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Research Article

Fixed Point Theorems For Two Paired Mappings In Fuzzy 2-Metric Spaces

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Abstract. In this paper, we prove common fixed-point theorems for two pairs of self-mappings on fuzzy 2-metric space using weaker condition of the compatibility of the maps. Our results improve and the results of Sharma [10] in the sense that the completeness of the fuzzy 2-metric space and continuity of the mappings have been dropped. Our results also extend the results of Cho [2] to fuzzy 2-metric spaces.

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1. Introduction.

Using the concept of fuzzy sets given by Zadeh [11], Kramosil and Michalek [7] developed the concept of fuzzy metric spaces. George and Veeramani [4] improved the concept of fuzzy metric spaces using t −norms. Gahler [3] introduced the concept of fuzzy 2-metric spaces. Further, Iseki et al [6] proved results for contractive type mappings in 2-metric spaces.

Cho [2] and Kutukcu et al [8] proved common fixed-point theorems for three mappings in fuzzy 2-metric spaces.

Many authors have studied common fixed-point theorems in fuzzy metric spaces. Some of interesting papers are Cho [1], George and Veeramani [4], Grabiec [5], Kramosil and Michalek [7] and Sharma [10].

Cho [1] proved a common fixed-point theorem for four mappings in fuzzy metric spaces and Sharma [10] proved a common fixed-point theorem for three mappings in fuzzy 2-metric spaces.

In this paper, we prove common fixed-point theorems for two pairs of compatible self-mappings. Our theorems improve the theorems of Sharma [10] and extend the results of Cho [2] to fuzzy 2-metric spaces.

2. Preliminaries.

Definition 2.1. [11] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a continuous t −norm if $*$ satisfies the following conditions for all $a, b, c, d \in [0,1]$.

(i) $*(a, 0) = 0, *(a, 1) = a$,
(ii) * is continuous,
(iii) $*(a, b) = *(b, a)$,
(iv) $*(a, b) \le *(c, d)$ if $a \le c, b \le d$,
(v) $*(*(a, b), c) = *\big(a, *(b, c)\big)$,

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [9] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ −

(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s)$,
(v) $M(x, y, .): [0, \infty] \to [0,1]$ is left continuous.

$M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Definition 2.3. [3] Let X be a nonempty set. A real valued function $d: X \times X \times X \to R$ is said to be a 2-metric on X if-

(i) given distinct elements $x, y \in X$, there exists an element $z \in X$ such that $d(x, y, z) \ne 0$,
(ii) $d(x, y, z) = 0$ if at least two of x, y and z are equal,
(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x) = d(y, x, z) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$, (Symmetry)
(iv) $d(x, y, z) \le d(x, y, w) + d(y, z, w) + d(z, x, w)$ for all $x, y, z, w \in X$, (Rectangle inequality)

Then pair (X, d) is called a 2-metric space.

Example 2.1. Let $X = R^3$ and $d(x, y, z) =$ the area of the triangle spanned by $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ which may be given explicitly by

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$$d(x, y, z) = |x_1(y_2z_3 - y_3z_2) - x_2(y_1z_3 - y_3z_1) + x_3(y_1z_2 - y_2z_1)$$

Then (X, d) is a 2-metric space.

Example 2.2 Let $d: X \times X \times X \to R$ be given by $d(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}$. Then (X, d) is a $2-$metric space.

Definition 2.4 [9]. An operation $*: [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous $t-$norm if $*$ satisfies the following conditions for all $a, b, c, d, e, f \in X -$

(i) $*(a, 0,0) = a, *(a, 1,1) = a$,

(ii) * is continuous,

(iii) $*(a, b, c) = *(b, a, c) = *(a, c, b) = *(c, a, b) = *(b, c, a) = *(c, b, a)$,

(iv) $*(a, b, c) \leq *(d, e, f)$ whenever $a \leq d, b \leq e, c \leq f$,

(v) $*(*(a, b, c), d, e) = *(a, *(b, c, d)), e) = *(a, b, *(c, d, e)))$,

Examples of *t*-norms are $a * b * c = abc$ and $a * b * c = \min\{a, b, c\}$.

