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Research Article

Fixed Point Theorems For Two Paired Mappings In Fuzzy 2-Metric Spaces

Rohit Pathak*
*Institute of Engineering & Technology, DAVV, Indore, M.P., India, E-Mail: rohitpathakres@yahoo.com

Abstract. In this paper, we prove common fixed-point theorems for two pairs of self-mappings on fuzzy 2-metric space using the property (E.A). Also, it is a generalization of a result of Sharma [10].

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Key words and phrases. Fuzzy metric space, coincidence point, common fixed point, property (E.A), t −norm.

1. Introduction.

Using the concept of fuzzy sets given by Zadeh [12], Kramosil and Michalek [7] developed the concept of fuzzy metric spaces. George and Veeramani [4] improved the concept of fuzzy metric spaces using t −norms. Gahler [3] introduced the concept of fuzzy 2-metric spaces. Further, Iseki et al [6] proved results for contractive type mappings in 2-metric spaces.
Cho [11] and Kutukcu et al [8] proved common fixed-point theorems for three mappings in fuzzy 2-metric spaces.
Many authors have studied common fixed point theorems in fuzzy metric spaces. Some of interesting papers are Cho [2], George and Veeramani [4], Grabiec [5], Kramosil and Michalek [7] and Sharma [10].
Cho [2] proved a common fixed point theorem for four mappings in fuzzy metric spaces and S. Sharma [10] proved a common fixed point theorem for three mappings in fuzzy 2-metric spaces.
Aamri and Moutawakil [1] generalized the concepts of non-compatibility by defining the notion of (E.A) property and proved common fixed point theorems under strict contractive condition.
In this paper we prove common fixed-point theorems for two pairs of self-mappings satisfying the property (E.A). Our theorems are extension of results of Cho [11] to fuzzy 2-metric spaces.

2. Preliminaries.

Definition 2.1. [9] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a continuous t −norm if $([0,1],*)$ is an Abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2.[7] The 3-tuple $(X, M,*)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ −

(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s)$,
(v) $M(x, y, .): [0, \infty] \to [0,1]$ is left continuous.

$M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Definition 2.3. [10] The 3-tuple $(X, M,*)$ is called a fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$ −

(i) $M(x, y, z, 0) = 0$,
(ii) $M(x, y, z, t) = 1$ for all $t > 0$ if and only if at least two of the three points are equal,
(iii) $M(x, y, z, t) = M(x, z, y, x, t) = M(y, z, x, t)$, for all $t > 0$,
(iv) $M(x, y, z, t_1 + t_2 + t_3) \ge M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$
(v) $M(x, y, z, .): [0,1] \to [0,1]$ is left continuous.

Definition 2.4.[1] A pair (S, T) of self-mappings on a fuzzy 2-metric space $(X, M,*)$ is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that-

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$$

for some $u \in X$.

Lemma 2.1. $M(p, q, r,.)$ is non decreasing for all for all $p, q, r \in X$.

Lemma 2.2. For fuzzy 2-metric space $(X, M,*)$, if $M(x, y, z, kt) \ge M(x, y, z, t)$, for all $x \ne y \ne z \in X$, $k \in [0,1], t > 0$, then x equal to y.

DOI:

3. Main Results.

Theorem 3.1: Let P, Q, R, S be four self-mappings of a fuzzy $2-$ metric space $(X, M,*)$ with continuous $t-$ norm $t * t > t$, where $t \in [0,1]$ such that-

(3.1.1) the pairs (P, R) and (Q, S) satisfy property (E.A),

(3.1.2) for all x, y, z ∈ X, $k \in (0,1)$ and $t > 0$

$M(Rx, Sy, z, kt) \geq \min\{M(Px, Qy, z, t), M(Px, Rx, z, t), M(Sy, Qy, z, t), M(Rx, Qy, z, t)\}$.

Then P, Q, R, S have a unique common fixed point in X.

Proof. Since the pairs (P, R) and (Q, S) satisfy property (E.A), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n = u$$
$$\lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Sy_n = v$$

for some $u, v \in X$ and all $t > 0$.

Therefore

$$\lim_{n\to\infty} M(PRx_n, RRx_n, z, t) = 1$$

which imply $M(Pu, Ru, z, t) = 1$.

Similarly,

$$\lim_{n\to\infty} M(QSy_n, SSy_n, z, t) = 1$$

which imply $M(Qv, Sv, z, t) = 1$.

