Limitations to the Growth Rate of a Disturbance in Maxwell Ferromagnetic Convection in a Densely Packed Porous Medium

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Abstract: The present paper describes the analytical limitations of the complex growth rate $\omega = \omega_r + \iota\omega_i$ of a randomly oscillating movement with an increasing amplitude in Maxwell ferromagnetic convection in a densely packed porous medium for the case of free-free and rigid-rigid boundaries. The eigen value problem is derived from the governing equations by employing normal mode analysis and linearized stability theory. It is found that, for free-free boundaries, the growth rate of disturbances must lie within a semicircle in the right half plane- $\omega_r\omega_i$ of ω, with its center located at origin and radius is $\sqrt{max\left\{\epsilon\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right), \frac{\epsilon}{k_1 G}\right\}}$. On the other hand, the upper bound for rigid-rigid boundaries is obtained as $|\omega|^2|\omega_i|^2 < max\left\{\epsilon^2\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right)^2, \left(\frac{\epsilon}{k_1 G}\right)^2\right\}$, where ω_r is real part of ω, ω_i is imaginary part of ω, M_1 denotes the magnetic number, R is Rayleigh number, G is stress relaxation parameter, P_r is Prandtl number, ϵ denotes the medium porosity and k_1 represents the medium permeability. The previous results are also recovered as special cases for classical ferromagnetic fluid flow.
Keywords: Maxwell ferrofluid, Linear stability, Growth rate, Upper bounds.

1. Introduction

Ferromagnetic fluids consist of magnetic nanosized particles dispersed in a carrier liquid, forming colloidal suspensions commonly known as magnetic fluids. These fluids respond to magnetic fields, generating distinct patterns and structures. To minimize agglomeration and increase stability, these nano-particles are coated with a surfactant. The carrier fluid can be any liquid that is compatible with the surfactant coating, such as water or oil. Superparamagnetism, magnetization saturation, and high magnetic susceptibility are all characteristics of ferromagnetic fluids. The potential future applications arise from the alterations in viscosity observed in magnetic fluids under the influence of magnetic field and shear dependence. These properties make ferromagnetic fluids valuable in many industrial and practical applications, such as pressure seals of blowers and compressors, dampers, shock absorbers, magnetic drug delivery, magnetic resonance imaging (MRI), soft robotics, microfluidics, avionics, and actuators [1-2].

Over the past few years, there has been increasing interest among researchers in the field of ferrofluids, particularly in examining the stability/instability in ferroconvective configurations. Finlayson [3] explored the convective instability within magnetic fluids subjected to heating from below while being exposed to a vertically acting magnetic field. In his analysis, he investigated the instability in the presence/absence of gravitational force, and found an accurate and approximate solutions for free and stiff boundaries respectively. Lalas and Carmi [4] examined the energy stability of Boussinesq ferromagnetic fluid in the presence of gravitational field, heat gradient and magnetic field gradient. Shliomis [5] investigated the thermo-convective instability exhibited by the free surface of a liquid when exposed to an external magnetic field. He also explored the relaxation processes of magnetization in a suspension, emphasizing the importance of Néel fluctuation mechanism and rotational Brownian motion of particles. Schwab et al. [6] explored the effect of a vertically oriented magnetic field on Bénard convectional problem in ferromagnetic fluid. Further, Stiles and Kagan [7] enhanced and expanded this study by considering the effect of colloid concentration and temperature under the influence of an intense magnetic field. Prakash et al. [8] explored convective instability in a ferrofluid layer through sparsely distributed porous medium in the presence of rotation and MFD viscosity. Sekar and Raju [9] investigated the thermal convective instability in micropolar ferrofluid under a transverse magnetic field. The study of Bénard convection problems in ferrofluid has been extensively explored by many researchers. Sharma [10] examined the thermosolutal convection occurring in a ferromagnetic fluid layer that saturates a porous medium, heated from below, and is subjected to a uniformly distributed magnetic field. To obtain more comprehensive information about these investigations, one may refer to [11-17]. N. Guerroudj et al. [18] investigated the convection of a ferromagnetic fluid in a vertical channel with different-shaped porous blocks by using numerical simulations.

Despite being extensive research on thermal convection in Newtonian ferrofluids, comparatively little emphasis has been placed on studying this phenomenon in non-Newtonian ferrofluids like Maxwell ferrofluid, Rivlin-Ericksen fluid, Jeffrey ferrofluid. These non-Newtonian ferrofluids display distinctive rheological properties, offering potential for significant advancements in diverse fields including advanced material science, biomedicine and microfluidics. Ijaz and Ayub [19] investigated the behavior of a Maxwell ferromagnetic fluid in the presence of magnetic dipole and dual stratification effects. A. Majeed et al. [20] examined how a rotating magnetic field influences the flow of Maxwell ferrofluid over a thermally stretching sheet with heat generation or absorption. Sudhir et al. [21] employed linear stability theory to investigate thermal convection within a Maxwell ferromagnetic fluid layer that saturates in a porous medium. Nabwey et al. [22] investigated that Maxwell ferromagentic fluid exhibits higher temperature fluctuations compared to Jeffrey ferromagnetic fluid but the heat diffusion in Maxwell ferromagnetic fluid is less gradual than Jeffrey ferromagnetic fluid.

