Effect of Ecological Fluctuations on a Commensal-Host Model with Harvesting

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Abstract

The focus of this study is the dynamics of interaction between commensal and host species in an aquatic ecological system with harvesting, and stochastic attributes. The progress and the stability of the system, as well as the dynamics of the commensal host relationship are examined. The stabilities at steady states are also examined. We discuss the possible harvesting strategies described by the various attributes. The possibility of existence of bio-economic equilibrium with optimal scheme is being discussed. We provided the analytical estimates of the population intensities of fluctuations by Fourier transform methods through stochastic perturbation. Some numerical simulations are also carried out to make lawful the analysis.

.**Key words:** Commensal -Host, steady states, Routh-Hurwitz criteria, Global stability, Bionomic harvesting, optimal harvesting, Pontriyagin's principle, Stochastic.

1. Introduction:

Ecology is the study of relationships between living organisms and their environment. Investigations in the discipline of theoretical ecology were initiated by Lotka [1] and by Volterra [2], several researchers contributed to the enlargement of this area of acquaintance has been expansively accounted in the dissertations of Meyer [3], Kushing [4], Paul colinvaux [5], Kapur [6,7] etc,. The biological dealings can be generally classified as Prey – predation, competition, commensalism, Ammensalism, Neutralism and so on.

The present investigation is devoted to the analytical study of commensalism between two species. A two species Commensalism is an ecological relation ship between two species where one species derives a benefit from the other which does not get affected by it. S_1 may be referred as the commensal species while S_2 the host. Some examples are Cattle Egret, Anemonetish, Barnacles etc.

The host species (S_2) supports the commensal species (S_1) which has a natural death rate in spite of a support other than from S_1. The commensal species (S_1) is assumed to be constitutionally so weak that it would not flourish, in fact it declines in spite of the support extended by the host (S_2).The present model is characterized by a coupled pair of first order non-linear differential equations. In all three equilibrium points of the system are identified and the stability analysis is carried out. It is noticed that the state in which host survives and the commensals are washed out and the co-existent state are stable states that too under the conditions, stated there in. The first equilibrium state is unstable. The linearised perturbed equations are solved and the trajectories are illustrated.

2. Basic Mathematical Model:

Consider a three species prey-predator-competitor model

$$\frac{\partial N_1}{dt} = N_1\left[-a_1 - \alpha_{11}N_1 + \alpha_{12}N_2 - q_1E_1\right] \tag{2.1}$$

$$\frac{\partial N_2}{dt} = N_2\left[a_2 - \alpha_{22}N_2 - q_2E_2\right] \tag{2.2}$$

where $N_1 = N_1(x,t)$ represents the biomass density of commensal species S_1, $N_2 = N_2(x,t)$ represents the biomass density of host species S_2, x denotes the space variable t denotes the time variable. a_1 represents the death rates of S_1, a_2 represents the natural growth rates of S_2, α_{11} represents the rate of decrease of S_1 due to limitations of natural resources, α_{12}: increase rate of the Commensal due to inhibition by the Host , q_1 represents the catch ability coefficient of S_1 species, q_2 represents the catch ability coefficient of S_2 species E_1 represents the effort applied to harvest the S_1 species, E_2 represents the effort applied to harvest the S_2 species. We are assuming that $a_2 - q_2E_2 > 0$.

3. Stability and equilibrium analysis:

The model equations of the structure with $D_i = 0; i = 1, 2$ are symbolized as set of three non linear differential equations as follows:

$$N_1' = -a_1N_1 - \alpha_{11}N_1^2 + \alpha_{12}N_1N_2 - q_1E_1N_1 \tag{3.1}$$

$$N_2' = a_2N_2 - \alpha_{22}N_2^2 - q_2E_2N_2 \tag{3.2}$$

The possible equilibrium points are $E_1(0,0)$, $E_2(N_1^\phi, 0)$, $E_3(0, N_2^\psi)$ and $E_4(N_1^*, N_2^*)$.

Case (i): $E_1(0,0)$: This equilibrium point always exist.

