Research Article

Singular kernel and Exponential Kernel Numerical method for FOPID Controller

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Abstract: Appropriate numerical method for solving the fractional domain problems is essential in order to find out solutions of various problems. Proportional Integral Derivative controller has the drawbacks of load dynamics and non-linearity hence the fractional order PID controller is introduced to overcome these problems but no appropriate numerical method is available for fractional type controller as in the case of traditional PID controller. The traditional PID controller could be implemented using any of numerical methods. Present work is carried out for investigating appropriate numerical method that have application in fractional order PID controller. Unfortunately, there is no unique definition of fractional derivative. In this work, two type of Caputo definition is used and numerical method is suggested for singular and exponential kernel of Caputo type definition. Results of Proposed numerical method is compared with analytical solution and also compared with Euler and ABC method. It is found that the singular type kernel numerical method shows minimum mean square error and also found suitable for fractional order PID controller. The fractional order PID is used to control the first kind of system G(S) with optimum FOPID parameters and sampling time is 2.56 ms. Response of the control strategy does not have oscillations and does not have underdamping response. Control loop settled down and achieved the set point in 128 ms.

Keywords: Numerical methods for fractional calculus, singular kernel Caputo derivative, exponential kernel Caputo derivative, Numerical method for FOPID controller

1. Introduction

Proportional Integral Derivative (PID) controllers are more popular in the control industries since from last decades. PID controllers have been attracted by many researcher and tuning rules are available for good performance of controller (**O'Dwyer. A.2006**). Although the PID controller shows non-linearity for non-linear loads and control loop become unstable due to improper settings of PID parameters. General PID controller is $y(t) = K_P e(t) + K_I \int_0^t e(t)\, dt + K_D \frac{de(t)}{dt}$ (1.1) and its Laplace domain form is,

$$Y(s) = K_P E(s) + \frac{K_I}{s} E(s) + K_D s\, E(s) \qquad (1.2)$$

Fractional Order PID (FOPID) controller overcomes the drawback of PID controller and FOPID shows better performance **(Podlubny, I. 1994)**. The Laplace domain FOPID controller is given as,

$$Y(s) = K_P E(s) + \frac{K_I}{s^{\lambda}} E(s) + K_D s^{\mu}\, E(s) \qquad (1.3)$$

Fractional derivative operator is defined as,

$$D^{\alpha} f(t) = \begin{cases} \frac{df(t)}{dt} & ; \ \alpha = 1 \\ f(t) & ; \ \alpha = 0 \\ \int_0^t f(t)\, dt & ; \ \alpha = -1 \end{cases} \qquad (1.4)$$

In this present work the Caputo derivative is used. The singular Kernel Caputo fractional derivative is defined as,

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f^1(\tau)\, d\tau \qquad (1.5)$$

The exponential kernel Caputo fractional derivative is defined as, (**Caputo, M., & Fabrizio, M. 2015)**

$$D^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t e^{\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]} f^1(\tau)\, d\tau \qquad (1.6)$$

Where, α, λ, μ are known as fractional order.

2. Significance Of The Study

Fractional order systems are difficult to implement due to the unavailability of appropriate numerical method for fractional calculus. Hence, the investigation of suitable numerical method is essential. Performance of FOPID controller not only depends on the FOPID parameters but also depend on which numerical method is used to solve the fractional derivative. This may happen due to unclear concepts of fractional calculus because some of definition of fractional derivative are misleading and inappropriate. The Caputo fractional definition is suitable for the engineering applications due to same physical initial values and due to this property, it is suitable for FOPID **(Caputo, M.1966).** Fractional calculus and its definition have singularity and non-locality issues. The non-local problem gives the memory effect and singularity introduce discontinuity. The memory effect could be managed with limiting resources and limiting past data but in order to address the singularity Caputo et. al. proposed exponential kernel fractional derivative which is avoiding singularity. Numerical method for singular and exponential kernel is essential for the control system applications.