Hence

$$Pu = Ru, Qv = Sv.$$

Therefore u is the coincidence point of P, R and v is the coincidence point of Q, S.

To show $u = v$, using (3.1.2),

$M(Rx_n, Sy_n, z, kt) \geq \min\{M(Px_n, Qy_n, z, t), M(Px_n, Rx_n, z, t), M(Sy_n, Qy_n, z, t), m(Rx_n, Qy_n, z, t)\}$.

As $n \to \infty$, we get

$M(u, v, z, kt) \geq \min\{M(u, v, z, t), M(u, u, z, t), M(v, v, z, t), M(u, v, z, t)\}$.

This implies $M(u, v, z, kt) \geq M(u, v, z, t)$ for all $t > 0$.

Using Lemma 2.2, we get $u = v$. Therefore P, Q, R, S have identical coincidence point in X.

Now to show that

$$Pu = Qu = Ru = Su = u.$$

Put $x = u$ and $y = y_n$ in (3.1.2), we get

$M(Ru, Sy_n, z, kt) \geq \min\{M(Pu, Qy_n, z, t), M(Pu, Ru, z, t), M(Sy_n, Qy_n, z, t), M(Ru, Qy_n, z, t)\}$.

As $n \to \infty$, we get

$M(Ru, v, z, kt) \geq \min\{M(Pu, v, z, t), M(Pu, Ru, z, t), M(v, v, z, t), M(Ru, v, z, t)\}$.

This implies $M(Ru, v, z, kt) \geq M(Ru, v, z, t)$ for all $t > 0$. Using Lemma 2.2, we get $Ru = v$.

Therefore

$$Ru = Pu = v.$$

Put $x = x_n$ and $y = v$ in (3.1.2), we get

$$M(Rx_n, Sv, z, kt) \geq min\{M(Px_n, Qv, z, t), M(Px_n, Rx_n, z, t), M(Sv, Qv, z, t), M(Rx_n, Qv, z, t)\}.$$

As $n \to \infty$, we get

$$M(u, Sv, z, kt) \geq min\{M(u, Qv, z, t), M(u, u, z, t), M(Sv, Qv, z, t), M(u, Qv, z, t)\}.$$

This implies $M(u, Sv, z, kt) \geq M(u, Sv, z, t)$ for all $t > 0$. Using Lemma 2.2, we get $Sv = u$.

Therefore

$$Sv = Qv = u.$$

Since $u = v$, we get

$$Pu = Qu = Ru = Su = u.$$

Hence P, Q, R, S have a unique common fixed point in X.

Theorem 3.2 Let A, B, P, Q, R, S be six self-mappings of a fuzzy $2-$ metric space $(X, M,*)$ with continuous $t-$ norm $t * t > t$, where $t \in [0,1]$ such that-

(3.2.1) the pairs (AP, R) and (BQ, S) satisfy property (E.A),

(3.2.2) $RB = BR, AP = PA, PB = BP, RQ = QR, AQ = QA, PQ = QP$,

(3.2.3) for all x, y, z ∈ X, $k \in (0,1)$ and $t > 0$

$M(Rx, Sy, z, kt) \geq \min\{M(APx, BQy, z, t), M(APx, Rx, z, t), M(BQy, Sy, z, t), M(Rx, BQy, z, t)\}$.

Then A, B, P, Q, R, S have a unique common fixed point in X.

Proof. Since the pairs (AP, R) and (BQ, S) satisfy property (E.A), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} APx_n = \lim_{n\to\infty} Rx_n = u$$

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$$\lim_{n\to\infty} BQy_n = \lim_{n\to\infty} Sy_n = v$$

for some $u, v \in X$ and all $t > 0$.
Therefore

$$\lim_{n\to\infty} M(APRx_n, RRx_n, z, t) = 1$$

which imply $M(APu, Ru, z, t) = 1$.
Similarly,

$$\lim_{n\to\infty} M(BQSy_n, SSy_n, z, t) = 1$$

which imply $M(BQv, Sv, z, t) = 1$.
Hence

$$APu = Ru, BQv = Sv.$$

Therefore u is the coincidence point of AP, R and v is the coincidence point of BQ, S.
To show $u = v$, using (3.2.3),
$M(Rx_n, Sy_n, z, kt) \geq \min\{M(APx_n, BQy_n, z, t), M(APx_n, Rx_n, z, t), M(BQy_n, Sy_n, z, t), M(Rx_n, BQy_n, z, t)\}$.
As $n \to \infty$, we get
$M(u, v, z, kt) \geq \min\{M(u, v, z, t), M(u, u, z, t), M(v, v, z, t), M(u, v, z, t)\}$.
This implies $M(u, v, z, kt) \geq M(u, v, z, t)$ for all $t > 0$.
Using Lemma 2.2, we get $u = v$. Therefore AP, R and BQ, S have identical coincidence point in X.
Now to show that