Case (ii) : $E_2(N_1^\phi, 0)$ (In the absence of host species): If N_1^ϕ is the positive solution of $N_1' = 0$, then $N_1^\phi = -(1/\alpha_{11})\{a_1 + q_1E_1\}$ provided $a_1 + q_1E_1 < 0$. But this is not possible. Since all parameters are assumed to be positive, therefore, $a_1 + q_1E_1$ never be negative. Hence this equilibrium point does not exist.

Case(iii): $E_3(0, N_2^\psi)$ (In the absence of commensal species): If N_2^ψ is the positive solution of $N_2' = 0$, then $N_2^\psi = -(1/\alpha_{22})\{a_2 - q_2E_2\}$. This point is said to be positive, if $q_2E_2 > a_2$ that is the product of the catchability coefficient and effort applied to harvest the host species must be greater than their natural growth rate.

Case (iv): $E_4\left(N_1^*, N_2^*\right)$: (The interior equilibrium): If N_1^* and N_2^* are positive solutions of $N_1' = 0$ and $N_2' = 0$, then $N_1^* = (1/\alpha_{11})\left\{(-a_1 - q_1 E_1) + (\alpha_{12}/\alpha_{22})(a_2 - q_2 E_2)\right\}$,   and $N_2^* = (1/\alpha_{22})(a_2 - q_2 E_2)$. N_1^* is said to be positive, if $\alpha_{12}(a_2 - q_2 E_2) > \alpha_{22}(a_1 + q_1 E_1)$

4. Swot of Local Stability:

To ascertain the local steadiness character of the interior equilibrium $E_2\left(N_1^*, N_2^*\right)$, we work

out the variational matrix about E_2 $J\left(N_1, N_2\right) = \begin{bmatrix} -\alpha_{11} N_1^* & \alpha_{12} N_1^* \\ 0 & -\alpha_{22} N_2^* \end{bmatrix}$ (4.1)

The characteristic equation of the given ecological scheme at the interior equilibrium $E_4\left(N_1^*, N_2^*\right)$ is $\lambda^2 + \left(\alpha_{11} N_1^* + \alpha_{22} N_2^*\right)\lambda + \alpha_{11}\alpha_{22} N_1^* N_2^* = 0$ (4.2)

The sum and product of roots of (4.2) are $\lambda_1 + \lambda_2 = -\left(\alpha_{11} N_1^* + \alpha_{22} N_2^*\right) < 0$ and $\lambda_1 \lambda_2 = \alpha_{11}\alpha_{22} N_1^* N_2^* > 0$ respectively. Hence the steady state is stable. $E_2\left(N_1^*, N_2^*\right)$ is locally asymptotically stable.

5. Testing of Global Stability:

Theorem: The equilibrium point $E_2\left(N_1^*, N_2^*\right)$ is globally asymptotically stable.

Proof: Let us consider the subsequent Lyapunov function

$$V(N_1, N_2) = \left[(N_1 - N_1^*) - N_1^* \ln(N_1/N_1^*)\right] + l_1\left[(N_2 - N_2^*) - N_2^* \ln(N_2/N_2^*)\right]$$ (5.1)

where l_1 is the positive constant.

$$\frac{dV}{dt} = \left(\frac{(N_1 - N_1^*)}{N_1}\right)\frac{dN_1}{dt} + l_1\left(\frac{(N_2 - N_2^*)}{N_2}\right)\frac{dN_2}{dt}$$

$$\frac{dV}{dt} = (N_1 - N_1^*)\left\{\alpha_{11} N_1^* - \alpha_{12} N_2^* - \alpha_{11} N_1 + \alpha_{12} N_2\right\} + l_1(N_2 - N_2^*)\left\{\alpha_{22} N_2^* - \alpha_{22} N_2\right\}$$

$$\frac{dV}{dt} \le -\left(\alpha_{11} - \frac{\alpha_{12}}{2}\right)\left(N_1 - N_1^*\right)^2 - \left(l_1\alpha_{22} - \frac{\alpha_{12}}{2}\right)\left(N_2 - N_2^*\right)^2$$

$$\frac{dV}{dt} \le 0 \text{ provided } l_1\alpha_{22} - \frac{\alpha_{12}}{2} > 0 \text{ and } \alpha_{11} - \frac{\alpha_{12}}{2} > 0$$

