 2k\lim_{n\to\infty}\int_0^{d(u,STy_n)}\emptyset(t)dt,$$

which is a contradiction, since $k\in\left[0,\frac{1}{8}\right)$. Hence $\lim_{n\to\infty}STy_n=u$. Since $ST(X)\subseteq I(X)$, so there exists a sequence $x_n\in X$ such that $STx_n=Ix_n$. Hence, we have

$$\lim_{n\to\infty}ABx_n=\lim_{n\to\infty}Jy_n=\lim_{n\to\infty}STy_n=\lim_{n\to\infty}Ix_n=u$$

Assume $I(X)$ is closed, then there exists $v\in X$ such that $Iv=u$. We claim that $ABv=u$. Now from condition (2),

$$\int_0^{d(ABv,STy_n)}\emptyset(t)dt\le k\int_0^{M(v,y_n)}\emptyset(t)dt,$$

where

$$M(v,y_n) = d(Jy_n, ABv) + d(Iv, Jy_n) + d(Iv, ABv)$$
$$+d(STy_n, Jy_n) + d(Iv, STy_n) \quad \text{...(9)}$$

Since $\lim_{n\to\infty} d(Jy_n, ABv) = d(u, ABv)$

$$\lim_{n\to\infty} d(Iv, Jy_n) = \lim_{n\to\infty} d(STy_n, Jy_n) = \lim_{n\to\infty} d(Iv, STy_n) = 0$$

So, taking limit as $n \to \infty$ in (9), we conclude that

$$\int_0^{d(ABv,u)} \emptyset(t)\,dt \le 2k \int_0^{d(u,ABv)} \emptyset(t)\,dt \quad \text{...(10)}$$

which is a contradiction, since $k \in \left[0, \frac{1}{8}\right)$. Hence, $d(ABv, u) = 0 \Rightarrow ABv = u$.

Now, we have

$$ABv = u = Iv \quad \text{...(11)}$$

This proves that v is the coincidence point of (AB, I).

Again $AB(X) \subseteq J(X)$ so there exists $w \in X$ such that

$$ABv = Jw = u$$

Now, we claim that $STw = u$. From condition (2), we have

$$\int_0^{d(u,STw)} \emptyset(t)\,dt = \int_0^{d(ABv,STw)} \emptyset(t)\,dt \le k \int_0^{M(v,w)} \emptyset(t)\,dt, \text{ where}$$

$$M(v,w) = d(Iv, Jw) + d(Iv, ABv) + d(Jw, STw), d(Iv, STw) + d(Jw, ABv)$$
$$= d(u,u) + d(u,u) + d(u, STw) + d(u, STw) + d(u,u)$$
$$= 3d(u,u) + 2d(u, STw)$$
$$\le 3\{d(u, STw) + d(STw, u)\} + 2d(u, STw)$$
$$\le 8d(u, STw)$$

Hence, $\int_0^{d(u,STw)} \emptyset(t)\,dt \le 8k \int_0^{d(u,STw)} \emptyset(t)\,dt$,

which is a contradiction, since $k \in \left[0, \frac{1}{8}\right)$.

Hence, $d(u, STw) = 0 \Rightarrow STw = u$.

Therefore, $STw = u = Jw$.

This represents that w is the coincidence point of the maps ST and J.

Hence,

$$u = STw = Jw = Iv = ABv$$

Since the pair (ST, J) and (AB, I) are weakly compatible so,

$$STJw = JSTw, IABv = ABIv$$

Since $Iu = IABv = ABIv = ABu$ and $Ju = JSTw = STJw = STu$, we claim that $STu = u$. From condition (2), we have

$$\int_0^{d(u,STu)} \emptyset(t)\,dt = \int_0^{d(ABv,STu)} \emptyset(t)\,dt \le k \int_0^{M(v,u)} \emptyset(t)\,dt, \text{ where}$$

$$M(v,u) = d(Iv, Ju) + d(Iv, ABv) + d(Ju, STu) + d(Iv, STu) + d(Ju, ABv)$$
$$= d(u, STu) + d(u,u) + d(STu, STu) + d(u, STu) + d(STu, u)$$
$$= 3d(u, STu) + d(u,u) + d(STu, STu)$$
$$\le 7d(u, STu)$$

Hence, $\int_0^{d(u,STu)} \emptyset(t)\,dt \le 7k \int_0^{d(u,STu)} \emptyset(t)\,dt$,

which is a contradiction, since $k \in \left[0, \frac{1}{8}\right)$.