Definition 2.5. [10] The 3-tuple $(X, M, *)$ is called a fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0 -$

(i) $M(x, y, z, 0) = 0$,

(ii) $M(x, y, z, t) = 1$ for all $t > 0$ if and only if at least two of the three points are equal,

(iii) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) = M(y, x, z, t) = M(z, y, x, t) = M(z, x, y, t)$ for all $t > 0$,

(iv) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$,

(v) $M(x, y, z, .): X^3 \times [0,1] \to [0,1]$ is left continuous.

Example 2.3. Let (X, d) be a 2-metric space and denote $a * b * c = abc$ (or $\min\{a, b, c\}$) for all $a, b, c \in [0,1]$. For all $x, y, z \in X$ and $t > 0$, define

$$M(x, y, z, t) = \frac{t}{t + d(x, y, z)}$$

Then $(X, M, *)$ is a fuzzy $2-$metric space. M is called the fuzzy metric induced by the 2-metric d.

Definition 2.6. [10] Let $(X, M, *)$ be a fuzzy 2-metric space.

(1) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) if

$$\lim_{n\to\infty} M(x_n, x, a, t) = 1 \text{ for all } a \in X, t > 0.$$

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M\left(x_{n+p}, x_n, a, t\right) = 1 \text{ for all } a \in X, t > 0, p > 0.$$

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.7. [9] Self mappings A and B of a fuzzy 2-metric space $(X, M, *)$ are said to be compatible if

$$\lim_{n\to\infty} M(ABx_n, BAx_n, a, t) = 1$$

for all $a \in X$ and $t > 0$, whenever $\{x_n\}$ is a sequence in X such that-

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

for some $z \in X$.

Lemma 2.1. [10] For fuzzy 2-metric space $(X, M, *)$, if $M(x, y, z, kt) \geq M(x, y, z, t)$, for all $x \neq y \neq z \in X$, $k \in [0,1], t > 0$, then $x = y$.

3. Main Results.

Theorem 3.1. Let P, Q, R, S be four self-mappings of a fuzzy $2-$ metric space $(X, M, *)$ such that-

(3.1.1) the pairs (P, R) and (Q, S) are compatible,

(3.1.2) for all $x, y, z \in X$, $k \in (0,1)$ and $t > 0$

$$M(Rx, Sy, z, kt) \geq \min\{M(Px, Qy, z, t), M(Px, Rx, z, t), M(Sy, Qy, z, t), M(Rx, Qy, z, t)\}.$$

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Then P, Q, R, S have a unique common fixed point in X.

Proof. Since the pairs (P, R) and (Q, S) are compatible maps pairs, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n = u \text{ and } \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Sy_n = v$$

for some $u, v \in X$, for all $t > 0$ *and*

$$\lim_{n\to\infty} M(PRx_n, RPx_n, a, t) = \lim_{n\to\infty} M(Pu, Ru, a, t) = 1$$
$$\lim_{n\to\infty} M(QSy_n, SQy_n, a, t) = \lim_{n\to\infty} M(Qv, Sv, a, t) = 1$$

for all $a \in X$. Hence

$$Pu = Ru, Qv = Sv.$$

Therefore u is the coincidence point of P, R and v is the coincidence point of Q, S.
To show $u = v$, using (3.1.2),

$$M(Rx_n, Sy_n, z, kt) \geq \min\{M(Px_n, Qy_n, z, t), M(Px_n, Rx_n, z, t), M(Sy_n, Qy_n, z, t), M(Rx_n, Qy_n, z, t)\}.$$

As $n \to \infty$, we get

$$M(u, v, z, kt) \geq \min\{M(u, v, z, t), M(u, u, z, t), M(v, v, z, t), M(u, v, z, t)\}.$$