3.Review Of Related Studies

Luchko, Y. (1999) proposed the methods for the fractional calculus of Caputo type definition. Numerical methods are suitable for solving fractional order differential equations and other important problems in fractional calculus. Euler's numerical method for Caputo type derivative is given as,

$$y_{n+1} = \sum_{j=0}^{m-1} \frac{t_{n+1}^{j}}{\Gamma(j+1)} y_0^{j} + h^{\alpha} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j) \tag{3.1}$$

$$b_{j,n+1} = \frac{(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}}{\Gamma(1+\alpha)} \tag{3.2}$$

Yadav, S., Pandey, R. K., & Shukla, A. K. (2019) conducted study numerical approximation of the Atangana–Baleanu-Caputo (ABC) fractional derivative. ABC numerical method for Atangana–Baleanu-Caputo fractional derivative is given as,

$$y_{n+1} = y_0 + \frac{(1-\alpha)}{ABC(\alpha)} f\big(t_n, y(t_n)\big) + \frac{(\alpha)}{ABC(\alpha)} \sum_{k=0}^{n} \Big(\frac{h^{\alpha}}{\Gamma(\alpha+2)} \quad \big((n+1-k)^{\alpha} \ (n-k+2+\alpha) - (n-k)^{\alpha} \ (n-k+2+2\alpha)\big) - \frac{h^{\alpha} f\big(t_{k-1}, y(t_{k-1})\big)}{\Gamma(\alpha+2)} \big((n+1-k)^{\alpha} \ - (n-k)^{\alpha} \ (n-k+1+\alpha)\big)\Big) \tag{3.3}$$

$$ABC(\alpha) = 1 - \alpha - \frac{\alpha}{\Gamma(\alpha)} \tag{3.4}$$

4. Objectives Of The Study

- To find out the appropriate numerical method for fractional calculus which is suitable for sampled-based control system in such a way that the controller could be implemented in real time.
- To find out whether there is any significance difference between proposed numerical method in terms of accuracy as compared with analytical solution and comparing with other numerical methods for fractional calculus suggested in literature.
- To find out the appropriate numerical method for the implementation of fractional order PID controller.

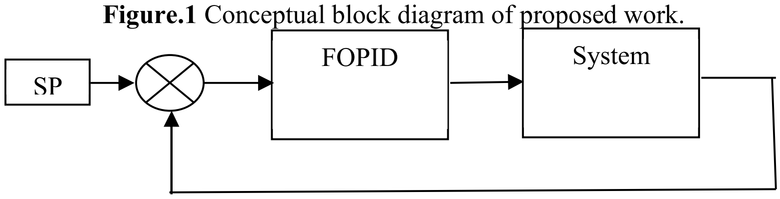
5. Hypotheses Of The Study

- System is Linear Time Invariant and causal system.
- Quantization error, sampling frame error, truncation and round off errors are neglected in this study.
- All type of disturbances and noises are considered as signal and distribution is normalized Gaussian type.
- All variables are bounded and does not extend to infinity.
- Constraints of memory and time stick is limited to cope up the proposed system in embedded environment.

6. Methodology

Experimental scheme of proposed method is given in the figure 1. Set Point (SP) is the desired output of system and the output of system is Process Value (PV). Controller is generating manipulation signal and excite the system to alter the output of system in order to achieve the set point. Controller function is to monitor continuously the system and maintain the output constant which is equal to set point. When there is change in the system output controller response to the difference between set point and process value known as error and make the change in manipulating signal which then bring back the desired process value. FOPID controller scheme is implemented in this present work.

Figure.1 Conceptual block diagram of proposed work.



6.1. Numerical Method for the present study

Appropriate suitable numerical method is investigated and results are comparing with analytical solution and also comparing with other numerical method for fractional calculus. Singular and exponential kernel has separate numerical schemes.

Numerical Method for Singular Kernel Caputo type.