Hence, $d(u, STu) = 0 \Rightarrow STu = u$.

Therefore, $u = STu = Ju$. Similarly, $ABu = u = Iu$.

Hence, $u = ABu = STu = Iu = Ju$.

This represents that u is a common fixed point of the mappings AB, ST, I and J.

Uniqueness:

If possible, let $z (\ne u)$ be other common fixed point of the mappings, then by the condition (2)

$$\int_0^{d(u,z)} \emptyset(t)\,dt = \int_0^{d(ABu,STz)} \emptyset(t)\,dt \le k \int_0^{M(u,z)} \emptyset(t)\,dt, \text{ where}$$

$$M(u,z) = d(Iu,Jz) + d(Iu,ABu) + d(Jz,STz) + d(Iu,STz) + d(Jz,ABu)$$
$$= d(u,z) + d(u,u) + d(z,z) + d(u,z) + d(z,u)$$
$$= 3d(u,z) + d(u,u) + d(z,z)$$
$$\leq 7d(u,z)$$

Hence, $\int_0^{d(u,z)} \emptyset(t)\,dt \leq 7k \int_0^{d(u,z)} \emptyset(t)\,dt$,

which is a contradiction, since $k \in \left[0, \frac{1}{8}\right)$.

Hence, $d(u,z) = 0 \Rightarrow u = z$. This establishes the uniqueness of the common fixed point of mappings AB, ST, I and J. Finally, we prove that u is also a common fixed point of A, B, S, T, I and J.

Let both the pairs (AB, I) and (ST, J) have a unique common fixed point u.

Then,

$$Au = A(ABu) = A(BAu) = AB(Au)$$
$$Az = A(Iu) = I(Au)$$
$$Bu = B(ABu) = B\big(A(Bu)\big) = BA(Bu) = AB(Bu)$$
$$Bu = B(Iu) = I(Bu)$$

which implies that (AB, I) has common fixed points which are Au and Bu. We get thereby $Au = u = Bu = Iu = ABu$.

Similarly, using the commutativity of $(S,T), (S,J)$ and (T,J), $Su = u = Tu = Ju = STu$ can be shown.

Now, we need to show that $Au = Su$ $(Bu = Tu)$.

By using condition (2), we have

$$\int_0^{d(Au,Su)} \varphi(t)\,dt = \int_0^{d(A(ABu),S(STu))} \varphi(t)\,dt = \int_0^{d(A(BAu),S(TSu))} \varphi(t)\,dt$$
$$= \int_0^{d(AB(Au),ST(Su))} \varphi(t)\,dt$$
$$\leq \int_0^{M(Au,Su)} \varphi(t)\,dt$$

where ,

$$M(Au,Su) = d(I(Au),J(Su)) + d(I(Au),AB(Au)) + d(J(Su),ST(Su)) + d(I(Au),ST(Su)) + d(J(Su),AB(Au))$$
$$= d(Au,Su) + d(Au,Au) + d(Su,Su) + d(Au,Su) + d(Su,Au)$$
$$\leq 3d(Au,Su) + d(Au,Au) + d(Su,Su)$$
$$\leq 7d(Au,Su)$$

Therefore, $\int_0^{d(Au,Su)} \varphi(t)\,dt \leq 7k \int_0^{d(Au,Su)} \varphi(t)\,dt$,

which is a contradiction, since $k \in \left[0, \frac{1}{8}\right)$.

Hence,

$$\int_0^{d(Au,Su)} \varphi(t)\,dt = 0$$
$$\Rightarrow d(Au,Su) = 0$$
$$\Rightarrow Au = Su.$$

Similarly, $Bu = Tu$ can be shown.

Consequently, z is a unique common fixed point of A, B, S, T, I and J.

If we put AB = A, ST = B in Theorem (3.1), we get the following, which generalize the result of Panthi and Subedi [20] in dislocated metric spaces.

