**MTBF of a Repairable Units with CCS Failures and M L Estimation**

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**ABSTRACT**

In this article, we discussed the method of maximum likelihood (M L) estimation to assess the estimates of mean time between failures (MTBF) of a redundant system. 2-unit identical system in case of series and parallel modes subject to individual, lethal and non-lethal common cause shock (LCCS & NCCS) failures is studied. System life times and repair times of the units are exponential. We developed the estimates of mean time between failures functions in case of series and parallel systems in the time off of analytical approach. We have presented the validity of the resulting estimates with the help of simulation.

**Keywords:** Reliability, Estimation, Mean Time between Failures, Common Cause Shock failures, Simulation

1. **INTRODUCTION**

Common Cause Shock (CCS) failures induce simultaneous failure in the system and drastically reduce reliability of the system. These events may be outside atmosphere such as fire, thunderstorm, lightning, flood, earthquake, error due to human interventions etc. Occur at random times producing simultaneous failure of several components. The reliability analysts and researchers were considered two types of CCS failures in literature. (i) Lethal common cause shocks results failure of all the components in the system and (ii) A non-lethal common cause shocks results failure of some components at random. Reliability analysts were discussed them in the evaluation of reliability measures and routine of the system very much. The CCS failure models can be found in the literature by Billinton and Allan [2]. Some other reliability models with CCS failures were analyzed by Chari et al [3], Verma [9], Sagar et al [6].

Mathematical modelling, estimation and life testing is vital interest in order to measure mean life of the system. In truth life testing experiments are intended to measure the average life of the units and also fascinated to counter such questions as ‘what is the probability that the system will down in the interval (0, t). Awgichew et al [1] derived M L estimates of availability measures for two unit system with CCS failures and also established simulation study on reliability estimates of a repairable system. Sreedhar et al [8] proposed maximum likelihood estimation approach for estimating reliability indices of a two unit system in the presence of CCS failures. Levitin [4] proposed the universal generating function of multi-state system dependability analysis to include CCS failures. A procedure is proposed to evaluate reliability functions of no repairable series-parallel multistate systems under the influence of CCS failures. Reddy [5] developed reliability measures with the effect of lethal and non-lethal common cause shock failures of a two unit non-identical system.

The present investigation is to integrate M L estimation approach in Sagar et al [7]. We estimate the M L estimates of mean time between failure functions for series system as well as parallel system.

2. **ASSUMPTIONS**

We consider a two component identical system.

i) The units fail individually and also simultaneously due to two kinds of CCS (lethal or non-lethal) failures in Poisson fashion.

ii) Individual, LCCS and NCCS failures are independent to each other.

iii) There is a single repair facility to repair failed units whether they are failed individually or simultaneously due to common cause shock.

iv) Repair times of failed units depend on the failure mode and are assumed exponentially distributed.

3. **NOTATIONS**

- $\lambda$: Failure rate (individual)
- $\omega$: LCCS failure rate
- $\beta$: NCCS failure rate
- $\mu_1$: Repair rate when one unit is down and other one is working
- $\mu_2$: Repair rate when second has failed whereas one unit was already down
- $\mu_c$: Repair rate when both units fail simultaneously
- $p(p)$: The probability of simultaneous failures of units due to NCCS (LCCS)

$R_{LS}(t)$: Reliability of the series system in (0, t) with LCCS and NCCS failures

$R_{LP}(t)$: Reliability of the parallel system in (0, t) with LCCS and NCCS failures
\( E_{\text{LCCS}}(T) \): MTBF of the system with LCCS and NCCS failures when units are in series

\( E_{\text{LNS}}(T) \): MTBF of parallel system in case of LCCS and NCCS failures

\( \hat{E}_{\text{LCCS}}(T) \): M L Estimate of MTBF function for series system under the influence of LCCS & NCCS failure mode

\( \hat{E}_{\text{LNS}}(T) \): Estimate of MTBF function for parallel system in the presence of LCCS and NCCS failures

\( \bar{X}, \bar{Y}, \bar{W} \): Sample means of individual, NCCS and LCCS failure occurrence

\( \bar{\tau} \): Average repair times

\( \hat{\tau}, \hat{\bar{Y}}, \hat{\bar{W}} \): Estimates for individual, NCCS and LCCS failure rates of occurrence respectively

\( \hat{\tau} \): Estimate of repair times

\( n \): Simple size

\( N \): Simulated samples

\( M S E \): Mean Square Error

### 4. STOCHASTIC MODEL

Under the stated assumptions Markov model can be formulated to derive the mean time between failure function \( E(T) \) under the influence of individual as well as CCS failures and the state transition diagram is given in Figure.1. The numerals in Figure.1 denote the system state.