The equilibrium point $E_2\left(N_1^*, N_2^*\right)$ is globally asymptotically stable

6. Bionomic equilibrium

The bionomic equilibrium is not anything but which is the grouping of the perceptions of biological as well as economic equilibriums. A biological equilibrium is given by

$N_1' = 0$; $N_2' = 0$. The economic equilibrium is said to be accomplished when the total income acquired by selling the harvested biomass equals the total cost for the effort devoted to harvesting. Let c_1, c_2 be harvesting cost per unit effort of the prey and predator respectively. Let p_1, p_1 be price per unit biomass of the commensal and host respectively. Consequently, net income or financial rent at any time given by $M = \sum_{i=1,2} M_i$. Where $M_i = (p_i q_i N_i - c_i) E_i$ are the net economic revenue for the commensal and host at any time t. The bionomic equilibrium $\left((N_1)_\infty, (N_2)_\infty, (E_1)_\infty, (E_2)_\infty\right)$ is given by the subsequent equations

$$-a_1 - \alpha_{11} N_1 + \alpha_{12} N_2 - q_1 E_1 = 0 \tag{6.1}$$

$$a_2 - \alpha_{22} N_2 - q_2 E_2 = 0 \tag{6.2}$$

$$M = \sum_{i=1,2} (p_i q_i N_i - c_i) E_i \tag{6.3}$$

With the intention of establishing the bionomic equilibrium we come across the subsequent cases.

Case (a): If $c_1 > p_1 q_1 N_1$, $c_2 > p_2 q_2 N_2$ then the cost is greater than revenues for both the species and the whole system will be closed.

Case (b): If for the host, harvesting cost is greater than the revenue $(c_2 > p_2 q_2 N_2)$, and then harvesting of host is not practicable. Hence harvesting of commensal population remains operational $(c_1 > p_1 q_1 N_1)$. Thus, when $E_2 = 0$ and $(c_1 > p_1 q_1 N_1)$ we have

$$(N_1)_\infty = \frac{c_1}{p_1 q_1}, \quad (N_2)_\infty = \frac{1}{\alpha_{12}} \left(a_1 + \alpha_{11} \frac{c_1}{p_1 q_1} + q_1 E_1 \right) \tag{6.4}$$

Case (c): If the cost is greater than the revenue in the commensal harvesting, then the commensal harvesting will be closed (i.e. $E_1 = 0$). Only host harvesting remains operational.

$$(N_2)_\infty = \frac{c_2}{p_2 q_2}, \quad (N_1)_\infty = \frac{1}{\alpha_{22}} (a_2 - q_2 E_2) \tag{6.5}$$

Case (iv): If $c_1 < p_1 q_1 N_1$, $c_2 < p_2 q_2 N_2$, then the revenues for both the species being positive, then the whole system will be in operation. In this case $(N_1)_\infty = \dfrac{c_1}{p_1 q_1}$ $\tag{6.6}$

and $\qquad (N_2)_\infty = \dfrac{c_2}{p_2 q_2}$ $\tag{6.7}$

Substitute (6.6) and (6.7) in (6.1), (6.2) we get

$$\left(E_1\right)_\infty = \frac{1}{q_1}\left(-a_1 - \alpha_{11}\frac{c_1}{p_1 q_1} + \alpha_{12}\frac{c_2}{p_2 q_2}\right) \tag{6.8}$$

$$\left(E_2\right)_\infty = \frac{1}{q_2}\left(a_2 - \alpha_{22}\frac{c_2}{p_2 q_2}\right) \tag{6.9}$$

$$\left(E_1\right)_\infty > 0 \quad \text{if} \quad \alpha_{12}\frac{c_2}{p_2 q_2} > \left(a_1 + \alpha_{11}\frac{c_1}{p_1 q_1}\right) \tag{6.10}$$

$$\left(E_2\right)_\infty > 0 \quad \text{if} \quad a_2 > \alpha_{22}\frac{c_2}{p_2 q_2} \tag{6.11}$$

The Non-trivial Bionomic equilibrium point $\left((N_1)_\infty,(N_2)_\infty,(E_1)_\infty,(E_2)_\infty\right)$ exists, if (6.10) and (6.11) must hold.

