Lagrange Formula Conjugate Third Order Differential Equation

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Abstract The paper considers a boundary value problem for a third order with constant coefficients and normal derivatives. Odds. This is because introduces the concept of the conjugate Green's function. It is tough to write the form of the conjugate differential operator based on the Lagrange equation. Therefore, in this work, using boundary conditions under value boundary problems conditions and conjugate, an explicit form is found as a conjugate operator since the boundary value problem for integral-differential equations has been studied based on introducing conjugate for third-order differential equation systems.

In this article, it can be said that Green's function is considered based on Lagrange's formula for the third-order differential equation with boundary conditions and its conjugate.

Key words: Conjugate, Green's Function, Lagrange's Formula, Normalization, Normalization of Boundary.

1. Introduction

In the function space $L_2(0,1)$ consider the operator in max generated by a fourth-order linear differential expression with smooth coefficients:

$$L(y) = y^{(3)}(x) + P_1(x)y^{(1)}(x) + P_0(x)y(x) \tag{1}$$

according to the formula $B_{\max}y(x) = Ly$ on the domain of definition $D(B_{\max}) = W_2^3[0,1]$. notice, that $Ran(B_{\max}) = L_2(0,1)$ and domain $Dom\ker(R_{\max}) = 3$. According to the Mikhailov- Keselman theorem there is an interval of two-point boundary conditions of the form

$$U_j(y) = \alpha_j y^{(\gamma_j)}(0) + \beta_j y^{(\gamma_j)}(1) = 0, \quad j = 1,2,3, \tag{2}$$

Such that if we choose the domain of definition of the operator S_U which represents the restriction $B_{\max}$) in the form

$$D(S_U) = \left\{ y \in W_2^n[0,1] : U_j(y) = \alpha_j y^{(\gamma_j)}(0) + \beta_j y^{(\gamma_j)}(1) = 0,\ j = 1,2,3 \right\} \tag{3}$$

then the system of root functions of the operator S_U forms a Riesz basis in $L_2(0,1)$. In what follows we assume that the interval of forms $\{U_j\}$. Replace with another form interval $\{V_j\}$ where

$$V_j = U_j(y) + \sum_{k=0}^{\gamma_{j-1}} \left(\alpha_{jk} y^{(\gamma_j)}(0) + \beta_{jk} y^{(\gamma_j)}(1) \right) + = \sum_{k=0}^{\gamma_j} \int_0^1 y^{(k)}(t)\ \delta_{jk}(t) dt$$

In this case, $\{\alpha_{jk}, \beta_{jk}\}$ is an arbitrary interval of complex numbers, $\{\delta_{jk}(t)\}$ - any of the $L_2(0,1)$ system of functions. Throughout we will assume that $\{S_V\}$,- the restriction of the maximal operator $B_{\max}$ is boundary invertible in the space $L_2(0,1)$.

Problem statement: given a certain interval of eigenvalues, uniquely reconstruct the interval of complex numbers $\{\alpha_{jk}, \beta_{jk}\}$ and the interval of boundary functions $\{\delta_{jk}(t)\}$. Below we will clarify which spectral data are used to reconstruct the region of determination of the reversible narrowing S_V. So, given the maximum operator $B_{\max}$

and a fixed interval of boundary forms $\{U_j\}$, as well as some spectral data it is necessary to reconstruct the domain of definition of the correct restriction S_V.

The opinion framework is based on third order differential equations with boundary conditions, Green's function, eigenfunctions, eigenvalues to obtain general solution and private solution on boundary conditions, Wronski determinant and differential operators.

This research is divided into six basic parts: introduction, elementary basic, normalization of boundary conditions, spectral data used for reconstruction, spectral properties of the operators and conclusion.

2. Literature review

Differential equations the relationship between the transformations of functions and the derivatives of functions have been developed for nearly 300 years. So its history naturally goes back to the discovery of the derivative by the English scientist Isaac Newton between the years (1642-1772) and the German Gottfried Leibniz worked on differential equations, including first-order differential equations, in the years (1646-1716). Jacob proposed Bernoulli's differential equation in 1674, but was unable to prove it until Euler proved it in 1705. Sturm-Liouville theorized the boundary problem with the first boundary in linear differential equations and its applications. The classical Sturm-Liouville theory, named after Jacques Francois Sturm and Joseph Liouville, was proposed between (1855-1803) and in (1809-1882), the theory of linear differential equations was formed in the second order. In 1969, the Russian scientist Naimark wrote in his book about the Linear Differential equations the Green's function for solving differential equations with boundary conditions. According to the theorems of Mikhailov and Keselman, the boundary conditions are often strictly regular and defined [4]. Therefore, the eigenvalues of the asymptotic operator are simple and distinct. There is a positive number such as δ which are separated from each other by a greater distance δ for both eigenvalues of the function [3]. It is also concluded from the works [1, 2] that the system of eigenfunctions and related functions form a basis Res in the space. In recent years, many pure mathematical scientists have worked in the field of obtaining Green's function for linear differential equations, including the Kazakh scientist Kanguzhin in 2019, who published an article entitled "Getting Green's function for second-order linear differential equations" [1,2,3,4,5,6,12,13].