The quantities in the above model are as follows:

\( \lambda_1 = 2(\lambda + \beta pq) \)

\( \lambda_2 = (\beta p^2 + \omega) \)

\( \lambda_3 = (\lambda + \beta \mu) \)

**Mathematical model**

Based on the arguments of stochastic theory, we can develop the set of differential equations associated with the existing mathematical model for the above mentioned state transition diagram.

\[
P_0(t) = -(\lambda_1 + \lambda_3)P_0(t) + \mu_1P_1(t) + \mu_2P_2(t) \quad (1)
\]

\[
P_1(t) = \lambda_1P_0(t) - (\lambda_2 + \mu_1)P_1(t) + \mu_2P_2(t) \quad (2)
\]

\[
P_2(t) = \lambda_2P_0(t) + \lambda_3P_1(t) - (\mu_2 + \mu_3)P_2(t) \quad (3)
\]

Initial conditions, \( P_0(0) = 1 \), and other state probabilities are zero at \( t = 0 \)

Taking Laplace transformation of equations (1) to (3) and using equation (4), we obtain

\[
P_0(t) = \frac{r_1^2 + r_1 t + m_1}{r_1 (r_1 - r_2)} \exp(r_1 t) - \frac{r_1^2 + r_2 t + m_1}{r_2 (r_1 - r_2)} \exp(r_2 t) + \frac{m_1}{r_2 r_1} \quad (5)
\]

\[
P_1(t) = \frac{r_1^2 + r_2 t + m_2}{r_1 (r_1 - r_2)} \exp(r_1 t) - \frac{r_2^2 + r_2 t + m_2}{r_2 (r_1 - r_2)} \exp(r_2 t) + \frac{m_2}{r_2 r_1} \quad (6)
\]

\[
P_2(t) = \frac{r_1^2 + r_1 t + m_3}{r_1 (r_1 - r_2)} \exp(r_1 t) - \frac{r_2^2 + r_2 t + m_3}{r_2 (r_1 - r_2)} \exp(r_2 t) + \frac{m_3}{r_2 r_1} \quad (7)
\]

Where

\[ l_1 = \lambda_2 + \mu_1 + \mu_2 + \mu_3 \]

\[ l_2 = \lambda_3 \]

\[ l_3 = \lambda_1 \]

\[ m_1 = \lambda_2 \mu_3 + \mu_1 \mu_4 + \mu_2 \mu_3 \]

\[ m_2 = \lambda_1 \mu_2 + \lambda_3 \mu_3 + \lambda_1 \mu_2 \]

\[ m_3 = \lambda_1 \mu_2 + \lambda_2 \mu_3 + \lambda_2 \mu_2 \]

\[ r_1, r_2 = -(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3) \pm \sqrt{[ (\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3)^2 - 4(\mu_1 (\lambda_1 + \lambda_3 + \mu_1) + \mu_2 (\lambda_2 + \lambda_3 + \mu_2) + \mu_3 (\lambda_1 + \lambda_2 + \lambda_3 + \mu_3)]} \quad (8) \]

### 5. ESTIMATION OF MEAN TIME BETWEEN FAILURES

In this section, we have used maximum likelihood estimation approach to estimate the reliability measures such as mean time between failures of two unit repairable systems in the presence of NCCS and LCCS failures for both series and parallel cases.

Let the samples \( x_1, x_2, \ldots, x_n \); \( y_1, y_2, \ldots, y_n \) and \( w_1, w_2, \ldots, w_n \) with size ‘\( n \)’ representing times between individual, NCCS and LCCS failures which will obey exponential population respectively.

Let \( z_{11}, z_{12}, \ldots, z_{1n}; \quad z_{21}, z_{22}, \ldots, z_{2n} \) & \( z_{31}, z_{32}, \ldots, z_{3m} \) be ‘\( n \)’ number of times between repairs of the units with exponential population law.