7. Analysis of environmental fluctuations:

The primary idea that leads us to widen the deterministic model (2.1)-(2.2) to a stochastic identical part is that it is practical to imagine the open system as noisy surrounding. There are a number of ways in which the located 'noise' may be included in the system (2.1)-(2.2). This reminds that the environmental noise should be distinguished from internal noise, for which the variation over time is due. External noise may arise either from random fluctuations of one or more model parameters around some known mean values or from stochastic fluctuations of the population densities around some constant values. In this part, we work out the population intensities of fluctuations (variances) around the positive equilibrium E_4 due to noise, according to the method introduced by R.M.Nisbet and W.S.C.Gurney [8] in 1982. Later many authors [8-9] studied about the effects of environmental fluctuations of various ecological models.Now we assume the presence of a randomly fluctuating driving force on the deterministic growth of the species S_1 and S_2 (commensal species with mortality rate and host species with intrinsic growth rate) at time t, so that the system (3.1)-(3.2) results in the stochastic system with 'additive noise' as follows:

$$\frac{dN_1}{dt} = -a_1 N_1 - \alpha_{11} N_1^2 + \alpha_{12} N_1 N_2 - q_1 E_1 N_1 + \alpha_1 \xi_1(t) \tag{7.1}$$

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 - q_2 E_2 N_2 + \alpha_2 \xi_2(t) \tag{7.2}$$

where α_1, α_2 are real constants and $\xi(t) = \left[\xi_1(t),\xi_2(t)\right]$ is a two dimensional Gaussian White noise process agreeable $E\left[\xi_i(t)\right] = 0; i = 1,2;\quad E\left[\xi_i(t)\xi_j(t')\right] = \delta_{ij}\delta(t-t'); i = j = 1,2.$ where δ_{ij} is the Kronecker symbol; δ is the delta-dirac function. In this analysis, we focus on the dynamics of the model (7.1)-(7.2) at the equilibrium point only. So we compute the population variances around $E_4\left(N_1^*,N_2^*\right)$.Let $N_1(t) = u_1(t) + S^*$; $N_2(t) = u_2(t) + P^*$; then we centre the system (7.1)-(7.2) on $E_4\left(N_1^*,N_2^*\right)$ and consider only the linear terms, so that we consider only the consequence of linear stochastic perturbations. Hence the model (7.1)-(7.2) reduces to the following linear system

$$\frac{du_1(t)}{dt} = -\alpha_{11}u_1(t)S^* + \alpha_{12}u_2(t)S^* + \alpha_1\xi_1(t) \tag{7.3}$$

$$\frac{du_2(t)}{dt} = -\alpha_{22}u_2(t)P^* + \alpha_2\xi_2(t) \tag{7.4}$$

Taking the Fourier transform on both sides of (7.3) and (7.4) we get,

$$\alpha_1\tilde{\xi}_1(\omega) = \left(i\omega + \alpha_{11}S^*\right)\tilde{u}_1(\omega) - \alpha_{12}S^*\tilde{u}_2(\omega) \tag{7.5}$$

$$\alpha_2\tilde{\xi}_2(\omega) = \left(i\omega + \alpha_{22}P^*\right)\tilde{u}_2(\omega) \tag{7.6}$$