Corollary 3.1. Let (X,d) be a complete dislocated metric space. Let $A, B, I, J: X \to X$ satisfying the following conditions

(i) $A(X) \subseteq J(X)$ and $B(X) \subseteq I(X)$

(ii) $\int_0^{d(Ax,By)} \emptyset(t)dt \leq k \int_0^{M(x,y)} \emptyset(t)dt,\ k \in \left[0, \frac{1}{8}\right)$

for all $(x,y) \in X \times X$ where $\emptyset: R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$\int_0^{\epsilon} \emptyset(t)\,dt > 0$ for all $\epsilon > 0$.

$M(x,y) = d(Ix,Jy) + d(Ix,Ax) + d(Jy,By) + d(Ix,By) + d(Jy,Ax)$

(iii) The pairs (A,I) or (B,J) satisfy E.A. property.

(iv) The pairs (A,I) and (B,J) are weakly compatible.

If $I(X)$ is closed then the mappings A,B,I and J have a unique common fixed point in X.
If we put A=B in the above Corollary 3.1, we can obtain the following corollary easily.
Corollary 3.2. Let (X,d) be a complete dislocated metric space. Let $A,I,J:X\to X$ satisfying the following conditions

(i) $A(X)\subseteq J(X)$ and $A(X)\subseteq I(X)$

(ii) $\int_0^{d(Ax,Ay)}\emptyset(t)dt\le k\int_0^{M(x,y)}\emptyset(t)dt,\ k\in\left[0,\frac{1}{8}\right)$

for all $(x,y)\in X\times X$ where $\emptyset:R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$\int_0^{\epsilon}\emptyset(t)dt>0$ for all $\epsilon>0$.
$M(x,y)=d(Ix,Jy)+d(Ix,Ax)+d(Jy,Ay)+d(Ix,Ay)+d(Jy,Ax)$

(iii) The pairs (A,I) or (A,J) satisfy E.A. property.

(iv) The pairs (A,I) and (A,J) are weakly compatible.

If $I(X)$ is closed then the mappings A,I and J have a unique common fixed point in X.
If we put I=J in the above Corollary 3.1,we can obtain the following corollaries easily.
Corollary 3.3. Let (X,d) be a complete dislocated metric space. Let $A,B,I:X\to X$ satisfying the following conditions

(i) $A(X)\subseteq I(X)$ and $B(X)\subseteq I(X)$

(ii) $\int_0^{d(Ax,By)}\emptyset(t)dt\le k\int_0^{M(x,y)}\emptyset(t)dt,\ k\in\left[0,\frac{1}{8}\right)$

for all $(x,y)\in X\times X$ where $\emptyset:R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that
$\int_0^{\epsilon}\emptyset(t)dt>0$ for all $\epsilon>0$.
$M(x,y)=d(Ix,Iy)+d(Ix,Ax)+d(Iy,By)+d(Ix,By)+d(Iy,Ax)$ …

(iii) The pairs (A,I) or (B,I) satisfy E.A. property.

(iv) The pairs (A,I) and (B,I) are weakly compatible.

If $I(X)$ is closed then the mappings A,B and I have a unique common fixed point in X.
If we put A=B and I=J in the above Corollary 3.1, we can obtain the following corollaries easily.
Corollary 3.4. Let (X,d) be a complete dislocated metric space. Let $A,I:X\to X$ satisfying the following conditions

(i) $A(X)\subseteq I(X)$

(ii) $\int_0^{d(Ax,Ay)}\emptyset(t)dt\le k\int_0^{M(x,y)}\emptyset(t)dt,\ k\in\left[0,\frac{1}{8}\right)$

for all $(x,y)\in X\times X$ where $\emptyset:R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that
$\int_0^{\epsilon}\emptyset(t)dt>0$ for all $\epsilon>0$.
$M(x,y)=d(Ix,Iy)+d(Ix,Ax)+d(Iy,Ay)+d(Ix,Ay)+d(Iy,Ax)$

(iii) The pairs (A,I) satisfy E.A. property.

(iv) The pairs (A,I) are weakly compatible.

If $I(X)$ is closed then the mappings A and I have a unique common fixed point in X.

Now, we establish the following theorem for six mappings involving Ciric's[5] type contractive condition in complete dislocated metric spaces.