Where, \( \hat{\lambda}, \hat{\beta}, \hat{\omega}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3 \) are the M L estimates of individual failure rate \( \lambda \), NCCS failure rate \( \beta \), LCCS failure rate \( \omega \) and repair rates of \( \mu_1, \mu_2, \mu_3 \) of the system respectively.
5.1 Series System
In this case, we have reliability function for two unit system with LCCS and NCCS failures as

\[ R_{LNS}(t) = P_0(t) = \frac{1}{(r_1 - r_2)} \left[ (\lambda_1 + r_1) \exp(r_1t) - (\lambda_2 + r_2) \exp(r_2t) \right] \]  
(9)

Where \( r_1, r_2 \) obtained in (8)

The mean time between failure function is

\[ E_{LNS}(T) = \int_0^\infty R_{LNS}(t) \, dt \]

\[ = \lambda_1 \left[ (\lambda_2 + r_2) \exp(r_2t) - (\lambda_1 + r_1) \exp(r_1t) \right] + \mu_1 \left[ (\lambda_2 + \lambda_1 + r_2) \exp(r_2t) - (\lambda_2 + \lambda_1 + r_1) \exp(r_1t) \right] \]  
(10)

Therefore, the M L estimate of MTBF function is given by

\[ \hat{E}_{LNS}(T) \]  
(14)

Where \( \hat{\lambda}, \hat{\beta}, \hat{\omega} \) and repairs \((\beta, \omega)\) from 5 to 30 at step 5

5.2 Parallel System
In this case, we have reliability function for two unit system with LCCS and NCCS failures as

\[ R_{LNP}(t) = \frac{1}{n!} \left[ (\lambda_1 + \lambda_2 + r_1) \exp(r_1t) - (\lambda_1 + \lambda_2 + r_2) \exp(r_2t) \right] \]  
(12)

\[ -(\lambda_1 + \lambda_2 + r_2) \exp(r_2t) \]
Where \( r_1 \) and \( r_2 \) are seen in (8)

The mean time between failure function is

\[ E_{LNP}(T) = \int_0^\infty R_{LNP}(t) \, dt \]

\[ = \lambda_1 \left[ (\lambda_2 + r_2) \exp(r_2t) - (\lambda_1 + r_1) \exp(r_1t) \right] \]  
(13)

In this case, the M L estimate of MTBF function is given by

\[ \hat{E}_{LNP}(T) \]  
(14)

6. SIMULATION
The suggested estimates of the MTBF functions by M LE method do not find analytic shape of density or distribution. Hence it is not likely to try or extend analytical verification properties of proposed M L estimates such as \( E_{LNS}(T), \hat{E}_{LNP}(T) \) of the present model. We have attempted empirical approach by Monte-Carlo simulation to generate large samples of different sizes and mean squared error of the estimates have computed. The various sample of sizes \( n \) from 5 to 30 at step 5 were developed empirically with specified parameter value of failure rate and used them to obtain MSE for various sizes of samples with simulations. The detailed analysis is tabulated and is seen in table 1 and table 2.

For illustration purpose by fixing a range of particular value of the rate of individual \((\lambda), \) NCCS failures \((\beta), \) LCCS failures \((\omega)\) and repairs \((\mu_1, \mu_2, \mu_c)\) for the sample size \( n=5 \) were simulated using C-Language and the sample estimates are computed for \( N=10,000,20,000,90,000 \) and mean squared error of the estimates are evolved for \( E_{LNS}(T), \hat{E}_{LNP}(T) \) which gives reasonably small even for the small samples of size \( n=5. \)

Therefore the research results established that maximum likelihood estimation approach is satisfactory to estimate reliability indices.

Table 1. Simulation results for MTBF function in the case of series system with \( \lambda = 0.1, \) \( \beta = 0.5, \) \( \omega = 0.1, \) \( \mu_1 = 1, \) \( \mu_2 = 1.5, \mu_c = 2, \) \( p = 0.7. \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Sample Size</th>
<th>( E_{LNS}(T) )</th>
<th>( \hat{E}_{LNP}(T) )</th>
<th>MSE</th>
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Table 2. Simulation results for MTBF function in case of parallel system with $\lambda = 0.1$, $\beta = 0.5$, $\omega = 0.1$, $\mu_1 = 1$, $\mu_2 = 1.5$, $p = 0.7$

<table>
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<tr>
<th>N</th>
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7. Discussion

This paper discusses the estimation method which could give appropriate estimation practice of the reliability indices with specific sign to lethal and non-lethal CCS failures. For this purpose, we considered a reliability model with repair of down system. We have derived maximum likelihood estimators of the reliability indices such as MTBF failures of the existing model. The estimates of MTBF function were derived for both series and parallel systems. The performance of the proposed estimates has been developed in provisions of MSE using simulation. The simulation result suggest that M L estimate is sensibly very good and give exact estimate even for n=5. When $n$ tending to large MSE is zero in all cases almost.

REFERENCES