The matrix form of (7.5) and (7.6) is $M(\omega)\tilde{u}(\omega) = \tilde{\xi}(\omega)$ \hfill (7.7)

where $M(\omega) = \begin{pmatrix} A(\omega) & B(\omega) \\ C(\omega) & D(\omega) \end{pmatrix}$; $\tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \end{bmatrix}$; $\tilde{\xi}(\omega) = \begin{bmatrix} \alpha_1\tilde{\xi}_1(\omega) \\ \alpha_2\tilde{\xi}_2(\omega) \end{bmatrix}$;

$$A(\omega) = i\omega + \alpha_{11}S^*; B(\omega) = -\alpha_{12}S^*; C(\omega) = 0; D(\omega) = i\omega + \alpha_{22}P^* \tag{7.8}$$

Hence the solution of (7.7) is given by $\tilde{u}(\omega) = \left[M(\omega)\right]^{-1}\tilde{\xi}(\omega)$

where $K(\omega) = \left[M(\omega)\right]^{-1} = \begin{bmatrix} \dfrac{D(\omega)}{|M(\omega)|} & -\dfrac{B(\omega)}{|M(\omega)|} \\ -\dfrac{C(\omega)}{|M(\omega)|} & \dfrac{A(\omega)}{|M(\omega)|} \end{bmatrix}$ \hfill (7.9)

we now describe the some of the fundamental results of random population function. If the function $Y(t)$ has a zero mean value, then the fluctuation intensity (variance) of its components in the frequency interval $[\omega, \omega + d\omega]$ is $S_Y(\omega)d\omega$, where $S_Y(\omega)$ is spectral density of Y and is defined as $S_Y(\omega) = \lim\limits_{T\to\infty} \dfrac{|\tilde{Y}(\omega)|^2}{\tilde{T}}$ \hfill (7.10)

If Y has a zero mean value, the inverse transform of $S_Y(\omega)$ is the auto covariance function

$$C_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)e^{i\omega\tau}d\omega \tag{7.11}$$

The related variance of fluctuations in $Y(t)$ is $\sigma_Y^2 = C_Y(0) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega$ \hfill (7.12)

and the auto correlation function is the normalized auto covariance $P_Y(\tau) = \dfrac{C_Y(\tau)}{C_Y(0)}$ \hfill (7.13)

For a Gaussian white noise process, it is $S_{\xi_i\xi_j}(\omega) = \lim\limits_{\hat{T}\to+\infty} \dfrac{E\left[\tilde{\xi}_i(\omega)\tilde{\xi}_j(\omega)\right]}{\hat{T}}$

$$= \lim_{\hat{T}\to+\infty} \frac{1}{\hat{T}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} E\left[\tilde{\xi}_i(t)\tilde{\xi}_j(t')\right] e^{-i\omega(t-t')} dt\,dt'$$

$$= \delta_{ij} \tag{7.14}$$

From (7.9), we have
$$\tilde{u}_i(\omega) = \sum_{j=1}^{2} K_{ij}(\omega)\tilde{\xi}_j(\omega);\ i=1,2 \tag{7.15}$$

From (7.10) we have
$$S_{u_i}(\omega) = \sum_{j=1}^{2} \alpha_j \left|K_{ij}(\omega)\right|^2;\ i=1,2 \tag{7.16}$$

Hence by (7.12) and (7.16), the intensities of fluctuations in the variable u_i; $i=1,2$ are given

by $\sigma_{u_i}^{\ 2} = \dfrac{1}{2\pi}\sum\limits_{j=1}^{2}\int\limits_{-\infty}^{\infty} \alpha_j \left|K_{ij}(\omega)\right|^2 d\omega;\ i=1,2 \hfill (7.17)$

and by (7.9), we obtain $\sigma_{u_1}^{\ 2} = \dfrac{1}{2\pi}\left\{\int\limits_{-\infty}^{\infty}\alpha_1\left\|\dfrac{D(\omega)}{M(\omega)}\right\|^2 d\omega + \int\limits_{-\infty}^{\infty}\alpha_2\left\|\dfrac{B(\omega)}{M(\omega)}\right\|^2 d\omega\right\}$

$$\sigma_{u_2}^{\ 2} = \frac{1}{2\pi}\left\{\int\limits_{-\infty}^{\infty}\alpha_1\left\|\frac{A(\omega)}{M(\omega)}\right\|^2 d\omega + \int\limits_{-\infty}^{\infty}\alpha_2\left\|\frac{C(\omega)}{M(\omega)}\right\|^2 d\omega\right\} \tag{7.18}$$

where $\left|M(\omega)\right| = R(\omega) + iI(\omega)$ $\hfill (7.19)$

Real part of $\left|M(\omega)\right| = R(\omega) = -\omega^2 + \alpha_{11}\alpha_{22}S^*P^*$ $\hfill (7.20)$