$$Au = Bu = Pu = Qu = Ru = Su = u.$$

Put $x = u$ and $y = y_n$ in (3.2.3), we get
$M(Ru, Sy_n, z, kt) \geq \min\{M(APu, BQy_n, z, t), M(APu, Ru, z, t), M(BQy_n, Sy_n, z, t), M(Ru, BQy_n, z, t)\}$.
As $n \to \infty$, we get
$M(Ru, v, z, kt) \geq \min\{M(Ru, v, z, t), M(Ru, Ru, z, t), M(v, v, z, t), M(Ru, v, z, t)\}$.
This implies $M(Ru, v, z, kt) \geq M(Ru, v, z, t)$ for all $t > 0$. Using Lemma 2.2, we get $Ru = v$. Therefore

$$Ru = APu = v.$$

Put $x = x_n$ and $y = v$ in (3.2.3), we get
$M(Rx_n, Sv, z, kt) \geq \min\{M(APx_n, BQv, z, t), M(APx_n, Rx_n, z, t), M(BQv, Sv, z, t), M(Rx_n, BQv, z, t)\}$.
As $n \to \infty$, using $BQv = Sv$, we get
$M(u, Sv, z, kt) \geq \min\{M(u, BQv, z, t), M(u, u, z, t), M(BQv, Sv, z, t), M(u, BQv, z, t)\}$.
This implies $M(u, Sv, z, kt) \geq M(u, Sv, z, t)$ for all $t > 0$. Using Lemma 2.2, we get $Sv = u$. Therefore

$$Sv = BQv = u.$$

Since $u = v$, we get

$$APu = BQu = Ru = Su = u.$$

Now we show $Qu = Bu = u$.
Put $x = Bu, y = y_n$ in (3.2.3), we get
$M(RBu, Sy_n, z, kt) \geq \min\{M(APBu, BQy_n, z, t), M(APBu, RBu, z, t), M(BQy_n, Sy_n, z, t), M(RBu, BQy_n, z, t)\}$.
As $n \to \infty$,
$M(RBu, u, z, kt) \geq \min\{M(APBu, u, z, t), M(APBu, RBu, z, t), M(u, u, z, t), M(RBu, u, z, t)\}$.
Since $RB = BR, AP = PA, PB = BP$, we get
$M(Bu, u, z, kt) \geq \min\{M(Bu, u, z, t), M(Bu, Bu, z, t), M(u, u, z, t), M(Bu, u, z, t)\}$.
This implies $M(Bu, u, z, kt) \geq M(Bu, u, z, t)$ for all $t > 0$. Using Lemma 2.2, we get $Bu = u$.
Now put
Put $x = Qu, y = y_n$ in (3.2.3), we get
$M(RQu, Sy_n, z, kt) \geq \min\{M(APQu, BQy_n, z, t), M(APQu, RQu, z, t), M(BQy_n, Sy_n, z, t), M(RQu, BQy_n, z, t)\}$.
As $n \to \infty$,
$M(RQu, u, z, kt) \geq \min\{M(APQu, u, z, t), M(APQu, RQu, z, t), M(u, u, z, t), M(RQu, u, z, t)\}$.
Since $RQ = QR, AQ = QA, PQ = QP$, we get
$M(Qu, u, z, kt) \geq \min\{M(Qu, u, z, t), M(Qu, Qu, z, t), M(u, u, z, t), M(Qu, u, z, t)\}$.
This implies $M(Qu, u, z, kt) \geq M(Qu, u, z, t)$ for all $t > 0$. Using Lemma 2.2, we get $Qu = u$.
Therefore, we have $Bu = Qu = Su = u$.
Similarly, we can show $Pu = u$ by substituting $x = x_n$ and $y = Pu$ and $Au = u$, by substituting $x = x_n$ and $y = Au$ in (3.2.3).
Hence, we obtain

$$Au = Bu = Pu = Qu = Ru = Su = u.$$

Hence A, B, P, Q, R, S have a unique common fixed point in X.

DOI:

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