Caputo fractional derivative is given as,

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} (t-\tau)^{-\alpha} f^{1}(\tau)\, d\tau$$

Where the gamma function is $\Gamma(\alpha) = \int_0^{\infty} e^{-t}\, t^{\alpha-1}\, dt$ and $f^1(\tau)$ is the first order derivative of function $f(\tau)$. Taylors series used to approximate the function $f^1(t)$, the general form of Taylors series is applicable when the function $f(t)$ is continuous and n times differentiable in the interval $[x, x+h]$, the points on the interval denoted by $(x+kh)$ and $k = 0,1,2,3,\dots,N$ such that $N = ceil\left(\frac{f(Nh) - f(0)}{h}\right)andh = \frac{1}{N}$;The Taylors series is

$$f(x+h) = \sum_{k=0}^{N} \frac{h^k}{\Gamma(k+1)} f^k(x) \qquad (6.1)$$

neglecting the higher order derivative and $k = 0,1$; $f(x+h) = f(x) + hf^1(x)$; first order approximated function $f^1(x)$ is $f^1(x) = \frac{f(x+h)-f(x)}{h}$; as $f^2(x) = f^3(x) = \cdots = f^N(x) = 0$ which is approaching to zero. The approximate $f^1(x)$ is equivalent to h-derivative function in quantum calculus. Now let $t = x_n = nh$ and any small interval is $[x_k, x_{k-1}]$, there are N number of subintervals starting from k=1 to n then the Caputo equation is written as $D^{\alpha} f(x_n) = \frac{1}{\Gamma(1-\alpha)} \int_0^{x_n} (x_n - \tau)^{-\alpha} f^1(\tau)\, d\tau$; as the subinterval is $[x_k, x_{k-1}]$ spaced in $[x, x+h]$, substituting in Caputo the function is,

$$D^{\alpha} f(x_n) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (x_n - \tau)^{-\alpha} \left\{\frac{f(x_k) - f(x_{k-1})}{h}\right\} d\tau \qquad (6.2)$$

by interchanging integration and summation and separating the variables,

$$D^{\alpha} f(x_n) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \left\{\frac{f(x_k) - f(x_{k-1})}{h}\right\} \int_{x_{k-1}}^{x_k} (x_n - \tau)^{-\alpha}\, d\tau \qquad (6.3)$$

as $x_k = kh$ and $x_{k-1} = (k-1)h$;

$$D^{\alpha} f(x_n) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \left\{\frac{f(x_k) - f(x_{k-1})}{h}\right\} \int_{(k-1)h}^{kh} (nh - \tau)^{-\alpha}\, d\tau \qquad (6.4)$$

after integration and simplification,

$$D^{\alpha} f(x_n) = \frac{1}{h^{\alpha}\Gamma(1-\alpha)} \sum_{k=1}^{n} [f(x_n) - f(x_{n-1})]\left((n-k+1)^{1-\alpha} - (n-k)^{1-\alpha}\right) \text{ and} \qquad (6.5)$$

$$D^{\alpha} f(x) = \sum_{n=0}^{N} D^{\alpha} f(x_n)$$

Above two equations are further modified by changing the summation limits and after simplification,

$$D^{\alpha} f(x_n) = \left[\frac{1}{h^{\alpha}\Gamma(1-\alpha)}\right] \left[\left(\left[f(x_n) G_{n,n}^{\alpha} - f(x_0) G_{n,1}^{\alpha}\right]\right) + \sum_{k=1}^{n-1} f(x_k)\, M_{n,k}^{\alpha}\right] \qquad (6.6)$$

$$G_{n,k}^{\alpha} = (n-k+1)^{1-\alpha} - (n-k)^{1-\alpha} \text{ and } M_{n,k}^{\alpha} = G_{n,k}^{\alpha} - G_{n,k-1}^{\alpha}\text{; as } G_{n,n}^{\alpha} = 1$$

$$D^{\alpha} f(x_n) = \left[\frac{1}{h^{\alpha}\Gamma(1-\alpha)}\right] \left[\left(\left[f(x_n) - f(x_0) G_{n,1}^{\alpha}\right]\right) + \sum_{k=1}^{n-1} f(x_k)\, M_{n,k}^{\alpha}\right] \qquad (6.7)$$

Numerical approximation for Caputo derivative is,

$$D^{\alpha} f(x) = \sum_{n=0}^{N} \left[\frac{1}{h^{\alpha}\Gamma(1-\alpha)}\right] \left[\left(\left[f(x_n) - f(x_0) G_{n,1}^{\alpha}\right]\right) + \sum_{k=1}^{n-1} f(x_k)\, M_{n,k}^{\alpha}\right] \qquad (6.8)$$