Theorem 3.2 Let (X,d) be a complete dislocated metric space. Let $A,B,S,T,I,J:X\to X$ satisfying the following conditions

(i) $AB(X)\subseteq J(X)$ and $ST(X)\subseteq I(X)$ …(12)

(ii) $\int_0^{d(ABx,STy)} \emptyset(t)\,dt \le k\int_0^{M(x,y)} \emptyset(t)\,dt,\ k \in \left[0,\frac{1}{2}\right)$ … (13)

for all $(x,y) \in X \times X$ where $\emptyset: R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$\int_0^{\epsilon} \emptyset(t)\,dt > 0$ for all $\epsilon > 0$. … (14)

$M(x,y) = max\{d(Ix,Jy), d(Ix,ABx), d(Jy,STy), d(Ix,STy), d(Jy,ABx)\}$, …(15)

(iii) The pairs (AB,I) or (ST,J) satisfy E.A. property.

(iv) The pairs (AB,I) and (ST,J) are weakly compatible.

If $I(X)$ is closed then the mappings AB,ST,I and J have a unique common fixed point in *X*. Furthermore, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point in X.

Proof:

Assume that the pair (AB,I) satisfy E. A. property, so there exists a sequence $\{x_n\} \in X$ Such that

$$\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} Ix_n = u \quad \ldots(16)$$

For some $u \in X$. Since $AB(X) \subseteq J(X)$, so there exists a sequence $\{y_n\} \in X$ such that $ABx_n = Jy_n$. Hence,

$$\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} Jy_n = u \quad \ldots(17)$$

From condition (13), we have

$$\int_0^{d(ABx_n,STy_n)} \emptyset(t)\,dt \le k\int_0^{M(x_n,y_n)} \emptyset(t)\,dt \quad \ldots (18)$$

where,

$M(x_n,y_n) = max\{d(Ix_n,Jy_n), d(Ix_n,ABx_n), d(Jy_n,STy_n), d(Ix_n,STy_n), d(Jy_n,ABx_n)\}$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_0^{d(ABx_n,STy_n)} \emptyset(t)\,dt \le k \lim_{n\to\infty} \int_0^{M(x_n,y_n)} \emptyset(t)\,dt \quad \ldots (19)$$

Since

$$\mathrm{Lim}_{n\to\infty}\, d(Ix_n,Jy_n) = \lim_{n\to\infty} d(Ix_n,ABx_n) = \lim_{n\to\infty} d(Jy_n,ABx_n) = 0$$

$$\lim_{n\to\infty} d(STy_n,Jy_n) = \lim_{n\to\infty} d(Ix_n,STy_n) = \lim_{n\to\infty} d(STy_n,u)$$

Therefore, we have

$$\lim_{n\to\infty} \int_0^{d(u,STy_n)} \emptyset(t)\,dt \le k \lim_{n\to\infty} \int_0^{d(u,STy_n)} \emptyset(t)\,dt, \quad \ldots(20)$$

which is a contradiction, since $k \in \left[0,\frac{1}{2}\right)$. Hence $\lim_{n\to\infty} STy_n = u$. Since $ST(X) \subseteq I(X)$, so there exists a sequence $x_n \in X$ such that $STx_n = Ix_n$. Hence, we have

$$\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} Jy_n = \lim_{n\to\infty} STy_n = \lim_{n\to\infty} Ix_n = u$$

Assume $I(X)$ is closed, then there exists $v \in X$ such that $Iv = u$. We claim that $ABv = u$. Now from condition (13)

$$\int_0^{d(ABv,STy_n)} \emptyset(t)\,dt \le k\int_0^{M(v,y_n)} \emptyset(t)\,dt \quad \ldots (21)$$

where

$$M(v,y_n) = max\{d(Jy_n,ABv), d(Iv,Jy_n), d(Iv,ABv), d(STy_n,Jy_n), d(Iv,STy_n)\} \quad \ldots(22)$$

Since $\lim_{n\to\infty} d(Jy_n,ABv) = d(u,ABv)$

$$\lim_{n\to\infty} d(Iv,Jy_n) = \lim_{n\to\infty} d(STy_n,Jy_n) = \lim_{n\to\infty} d(Iv,STy_n) = 0$$

So, taking limit as $n \to \infty$ in (22), we conclude that

$$\int_0^{d(ABv,u)} \emptyset(t)\,dt \le k\int_0^{d(u,ABv)} \emptyset(t)\,dt \quad \ldots(23)$$

which is a contradiction, since $k \in \left[0, \frac{1}{2}\right)$. Hence, $d(ABv, u) = 0 \Rightarrow ABv = u$.

Now, we have

$$ABv = u = Iv \qquad \dots(24)$$

This proves that v is the coincidence point of (AB, I).