Imaginary part of $\left|M(\omega)\right| = I^2(\omega) = \omega(\alpha_{11}S^* + \alpha_{22}P^*)$ $\hfill (7.21)$

Finally from (7.8), we get

$$\left|A(\omega)\right|^2 = \omega^2 + (\alpha_{11}S^*)^2;\left|B(\omega)\right|^2 = (\alpha_{12}S^*)^2;\left|C(\omega)\right|^2 = 0;\left|D(\omega)\right|^2 = \omega^2 + (\alpha_{22}P^*)^2 \tag{7.22}$$

$$\sigma_{u_1}^{\ 2} = \frac{1}{2\pi}\left\{\int\limits_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[\alpha_1\left\{\omega^2 + (\alpha_{22}P^*)^2\right\} + \alpha_2(\alpha_{12}S^*)^2\right]d\omega\right\} \tag{7.23}$$

$$\sigma_{u_2}^{\ 2} = \frac{1}{2\pi}\left\{\int\limits_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[\alpha_1\left\{\omega^2 + (\alpha_{11}S^*)^2\right\}\right]d\omega\right\} \tag{7.24}$$

If we are interested in the dynamics of system (7.1)-(7.2) with either $\alpha_1 = 0$ or $\alpha_2 = 0$ then

the population variances are $\quad \sigma_{u_1}^{\ 2} = \dfrac{\alpha_2 (\alpha_{12} S^*)^2}{2\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{1}{R^2(\omega) + I^2(\omega)} d\omega$; $\sigma_{u_2}^{\ 2} = 0$ if $\alpha_1 = 0$ $\qquad$ (7.25)

and $\sigma_{u_1}^{\ 2} = \dfrac{\alpha_1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{1}{R^2(\omega) + I^2(\omega)} \Big[\omega^2 + (\alpha_{22} P^*)^2 \Big] d\omega$; $\sigma_{u_2}^{\ 2} = \dfrac{\alpha_1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{1}{R^2(\omega) + I^2(\omega)} \Big[\omega^2 + (\alpha_{11} S^*)^2 \Big] d\omega$ (7.26)

if $\alpha_2 = 0$.

The population variances given in (7.18) point out the stability of population for smaller values of mean square fluctuations, while the larger values of population variances indicate the instability of the populations. The integrals in (7.18) can be evaluated both analytically and numerically.

9. Computer simulations:

Evaluation of integrals in (7.18) is very complex, but it can establish numerically for diverse set of parameters. The three variances in (7.23-7.24) stand for the mean square fluctuations of the population. When the variances are very not as much of, it can be easily observed that the system is stable, otherwise unstable. We visualized these results in the computer simulation. The exclusive of numerical authentication of the outcomes in view of stochastic (fig: 1-5), are portrayed for the given set of parameters.
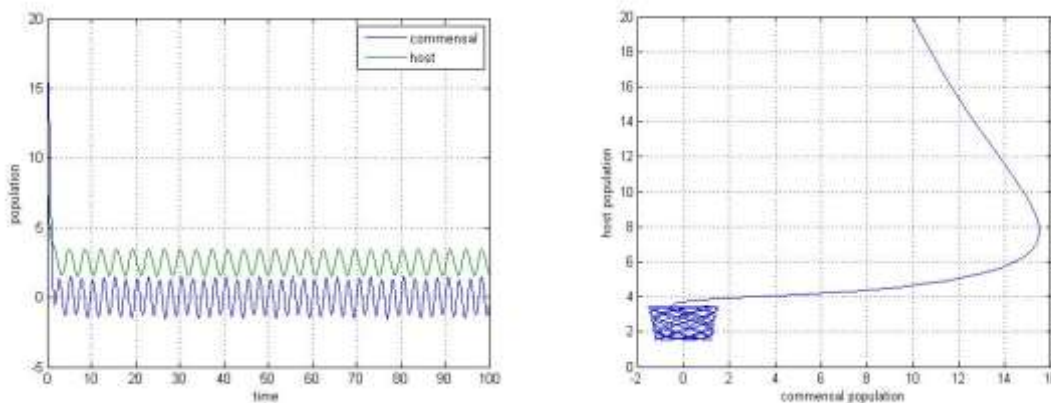