Numerical Method for exponential kernel Caputo type

Exponential kernel Caputo derivative is,

$$D^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t e^{\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]} f^1(\tau)\, d\tau \; ; M(\alpha) = \frac{2}{2-\alpha} \quad (6.9)$$

Similarly, the approximation of exponential kernel Caputo derivative is carried out using above method mentioned in Numerical method for singular kernel. The final result is,

$$D^{\alpha} f(x) = \left[\frac{M(\alpha)}{\alpha h}\right] \sum_{n=0}^{N} \left[\left(\left[f(x_n) MG_{n,n}^{\alpha,h} - f(x_0) MG_{n,1}^{\alpha,h}\right]\right) + \sum_{k=1}^{n-1} f(x_k)\, MGG_{n,k}^{\alpha,h}\right] \quad (6.10)$$

$$MGG_{n,k}^{\alpha,h} = MG_{n,n}^{\alpha,h} - MG_{n,k+1}^{\alpha,h} \quad \text{and } MG_{n,k}^{\alpha,h} = e^{\left[\frac{\alpha h}{1-\alpha}(k-n)\right]} - e^{\left[\frac{\alpha h}{1-\alpha}(k-n-1)\right]}$$

Present numerical method is validated using the analytical solution of function. The mean square error is calculated using the values of exact solution and values obtained from proposed numerical method. Results are also compared with other numerical methods such as Euler's method and ABC method.

6.2. Validation of Numerical Method

Let, the function $f(t) = t^P$ is continuous and exact analytical solution is $D^{\alpha}\{f(t)\} = \frac{t^{P-\alpha}\,\Gamma(P+1)}{\Gamma(P+1-\alpha)}$; Now P =3 and h= $\frac{1}{N}$;

Table.1. showing the mean square error of exact solution and numerical method for singular kernel.

N	α =0.1	α =0.3	α =0.5	α =0.7	α =0.9
10	2.3033e-06	5.9414e-05	4.8328e-04	0.0028	0.0136
20	2.1032e-07	6.5540e-06	6.6288e-05	4.8746e-04	0.0031
40	1.8555e-08	6.9930e-07	8.8425e-06	8.3187e-05	6.7596e-04
80	1.5910e-09	7.2732e-08	1.1569e-06	1.4013e-05	1.4850e-04
160	1.3326e-10	7.4168e-09	1.4932e-07	2.3414e-06	3.2482e-05
320	1.0947e-11	7.4485e-10	1.9090e-08	3.8930e-07	7.0885e-06
640	8.8469e-13	7.3914e-11	2.4245e-09	6.4530e-08	1.5450e-06

Interpretation of table-1.

It is observed from the above table that the mean square error is 8.8469e-13 when $\alpha = 0.1$, N=640 among the all combination having minimum value, 0.0136 have maximum value of mean square error when $\alpha = 0.9$, N=10. Mean square error increases with increase in the value of α and it is decreasing as increasing value of N. Figure 2 shows the graph of exact solution and proposed method at $\alpha = 0.9$ and N=10 at its maximum mean square value. Figure 3 shows the graph of exact solution and proposed method at $\alpha = 0.1$ and N=640 at its minimum mean square value.

Figure. 2 Comparison graph of exact analytical solution with the proposed singular kernel type numerical method at maximum mean square value