This implies $M(u, v, z, kt) \geq M(u, v, z, t)$ for all $t > 0$.
Using Lemma 2.1, we get $u = v$. Therefore P, Q, R, S have identical coincidence point in X.
Now to show that

$$Pu = Qu = Ru = Su = u.$$

Put $x = u$ and $y = y_n$ in (3.1.2), we get

$$M(Ru, Sy_n, z, kt) \geq \min\{M(Pu, Qy_n, z, t), M(Pu, Ru, z, t), M(Sy_n, Qy_n, z, t), M(Ru, Qy_n, z, t)\}.$$

As $n \to \infty$, we get

$$M(Ru, v, z, kt) \geq \min\{M(Pu, v, z, t), M(Pu, Ru, z, t), M(v, v, z, t), M(Ru, v, z, t)\}.$$

This implies $M(Ru, v, z, kt) \geq M(Ru, v, z, t)$ for all $t > 0$. Using Lemma 2.1, we get $Ru = v$.
Therefore

$$Ru = Pu = v.$$

Put $x = x_n$ and $y = v$ in (3.1.2), we get

$$M(Rx_n, Sv, z, kt) \geq min\{M(Px_n, Qv, z, t), M(Px_n, Rx_n, z, t), M(Sv, Qv, z, t), M(Rx_n, Qv, z, t)\}.$$

As $n \to \infty$, we get

$$M(u, Sv, z, kt) \geq min\{M(u, Qv, z, t), M(u, u, z, t), M(Sv, Qv, z, t), M(u, Qv, z, t)\}.$$

This implies $M(u, Sv, z, kt) \geq M(u, Sv, z, t)$ for all $t > 0$. Using Lemma 2.1, we get $Sv = u$.
Therefore

$$Sv = Qv = u.$$

Since $u = v$, we get

$$Pu = Qu = Ru = Su = u.$$

Hence P, Q, R, S have a unique common fixed point in X.

Theorem 3.2. Let A, B, P, Q, R, S be six self-mappings of a fuzzy $2-$metric space $(X, M, *)$ such that-

(3.2.1) the pairs (AP, R) and (BQ, S) are compatible,

(3.2.2) $RB = BR, AP = PA, PB = BP, RQ = QR, AQ = QA, PQ = QP$,

(3.2.3) for all $x, y, z \in X$, $k \in (0,1)$ and $t > 0$

$$M(Rx, Sy, z, kt) \geq \min\{M(APx, BQy, z, t), M(APx, Rx, z, t), M(BQy, Sy, z, t), M(Rx, BQy, z, t)\}.$$

Then A, B, P, Q, R, S have a unique common fixed point in X.

Proof. Since the pairs (AP, R) and (BQ, S) are compatible maps pairs, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} APx_n = \lim_{n\to\infty} Rx_n = u \text{ and } \lim_{n\to\infty} BQy_n = \lim_{n\to\infty} Sy_n = v$$

for some $u, v \in X$ and all $t > 0$ *and*

$$\lim_{n\to\infty} M(BQSy_n, SBQy_n, a, t) = \lim_{n\to\infty} M(BQv, Sv, a, t) = 1$$

for all $a \in X$.
Hence

$$APu = Ru, BQv = Sv.$$

Therefore u is the coincidence point of AP, R and v is the coincidence point of BQ, S.
To show $u = v$, using (3.2.3),

$$M(Rx_n, Sy_n, z, kt) \geq \min\{M(APx_n, BQy_n, z, t), M(APx_n, Rx_n, z, t), M(BQy_n, Sy_n, z, t), M(Rx_n, BQy_n, z, t)\}.$$

As $n \to \infty$, we get

$$M(u, v, z, kt) \geq \min\{M(u, v, z, t), M(u, u, z, t), M(v, v, z, t), M(u, v, z, t)\}.$$

This implies $M(u, v, z, kt) \geq M(u, v, z, t)$ for all $t > 0$.
Using Lemma 2.1, we get $u = v$. Therefore AP, R *and* BQ, S have identical coincidence point in X.
Now to show that

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$$Au = Bu = Pu = Qu = Ru = Su = u.$$