Figure (1)
a1=3.5;a11=0.01;a12=0.5;q1=0.02;E1=25;omga=2.5;
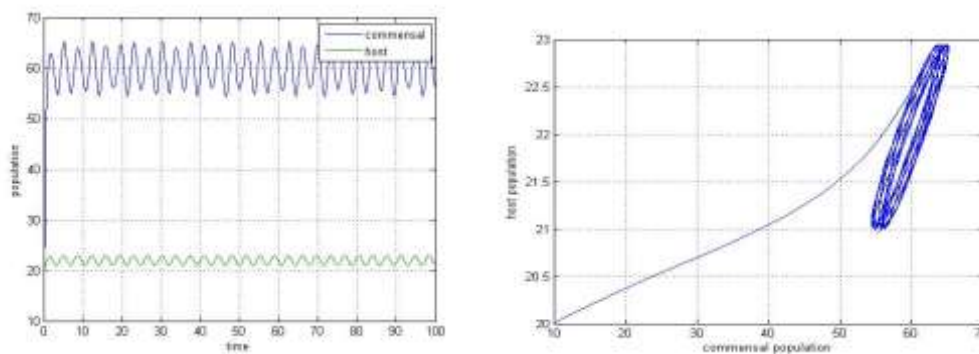a2=1.75;a22=0.5;q2=0.02;E2=20;gama=1.75;

Figure (2)
a1=4.5;a11=0.1;a12=0.5;q1=0.02;E1=25;omga=2.5;
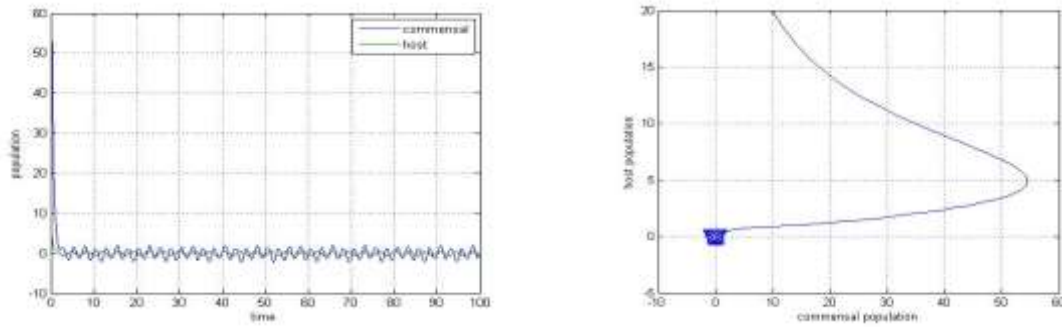
```
a2=1.5;a22=0.05;q2=0.02;E2=20;gama=1.75;
```