Again $AB(X) \subseteq J(X)$ so there exists $w \in X$ such that

$$ABv = Jw = u$$

Now, we claim that $STw = u$. From condition (13), we have

$$\int_0^{d(u,STw)} \emptyset(t)\,dt = \int_0^{d(ABv,STw)} \emptyset(t)\,dt \le k\int_0^{M(v,w)} \emptyset(t)\,dt$$

where

$$\begin{aligned} M(v,w) &= \max\{d(Iv,Jw), d(Iv,ABv), d(Jw,STw), d(Iv,STw), d(Jw,ABv)\} \\ &= \max\{d(u,u), d(u,u), d(u,STw), d(u,STw), d(u,u)\} \\ &= max\{d(u,u), d(u,STw)\} \end{aligned}$$

Hence $\int_0^{d(u,STw)} \emptyset(t)\,dt \le k\int_0^{max\{d(u,u),d(u,STw)\}} \emptyset(t)\,dt$,

Since $d(u,u) \le 2d(u,STw)$

So if $max\{d(u,u), d(Bw,u)\} = d(u,u)$ or $d(STw,u)$ we get the contradiction, since

$$\int_0^{d(u,STw)} \emptyset(t)\,dt \le 2k\int_0^{d(u,STw)} \emptyset(t)\,dt$$

or $\int_0^{d(u,STw)} \emptyset(t)\,dt \le k\int_0^{d(u,STw)} \emptyset(t)\,dt$ and $k \in \left[0, \frac{1}{2}\right)$.

We have, $d(u,STw) = 0 \Rightarrow STw = u$.

Therefore, $STw = u = Jw$.

This represents that w is the coincidence point of the maps ST and J.

Hence,

$$u = STw = Jw = Iv = ABv$$

Since the pair (ST, J) and (AB, I) are weakly compatible so,

$$STJw = JSTw, IABv = ABIv$$

Since $Iu = IABv = ABIv = ABu$ and $Ju = JSTw = STJw = STu$, we claim that $STu = u$. From condition (13), we have

$\int_0^{d(u,STu)} \emptyset(t)\,dt = \int_0^{d(ABv,STu)} \emptyset(t)\,dt \le k\int_0^{M(v,u)} \emptyset(t)\,dt$, where

$$\begin{aligned} M(v,u) &= max\{d(Iv,Ju), d(Iv,ABv), d(Ju,STu), d(Iv,STu), d(Ju,ABv)\} \\ &= \max\{d(u,STu), d(u,u), d(STu,STu), d(u,STu), d(STu,u)\} \\ &= max\{d(u,STu), d(u,u), d(STu,STu)\} \end{aligned}$$

Hence $\int_0^{d(u,STu)} \emptyset(t)\,dt \le k\int_0^{max\{d(u,STu),d(u,u),d(STu,STu)\}} \emptyset(t)\,dt$

Since

$$d(u,u) \le 2d(u,STu) \text{ and } d(STu,STu) \le 2d(u,STu)$$

So if $max\{d(u,STu), d(u,u), d(STu,STu)\} = d(u,STu)$, $d(STu,STu)$ or $d(u,u)$, we get the contradiction, since

$$\int_0^{d(u,STu)} \emptyset(t)\,dt \le 2k\int_0^{d(u,STu)} \emptyset(t)\,dt$$

or $\int_0^{d(u,STu)} \emptyset(t)\,dt \le k\int_0^{d(u,STu)} \emptyset(t)\,dt$ and $k \in \left[0, \frac{1}{2}\right)$.

We have, $d(u,STu) = 0 \Rightarrow STu = u$.

Therefore, $u = STu = Ju$. Similarly, $ABu = u = Iu$.

Hence, $u = ABu = STu = Iu = Ju$.

This represents that u is a commpn fixed point of the mappings AB, ST, I and J.