3. Normalization of boundary conditions

When studying spectral properties asymptotic of eigenvalues and convergence of spectral expansions narrowing S_V. first the so-called normalization of the boundary conditions is performed [4]. Kirchhoff-regular boundary conditions are identified and subsequently strongly regular boundary conditions are determined when solving the inverse problem of spectral analysis. It is also necessary to first normalize the interval of boundary forms $\{V_j\}$ and only proceed to solve the inverse problem. It turns out that in this case a new canonical form of writing the boundary conditions is necessary. In this section we present the necessary equivalent form of writing boundary conditions. In other words, we replace the interval of boundary forms $\{V_j\}$ with another equivalent interval of boundary forms. We consider the interval of equivalent forms given by us to be normalized boundary forms. Let us introduce a system of solutions $\{\psi_j\}$ of the homogeneous equation $Ly=0,$ satisfying the inhomogeneous boundary conditions $V_k(\psi_j)=\delta_{kj}$. Such a system of solutions is determined uniquely, since the operator $S_V -$ is limitedly invertible. Then by direct verification one can verify the validity of the formula:

$$S_V^{-1} f(x) = S_U^{-1} f(x) - \sum_{k=1}^{3} \psi_k(x) V_k\,(S_U^{-1} f) \tag{4}$$

According to the monograph [12], the inverse operator S_U^{-1}, corresponding to the two-point boundary value problem, have the form

$$S_U^{-1} f(x) = \frac{H_U(x, f)}{\Delta_U} \tag{5}$$

Where $\Delta_U = \det\left[U_j(y_k)\right]$

$$H_U(x,f)=\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & \int_0^x g(x,t)f(t)dt \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & \beta_1\int_0^1 D^{\gamma_1}g(x,t)|_{x=1}\ f(t)dt \\ U_2(y_1) & U_2(y_2) & U_3(y_3) & \beta_2\int_0^x D^{\gamma_2}g(x,t)|_{x=1}\ f(t)dt \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & \beta_3\int_0^x D^{\gamma_3}g(x,t)|_{x=1}\ f(t)dt \end{vmatrix},\quad D=\frac{\partial}{\partial x},$$

there $\{y_j\}$ is a system of solutions to the homogeneous equation $Ly=0,$ satisfying the inhomogeneous boundary conditions $y_j^{(k-1)}(0)=\delta_{kj}$. The function $g(x,t)$ is determined by the formula

$$g(x,t)=\frac{p(x,t)}{W(t)}$$

Where $p(x,t)=\begin{vmatrix} y_1(x,t) & y_2(x,t) & y_3(x,t) \\ y_1(t) & y_2(t) & y_3(t) \\ y_1(t) & y_2(t) & y_3(t) \end{vmatrix}$ and $W(t)=\begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{vmatrix}.$

Using the above representation of the inverse operator S_U^{-1}, we calculate $V_k\left(S_U^{-1}f\right)$. for doing this, we introduce the following notation:

$$T_{sj}=\sum_{k=0}^{\gamma_{j-1}}\alpha_{jk}y_s^{(k)}(0)+\sum_{k=0}^{\gamma_{j-1}}\beta_{jk}y_s^{(k)}(1)+\sum_{k=0}^{\gamma_j}\int_0^1 y_s^{(k)}(t)\delta_{jk}(t)dt \tag{6}$$

$$R_j(t)=\sum_{k=0}^{\gamma_{j-1}}\beta_{jk}\,D^k g(x,t)|_{x=1}+\sum_{k=0}^{\gamma_j}\int_0^1\delta_{jk}(t)D^k g(x,t)dx \tag{7}$$

Then we can write the equality

$$V_k(S_U^{-1}f)=\frac{V_k\left(H_U(x,f)\right)}{\Delta_U}=$$

$$=\frac{1}{\Delta_U}\begin{vmatrix} T_{1k} & T_{2k} & T_{3k} & \int_0^1 R_k(t)f(t)dt \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & \beta_1\int_0^1 D^{\gamma_1}g(x,t)|_{x=1}\ f(t)dt \\ U_2(y_1) & U_2(y_2) & U_3(y_3) & \beta_2\int_0^x D^{\gamma_2}g(x,t)|_{x=1}\ f(t)dt \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & \beta_3\int_0^x D^{\gamma_3}g(x,t)|_{x=1}\ f(t)dt \end{vmatrix}$$

In the next theorem, the interval of boundary forms $\{V_j\}$ are replaced by another equivalent interval of forms.

Theorem. Domain

$$D(S_V)=\left\{y\in W_2^2[0,1]:V_j(y)=0,\ j=1,2,3\right\} \tag{8}$$

operator S_V can be written as

$$D(S_V)=\left\{ y\in W_2^2[0,1]: U_j(y)+\int_0^1 \delta_j(t)Ly\,dt=0,\ j=1,2,3 \right\} \quad (9)$$

So, instead of reconstructing the interval of complex numbers $\{\alpha_{jk},\beta_{jk}\}$ and the interval of boundary functions $\{\delta_{jk}(t)\}$ it is sufficient to use the maximum operator B_{Max} and a fixed interval of boundary form $\{U_j\}$, and also, from some spectral data, be able to find an interval of functions $\{\delta_j(t)\}$.