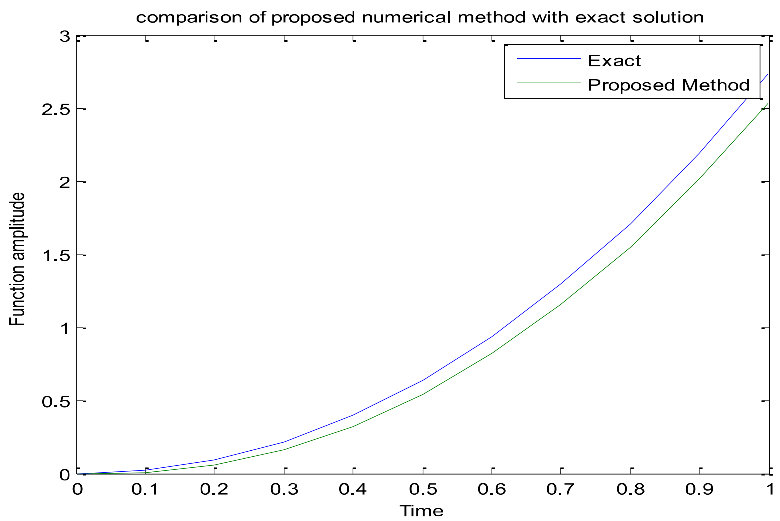


Figure. 3 Comparison graph of exact analytical solution with the proposed singular kernel type numerical method at minimum mean square value

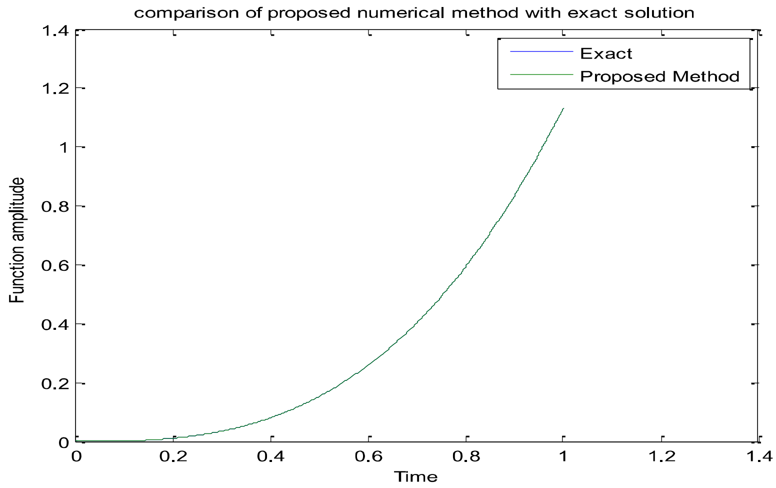


Table.2. showing the mean square error of exact solution and numerical method for exponential kernel.

N	α =0.1	α =0.3	α =0.5	α =0.7	α =0.9
10	3.7350e-05	0.0019	0.0296	0.3141	3.6365
20	2.8145e-05	0.0011	0.0190	0.2102	2.5173
40	2.7788e-05	7.5835e-04	0.0137	0.1535	1.6703
80	2.9179e-05	6.1799e-04	0.0112	0.1255	1.2132
160	3.0338e-05	5.5656e-04	0.0100	0.1119	0.9868
320	3.1044e-05	5.2830e-04	0.0094	0.1052	0.8762
640	3.1429e-05	5.1482e-04	0.0091	0.1018	0.8217

Interpretation of table-2.

It is observed from the above table that the mean square error is 2.7788e-05 when $\alpha = 0.1$, N=40 among the all combination having minimum value, 3.6365 have maximum value of mean square error when $\alpha = 0.9$, N=10. Mean square error increases rapidly than earlier method (table 1) with increase in the value of α and it is decreasing as increasing value of N from the value of $\alpha = 0.5$ and above. Figure 4 shows the graph of exact solution and proposed method at $\alpha = 0.9$ and N=10 at its maximum mean square value. Figure 5 shows the graph of exact solution and proposed method at $\alpha = 0.1$ and N=40 at its minimum mean square value.

Figure. 4 shows the graph of exact solution and proposed method at $\alpha = 0.9$ and N=10 at its maximum mean square value.