Uniqueness:

If possible, let $z(\neq u)$ be other common fixed point of the mappings, then by the condition (13)

$\int_0^{d(u,z)} \emptyset(t)\,dt = \int_0^{d(Au,Bz)} \emptyset(t)\,dt \le k\int_0^{M(u,z)} \emptyset(t)\,dt$, where

$$\begin{aligned} M(u,z) &= max\{d(Iu,Jz), d(Iu,ABu), d(Jz,STz), d(Iu,STz), d(Jz,ABu)\} \\ &= \max\{d(u,z), d(u,u), d(z,z), d(u,z), d(z,u)\} \\ &= max\{d(u,z), d(u,u), d(z,z)\} \end{aligned}$$

Hence $\int_0^{d(u,z)} \emptyset(t)\,dt \le k\int_0^{max\{d(u,z),d(u,u),d(z,z)\}} \emptyset(t)\,dt$

Since

$d(u,u) \le 2d(u,z)$ and $d(z,z) \le 2d(z,u)$

So if $max\{d(u,z), d(u,u), d(z,z)\} = d(u,z)$ or $d(u,u)$ or $d(z,z)$ we get the contradiction, since

$\int_0^{d(u,z)} \emptyset(t)\,dt \le 2k \int_0^{d(u,z)} \emptyset(t)\,dt$

or $\int_0^{d(u,z)} \emptyset(t)\,dt \le k \int_0^{d(u,z)} \emptyset(t)\,dt$ and $k \in \left[0, \frac{1}{2}\right)$.

We have, $d(u,z) = 0 \Rightarrow u = z$.

This establishes the uniqueness of the common fixed point of mappings AB, ST, I and J. Finally, we prove that u is also a common fixed point of A, B, S, T, I and J.

Let both the pairs (AB, I) and (ST, J) have a unique common fixed point u.

Then,

$$Au = A(ABu) = A(BAu) = AB(Au)$$
$$Az = A(Iu) = I(Au)$$
$$Bu = B(ABu) = B\big(A(Bu)\big) = BA(Bu) = AB(Bu)$$
$$Bu = B(Iu) = I(Bu)$$

which implies that (AB, I) has common fixed points which are Au and Bu. We get thereby $Au = u = Bu = Iu = ABu$.

Similarly, using the commutativity of $(S,T), (S,J)$ and (T,J), $Su = u = Tu = Ju = STu$ can be shown.

Now, we need to show that $Au = Su$ $(Bu = Tu)$.

By using condition (13), we have

$$\int_0^{d(Au,Su)} \varphi(t)\,dt = \int_0^{d(A(ABu),S(STu))} \varphi(t)\,dt = \int_0^{d(A(BAu),S(TSu))} \varphi(t)\,dt$$
$$= \int_0^{d(AB(Au),ST(Su))} \varphi(t)\,dt$$
$$\le \int_0^{M(Au,Su)} \varphi(t)\,dt$$

where , $M(Au,Su) = \max\{d\big(I(Au), J(Su)\big), d\big(I(Au), AB(Au)\big),$
$d\big(J(Su), ST(Su)\big), d\big(I(Au), ST(Su)\big), d\big(J(Su), AB(Au)\big)\}$
$= \max\{d(Au,Su), d(Au,Au), d(Su,Su), d(Au,Su), d(Su,Au)\}$
$= \max\{d(Au,Su), d(Au,Au), d(Su,Su)\}$

Hence, $\int_0^{d(Au,Su)} \varphi(t)\,dt \le k \int_0^{\max\{d(Au,Su), d(Au,Au), d(Su,Su)\}} \varphi(t)\,dt$,

Since

$d(Au,Au) \le 2d(Au,Su)$ and $d(Su,Su) \le 2d(Au,Su)$

So if $\max\{d(Au,Su), d(Au,Au), d(Su,Su)\} = d(Au,Su)$ or $d(Au,Au)$ or $d(Su,Su)$ we get the contradiction, since

$\int_0^{d(Au,Su)} \emptyset(t)\,dt \le 2k \int_0^{d(Au,Su)} \emptyset(t)\,dt$

or $\int_0^{d(Au,Su)} \emptyset(t)\,dt \le k \int_0^{d(Au,Su)} \emptyset(t)\,dt$ and $k \in \left[0, \frac{1}{2}\right)$.

We have, $d(Au,Su) = 0 \Rightarrow Au = Su$.

Similarly, $Bu = Tu$ can be shown.

Consequently, u is a unique common fixed point of A, B, S, T, I and J.

If we put AB = A, ST = B in Theorem (3.2), we get the following, which generalize the result of Panthi and Kumari [20] in dislocated metric spaces.