4. Spectral data used for reconstruction

It is convenient for us to redesign the operators S_U and S_V as Reassign via S_0 and S_3, and also introduce the operators S_0, S_1, S_2. The operator S_k is entered according to the formulas:

$$S_k y(x)=B_{\max} y(x)$$

$$U_j(y)+\int_0^1 \delta_j(t)B_{\max}y(t)dt=0,\quad j=1,2,...,k$$

$$U_j(y)=0,\quad j=k+1,k+2,...,3$$

Refined formulation of the problem of restoring boundary conditions using the maximum operator $B_{\max}$, boundary forms $\{U_j\}$ and the set of spectra $\{\delta(S_k),k=1,2,3\}$ of operators S_1,S_2,S_3, it is required to restore the boundary functions $\{\delta_1(t),\delta_2(t),\delta_3(t),\delta_4(t)\}$ the set of spectra $\{\delta(S_k),k=1,2,3\}$ of operators S_1,S_2,S_3 plays the role of spectral data from which the boundary functions are reconstructed $\{\delta_1(t),\delta_2(t),\delta_3(t),\delta_4(t)\}$.

5. Spectral properties of the operators S_1,S_2,S_3

In this section, we recall the well-known properties of these operators. Boundary conditions:

$$U_j(y)=\alpha_j y^{(\gamma_j)}(0)+\beta_j y^{(\gamma_j)}(1)=0,\quad j=1,2,3$$

chosen according to the Mikhailov-Keselman theorem are often called strongly regular boundary conditions [12]. Therefore, the eigenvalues of the operator S_0 are asymptotically simple and separated [9], there is a positive number $\delta.$ for which any two eigenvalues of the operator S_0 are separated from each other by a distance greater than $\delta.$ It also follows from works [7,8,9,11] that the system of eigenvalues and associated eigenfunctions of the operator S_0 forms a Riesz basis in the space $L_2(0,1)$.

Let us write down the conjugate boundary conditions to the Mikhailov-Keselman boundary conditions. We will assume that

$$\gamma_3\ge\gamma_2\ge\gamma_1$$

Case 1. If everything γ_j, $j=1,2,3,...,n$ different then

$$\gamma_1=0,\ \gamma_2=1,\ \gamma_2=2$$

Case 2. If $\gamma_j=\gamma_{j+1}$, then we choose $U_j(y)=y^{(\gamma_j)}(0),\ U_{j+1}(y)=y^{(\gamma_j)}(1)$ in what follows, case 1 is studied in detail. Case 2 can be studied similarly. So let $\gamma_1=0,\ \gamma_2=1,\ \gamma_2=2$ in this case, we introduce a set of forms $U_{j+n}(y)=-\beta_j\, y^{(j-1)}(0)+\alpha_j y^{(j-1)}(1),\ j=1,2,3,...,n.$ It can be considered that $\alpha_j^2+\beta_j^2=1.$ Then the equalities are true

$$y^{(\gamma_j)}(0)=\alpha_j U_j(y)-\beta_j U_{j+n}(y)$$

$$y^{(\gamma_j)}(1)=\beta_j U_j(y)+\alpha_j U_{j+n}(y)$$

For an arbitrary smooth function $V(x)$ we introduce the notation

$$Q_1(x,V) = V(x),$$
$$Q_2(x,V) = -V^{(1)}(x),$$
$$Q_3(x,V) = V^{(2)}(x) + \overline{p_2(x)}V(x)$$

Now you can enter conjugate boundary forms using the formulas

$$W_{4-s}(V) = \overline{\beta_{4-s}Q_s(1,V) - \alpha_{4-s}Q_s(0,V)}$$
$$W_{8-s}(V) = \overline{\alpha_{4-s}Q_s(1,V) + \beta_{4-s}Q_s(0,V)}$$

Thus, for any two smooth functions $V(x),\ y(x)$ the Lagrange formula is valid [9]

$$\int_0^1 L(y)\,\overline{V(x)}dx - \int_0^1 y(x)\,\overline{L^+(V)}\,dx = \sum_{s=1}^{3} U_{4-s}(y)\overline{W_{4-s}(V)} + \sum_{s=1}^{3} U_{8-s}(y)\overline{W_{8-s}(V)}$$

where $L^+(\cdot)$ is a formally conjugate differential expression. For any $U_1(\cdot),\ldots,U_6(\cdot)$ there are linearly independent forms $W_1(\cdot),\ldots,W_6(\cdot)$.

6. Conclusion

As a result, it can be said that the obtained formula is actually the solution of the third order differential equation with the conditions of the boundary problem. This conjugate formula is obtained based on Green's function and Lagrange's formula for the third order differential equation and it is proposed to solve the problems of dynamic systems to improve the efficiency of linear and dynamic system in two-dimensional space.

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