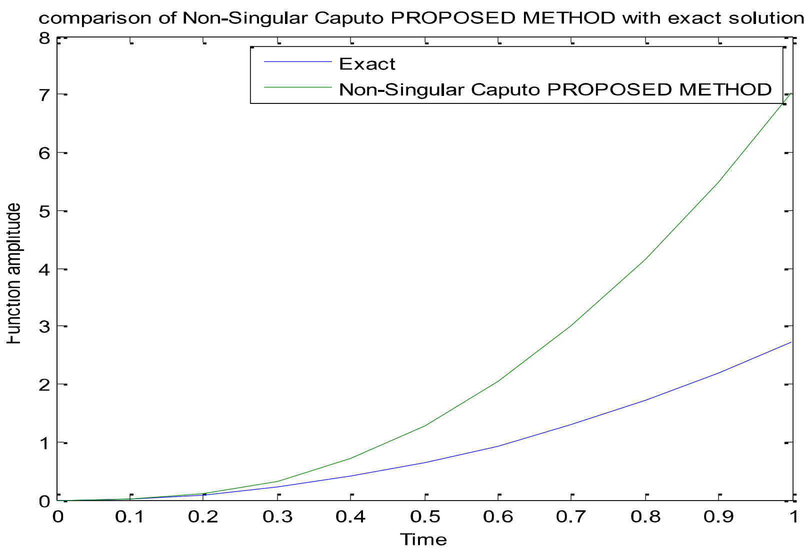


Figure. 5 shows the graph of exact solution and proposed method at $\alpha = 0.1$ and N=40 at its minimum mean square value.

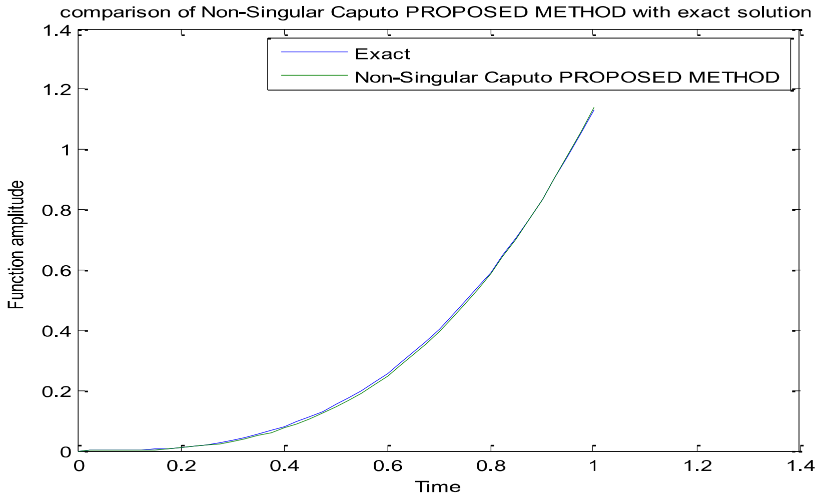


Table.3. showing the mean square error of exact solution and Euler method

N	α =0.1	α =0.3	α =0.5	α =0.7	α =0.9
10	0.0100	0.1034	0.3246	0.7165	1.3360
20	0.0105	0.1039	0.3178	0.6908	1.2787
40	0.0108	0.1043	0.3140	0.6769	1.2482
80	0.0110	0.1046	0.3122	0.6697	1.2326
160	0.0111	0.1049	0.3113	0.6660	1.2246
320	0.0112	0.1051	0.3108	0.6642	1.2206
640	0.0113	0.1052	0.3106	0.6633	1.2186

Interpretation of table-3.
It is observed from the above table that the mean square error is 0.0100 when $\alpha = 0.1$, N=10 among the all combination having minimum value, 1.3360 have maximum value of mean square error when $\alpha = 0.9$, N=10. Mean square error increases rapidly than earlier method (table 1) with increase in the value of α and it is decreasing as increasing value of N from the value of $\alpha = 0.5$ and above.

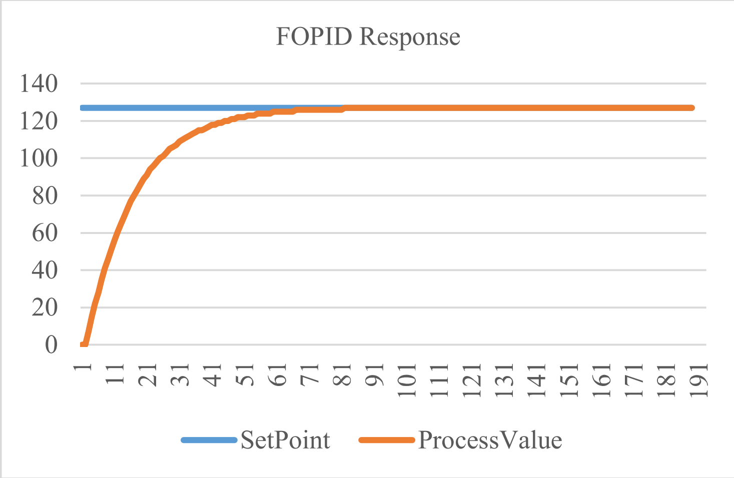
Table.4. showing the mean square error of exact solution and ABC method