Corollary 3.5. Let (X,d) be a complete dislocated metric space. Let $A, B, I, J: X \to X$ satisfying the following conditions

(i) $A(X) \subseteq J(X)$ and $B(X) \subseteq I(X)$

(ii) $\int_0^{d(Ax,By)} \emptyset(t)\,dt \le k \int_0^{M(x,y)} \emptyset(t)\,dt$, $k \in \left[0, \frac{1}{2}\right)$

for all $(x,y) \in X \times X$ where $\emptyset: R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$\int_0^{\epsilon} \emptyset(t)\,dt > 0$ for all $\epsilon > 0$.

$M(x,y) = max\{d(Ix,Jy), d(Ix,Ax), d(Jy,By), d(Ix,By), d(Jy,Ax)\}$

(iii) The pairs (A,I) or (B,J) satisfy E.A. property.

(iv) The pairs (A,I) and (B,J) are weakly compatible.

If $I(X)$ is closed then the mappings A,B,I and J have a unique common fixed point in *X*.

If we put A=B in the above Corollary 3.5, we can obtain the following corollary easily.

Corollary 3.6. Let (X,d) be a complete dislocated metric space. Let $A,I,J:X\to X$ satisfying the following conditions

(i) $A(X)\subseteq J(X)$ and $A(X)\subseteq I(X)$

(ii) $\int_0^{d(Ax,Ay)}\emptyset(t)dt \le k\int_0^{M(x,y)}\emptyset(t)\,dt,\ k\in\left[0,\frac{1}{2}\right)$

for all $(x,y)\in X\times X$ where $\emptyset: R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$\int_0^{\epsilon}\emptyset(t)\,dt>0$ for all $\epsilon>0$.

$M(x,y) = max\{d(Ix,Jy),d(Ix,Ax),d(Jy,Ay),d(Ix,Ay),d(Jy,Ax)\}$

(iii) The pairs (A,I) or (A,J) satisfy E.A. property.

(iv) The pairs (A,I) and (A,J) are weakly compatible.

If $I(X)$ is closed then the mappings A,I and J have a unique common fixed point in *X*.

If we put I=J in the above Corollary 3.5, we can obtain the following corollaries easily.

Corollary 3.7. Let (X,d) be a complete dislocated metric space. Let $A,B,I:X\to X$ satisfying the following conditions

(i) $A(X)\subseteq I(X)$ and $B(X)\subseteq I(X)$

(ii) $\int_0^{d(Ax,By)}\emptyset(t)dt \le k\int_0^{M(x,y)}\emptyset(t)dt,\ k\in\left[0,\frac{1}{2}\right)$

for all $(x,y)\in X\times X$ where $\emptyset: R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$\int_0^{\epsilon}\emptyset(t)\,dt>0$ for all $\epsilon>0$.

$M(x,y) = max\{d(Ix,Iy),d(Ix,Ax),d(Iy,By),d(Ix,By),d(Iy,Ax)\}$

(iii) The pairs (A,I) or (B,I) satisfy E.A. property.

(iv) The pairs (A,I) and (B,I) are weakly compatible.

If $I(X)$ is closed then the mappings A,B and I have a unique common fixed point in *X*.

If we putA=B and I=J in the above Corollary 3.5, we can obtain the following corollaries easily.

Corollary 3.8. Let (X,d) be a complete dislocated metric space. Let $A,I:X\to X$ satisfying the following conditions

(i) $A(X)\subseteq I(X)$

(ii) $\int_0^{d(Ax,Ay)}\emptyset(t)dt \le k\int_0^{M(x,y)}\emptyset(t)\,dt,\ k\in\left[0,\frac{1}{8}\right)$

for all $(x,y)\in X\times X$ where $\emptyset: R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$\int_0^{\epsilon}\emptyset(t)\,dt>0$ for all $\epsilon>0$.

$M(x,y) = max\{d(Ix,Iy),d(Ix,Ax),d(Iy,Ay),d(Ix,Ay),d(Iy,Ax)\}$

(iii) The pairs (A,I) satisfy E.A. property.

(iv) The pairs (A,I) are weakly compatible.

If $I(X)$ is closed then the mappings A and I have a unique common fixed point in *X*.