Put $x = u$ and $y = y_n$ in (3.2.3), we get

$$M(Ru, Sy_n, z, kt) \geq \min\{M(APu, BQy_n, z, t), M(APu, Ru, z, t), M(BQy_n, Sy_n, z, t), M(Ru, BQy_n, z, t)\}.$$

As $n \to \infty$, we get

$$M(Ru, v, z, kt) \geq \min\{M(Ru, v, z, t), M(Ru, Ru, z, t), M(v, v, z, t), M(Ru, v, z, t)\}.$$

This implies $M(Ru, v, z, kt) \geq M(Ru, v, z, t)$ for all $t > 0$. Using Lemma 2.1, we get $Ru = v$. Therefore

$$Ru = APu = v.$$

Put $x = x_n$ and $y = v$ in (3.2.3), we get

$$M(Rx_n, Sv, z, kt) \geq \min\{M(APx_n, BQv, z, t), M(APx_n, Rx_n, z, t), M(BQv, Sv, z, t), M(Rx_n, BQv, z, t)\}.$$

As $n \to \infty$, using $BQv = Sv$, we get

$$M(u, Sv, z, kt) \geq \min\{M(u, BQv, z, t), M(u, u, z, t), M(BQv, Sv, z, t), M(u, BQv, z, t)\}.$$

This implies $M(u, Sv, z, kt) \geq M(u, Sv, z, t)$ for all $t > 0$. Using Lemma 2.1, we get $Sv = u$. Therefore

$$Sv = BQv = u.$$

Since $u = v$, we get

$$APu = BQu = Ru = Su = u.$$

Now we show $Qu = Bu = u$.

Put $x = Bu, y = y_n$ in (3.2.3), we get

$$M(RBu, Sy_n, z, kt) \geq \min\{M(APBu, BQy_n, z, t), M(APBu, RBu, z, t), M(BQy_n, Sy_n, z, t), M(RBu, BQy_n, z, t)\}.$$

As $n \to \infty$,

$$M(RBu, u, z, kt) \geq \min\{M(APBu, u, z, t), M(APBu, RBu, z, t), M(u, u, z, t), M(RBu, u, z, t)\}.$$

Since $RB = BR, AP = PA, PB = BP$, we get

$$M(Bu, u, z, kt) \geq \min\{M(Bu, u, z, t), M(Bu, Bu, z, t), M(u, u, z, t), M(Bu, u, z, t)\}.$$

This implies $M(Bu, u, z, kt) \geq M(Bu, u, z, t)$ for all $t > 0$. Using Lemma 2.1, we get $Bu = u$.

Now put $x = Qu, y = y_n$ in (3.2.3), we get

$$M(RQu, Sy_n, z, kt) \geq \min\{M(APQu, BQy_n, z, t), M(APQu, RQu, z, t), M(BQy_n, Sy_n, z, t), M(RQu, BQy_n, z, t)\}.$$

As $n \to \infty$,

$$M(RQu, u, z, kt) \geq \min\{M(APQu, u, z, t), M(APQu, RQu, z, t), M(u, u, z, t), M(RQu, u, z, t)\}.$$

Since $RQ = QR, AQ = QA, PQ = QP$, we get

$$M(Qu, u, z, kt) \geq \min\{M(Qu, u, z, t), M(Qu, Qu, z, t), M(u, u, z, t), M(Qu, u, z, t)\}.$$

This implies $M(Qu, u, z, kt) \geq M(Qu, u, z, t)$ for all $t > 0$. Using Lemma 2.1, we get $Qu = u$.

Therefore, we have $Bu = Qu = Su = u$.

Similarly, we can show $Pu = u$ by substituting $x = x_n$ and $y = Pu$ and $Au = u$, by substituting $x = x_n$ and $y = Au$ in (3.2.3).

Hence, we obtain

$$Au = Bu = Pu = Qu = Ru = Su = u.$$

Hence A, B, P, Q, R, S have a unique common fixed point in X.

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