Figure (3)
```
a1=1.5;a11=0.1;a12=1.5;q1=0.02;E1=25;omga=2.5;
   a2=1.5;a22=0.5;q2=0.2;E2=20;gama=1.75;
```
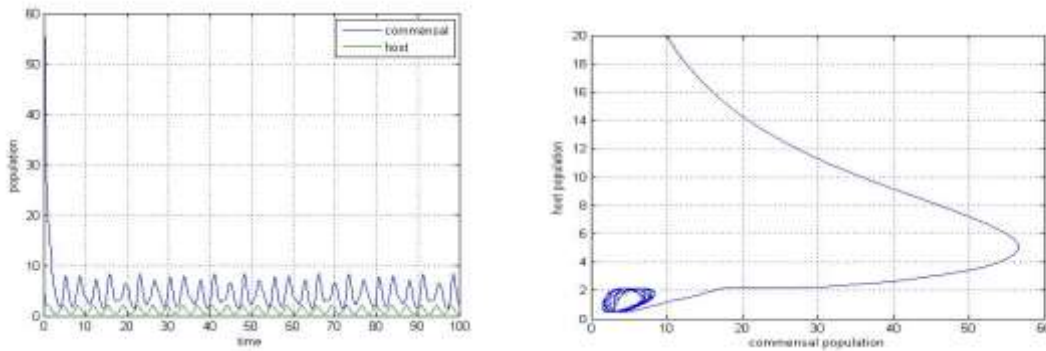

Figure (4)
```
a1=1.5;a11=0.1;a12=1.5;q1=0.02;E1=25;omga=2.5;
   a2=1.5;a22=0.5;q2=0.2;E2=20;gama=1.75;
```
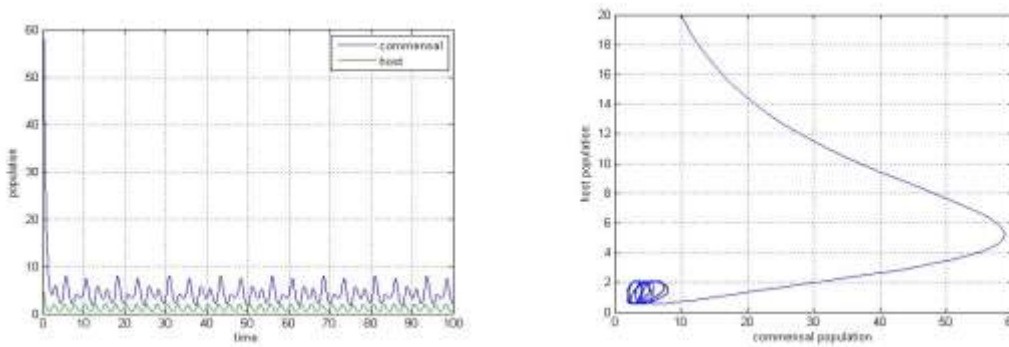

Figure (5)
```
a1=1.5;a11=0.1;a12=1.5;q1=0.02;E1=25;omga=1.5;
   a2=1.5;a22=0.5;q2=0.2;E2=20;gama=2.5;
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References

[1] Lotka A.J., Elements of Physical Biology, Williams & Wilking, Baltimore, 1925

[2] Volterra V., Leconssen La Theorie Mathematique De La Leitte Pou Lavie, Gauthier

   Villars, Paris, 1931.

[3] Meyer W.J., Concepts of Mathematical Modeling Mc. Grawhill, 1985

[4] [KushingJ.M., Integro-Differential Equations and Delay Models in Population Dynamics, Lecture Notes in Bio-Mathematics, Vol.20, Springer Verlag, 1977

[5] Paul Colinvaux A., Ecology, John Wiley, New York, 1986.

[6] KapurJ.N.,Mathematical Modellingin Biology andMedicine,AffiliatedEastWest,1985.

[7] KapurJ.N,MathematicalModelling,Wiley Easter,1985

[8] R.M. Nisbet, W.S.C. Gurney, Modelling Fluctuating Populations, John Wiley, New York, 1982

[9] Kalyan Das, M.N. Srinivas, M.A.S. Srinivas, N.H. Gazi. Chaotic dynamics of a three species prey-predator competition model with bionomic harvesting due to delayed environmental Noise as external driving force. C R Biologies-Elsevier, 2012, 335: 503 - 513.

[10] M.Carletti, Numerical solution of stochastic differential problems in the biosciences, J. Comput. Appl. Math. 185 (2006) 422–440.