N	α =0.1	α =0.3	α =0.5	α =0.7	α =0.9
10	0.0107	0.1108	0.3468	0.7555	1.3935
20	0.0095	0.0959	0.3052	0.6838	1.2905
40	0.0087	0.0868	0.2818	0.6468	1.2405
80	0.0082	0.0806	0.2673	0.6265	1.2156
160	0.0078	0.0760	0.2579	0.6150	1.2029
320	0.0075	0.0725	0.2515	0.6083	1.1963
640	0.0072	0.0697	0.2471	0.6043	1.1929

Interpretation of table-4.
It is observed from the above table that the mean square error is 0.0072 when $\alpha = 0.1$, N=640 among the all combination having minimum value, 1.3935 have maximum value of mean square error when $\alpha = 0.9$, N=10. Mean square error increases with increase in the value of α and it is decreasing as increasing value of N.

6.3. Fractional order PID controller
Fractional order PID controller is implemented using the proposed singular type numerical method for the system $G(s) = \frac{1}{1+0.1s}$ and using the FOPID parameters K_P=1.3935, K_I=0.01, K_D=0.02, $\lambda = 0.8$ and $\mu = 0.5$. The sampling time is 2.56 ms. Root means square error between set point and process value is 31.99 and manipulation effort is 34.47. Settling time is 128 ms. No peak overshoot is observed on the response as shown in the figure 6.

Figure. 6 shows the graph of FOPID response that is set point and process value.



7. Recommendations

- Among the four numerical method the singular type proposed numerical method (table 1) is best suitable for fractional calculus.
- Fractional order PID controller successfully implemented using the proposed numerical method. Hence, this numerical method is suitable for FOPID controller.
- Sampling time selection is crucial and depends on order of system or settling time of unit step response of system. Select the sampling time slightly greater than the settling time of unit step response of system.
- FOPID proportional gain if near to unity then other FOPID parameters should be less than unity and select small values.
- Value of λ should be near to 0.9 and value of μ should be near to 0.5 for better performance of FOPID.

8. Conclusion

In this present work, numerical method for fractional Caputo derivative is investigated for both singular and exponential kernel. Mean square error is minimum for small value of α and larger value of N whereas the mean square error is maximum for smaller value of N and larger value of α. Present numerical method shows minimum mean square error as compared with Eulers method and ABC method. Hence, the singular type kernel numerical method is advocated for fractional order PID controller. The FOPID controller is implemented successfully using proposed numerical method. It is observed from experimental work that the FOPID controller shows the smoother response and settled down quickly.

References

[1].Caputo, M. (1966). Linear models of dissipation whose Q is almost frequency independent. Annals of Geophysics, 19(4), 383-393.
[2].Caputo, M., & Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Progress in Fractional Differentiation & Applications, 1(2), 73-85.
[3].Luchko, Y. (1999). Operational method in fractional calculus. Fract. Calc. Appl. Anal, 2(4), 463-488.
[4].O'Dwyer, A. (2006). PI and PID controller tuning rules: an overview and personal perspective. Proceedings of the IET Irish Signals and Systems Conference, pp. 161-166, Dublin Institute of Technology, June, 2006. doi:10.21427/ekry-ap03
[5].Podlubny, I. (1994). Fractional-order systems and fractional-order controllers. Institute of Experimental Physics, Slovak Academy of Sciences, Kosice, 12(3), 1-18.
[6].Yadav, S., Pandey, R. K., & Shukla, A. K. (2019). Numerical approximations of Atangana–Baleanu Caputo derivative and its application. Chaos, Solitons & Fractals, 118, 58-64.