REFERENCES

1. Banach, S., "Sur les operations dans les ensembles abstraits et leur applications aux equations integrals", Fundamental Mathematicae, 3(7), 133–181, 1922.
2. Branciari, A., "A fixed point theorem for mapping satisfying a general contractive condition of integral type", Internat. J. Math. Math.Sci., 29, 531–536, 2002.
3. Aliouche, A., "A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type", J. Math. Anal. and Appl., 332(2), 796–802, 2006.
4. Aamari, M. and Moutawakil, D.EI., "Some new common fixed point theorem under strict contractive conditions", J. Math. Anal. Appl., 270(1), 181–188, 2002.
5. Ciric, Lj.B., "Generalized contractions and fixed point theorems", Publ. Inst. Math., 26, 115-148, 1963.
6. Djoudi, A. and Merghadi, F., "Common fixed point theorems for maps under a contractive condition of integral type", J. Math. Anal. Appl., 341, 953–960, 2008.
7. Jungck, G., "Commuting mappings and fixed points", Amer. Math. Monthly., 83, 261–263, 1976.
8. Jungck, G., "Compatible mappings and Common fixed points", Internat. J. Math. and Math. Sci., 2, 285–288, 1986.
9. Jungck, G., Murthy, P.P. and Cho, Y.J., "Compatible mappings of type (A) and Common fixed point theorems", Math. Japan., 38, 381–390, 1993.
10. Jungck, G. and Rhoades, B.E., "Fixed points for set–valued function without continuity", Ind. J. Pure Appl. Math., 29(3), 227–238, 1998.
11. Matthews, S.G., "Metric Domains for Completeness",. Technical Report 76, Ph.D. Thesis, Department of Computer Science, University of Warwick, Coventry,1986.
12. Hitzler, P. and Seda, "A.K. Dislocated Topologies. *Journal of Electrical Engineering*",**51**, 3-7,2000.
13. Kumari,P.S. Zoto,K. and Panthi, D., "d-Neighborhood System and generalized F- Contraction in Dislocated Metric Space",. *Springer Plus*, **4**, 1-10,2015
14. Kumari, P.S. "Common Fixed Point Theorems on Weakly Compatible Maps on Dis located Metric Spaces",. *Mathematical Sciences*, **6**, 71,2012.
15. Kumari, P.S. and Panthi, D., " Cyclic Contractions and Fixed Point Theorems on Various Generating Spaces",. *Fixed Point Theory and Applications*, 153,**2015**.
16. Kumari, P.S., Ramana, C.V., Zoto, K. and Panthi, D., "Fixed Point Theorems and Generalizations of Dislocated Metric Spaces",. *Indian Journal of Science and Technology*, **8**, 154-158,**2015**.
17. Panthi, D., "Common Fixed Point Theorems for Compatible Mappings in Dislocated Metric Space",. *International Journal of Mathematical Analysis*, **9**, 2235-2242,**2015**.
18. Panthi, D.,"Fixed Point Results in Cyclic Contractions of Generalized Dislocated Metric Spaces",. *Annals of Pure and Applied Mathematics*, **5**, 192-197,2014.
19. Panthi, D. and Kumari, P.S., "Some Integral Type Fixed Point Theorems in Dislocated Metric Space",. *American Journal of Computational Mathematics*, **6**, 88-97,**2016**.
20. Panthi, D. and Kumari, P.S., "Common Fixed Point Theorems for Mappings of Compatible Type(A) in Dislocated Metric Space",. *Nepal Journal of Science and Technology*, **16**, 79-86,**2015**.
21. Panthi, D. and Jha, K., "A Common Fixed Point of Weakly Compatible Mappings in Dislocated Metric Space", *Kathmandu University Journal of Science, Engineering and Technology*, **8**, 25-30,**2012**.
22. Rhoades, B. E.,"Two Fixed–Point Theorems For Mappings Satisfying a General Contractive Condition of Integral Type", Int. J. of Math. and Math. Sci., 63, 4007–4013, **2003**.
23. Sarma, I.R., Rao, J.M., Kumari, P.S. and Panthi, D. "Convergence Axioms on Dislocated Symmetric Spaces" *Abstract and Applied Analysis*, **2014**.
24. Sessa, S.,"On a weak commutativity condition of mappings in fixed point considerations", Publ. Inst. Math. (Beograd), 32, 149–153, **1982**.
25. Sintunavarat, W. and Kumam, P. "Common Fixed Points for a Pair of Weakly Compatible Maps in Fuzzy Metric Spaces",. *Journal of Applied Mathematics*, 1-14, **2011**.