The task of establishing the upper bounds is equally important in order to provide experimenters and computational analysts with improved estimations of growth rate of any randomly oscillating movement of rising or neutral amplitude. Banerjee et al. [23] established the novel technique to determine the upper limits for the growth rate of disturbances in thermohaline convection. Prakash [24], Prakash and Gupta [25] also determined upper limits for growth rates of perturbations in various ferroconvective configurations. Recently, K. Ram et al. [26] investigated the complex growth rate of perturbations in a thermohaline ferroconvection within a densely packed porous medium, considering that the viscosity is dependent on magnetic field strength. To the best of our knowledge, the complex growth rate of disturbances in Maxwell ferromagnetic convection through a densely packed porous medium has not been examined yet. The aim of the present communication is to analytically investigate the complex growth rate in Maxwell ferromagnetic convection. This investigation will provide deep insights into the stability of these perturbations and drive the development of more efficient and reliable engineering solutions.

2. Problem Description

Consider an incompressible and electrically non conducting Maxwell ferrofluid layer confined statically in a densely packed porous medium, with infinite horizontal extension. The ferrofluid layer has a thickness 'd' and is subjected to heating from below. The lower and upper boundary surfaces are kept at constant temperatures, denoted as T_0 and T_1 respectively, creating a uniform temperature gradient $\beta = \left|\frac{dT}{dz}\right|$. The entire system is influenced by the gravitational field, represented as $\vec{g} = -g\hat{k}$. Figure 1 displays the geometrical setup of the problem

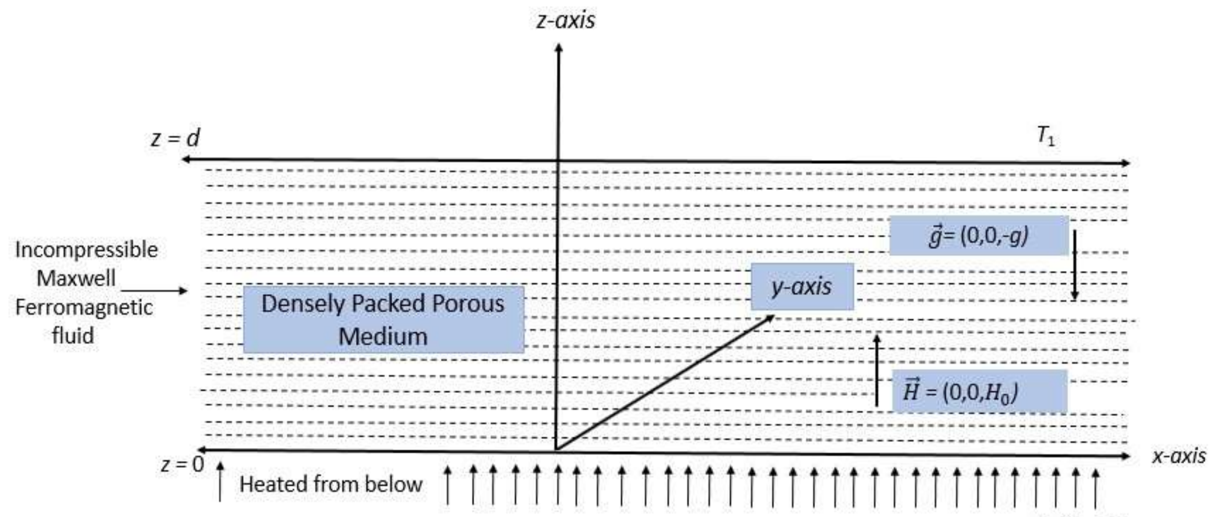

Figure 1 : Geometrical Configuration

The basic equations which govern the Maxwell ferroconvection in a densely packed porous medium are:

The equation of continuity is

$$\nabla \cdot \vec{q} = 0. \tag{1}$$

The equation of motion is

$$\frac{\rho_0}{\epsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\vec{q}}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left(-\nabla p+\rho\vec{g}+\nabla\cdot\left(\vec{H}\vec{B}\right)\right)-\frac{\mu}{k_1}\vec{q}. \tag{2}$$

The temperature equation is

$$\epsilon\left[\rho_0 C_{H,V}-\mu_0\vec{H}\cdot\left(\frac{\partial\vec{M}}{\partial T}\right)_{H,V}\right]\frac{dT}{dt}-(\epsilon-1)C_s\rho_s\frac{\partial T}{\partial t}+T\mu_0\left(\frac{\partial\vec{M}}{\partial T}\right)_{H,V}\cdot\frac{d\vec{H}}{dT}=K_1\nabla^2 T, \tag{3}$$

where $\vec{q}$ is velocity, α referred to coefficient of volume expansion, $\vec{g}=(0,0,-g)$ is gravitational acceleration, ϵ is medium porosity, $\rho=\rho_0[1+(T_0-T_1)\alpha]$ is fluid's density, k_1 is medium permeability, μ_0 is magnetic permeability, $\vec{H}$ represents magnetic field, p denoted pressure, λ is stress relaxation time, ρ_0 fluid's density at temperature T_0, μ denotes viscosity, $C_{H,V}$ is heat capacity at constant magnetic field and volume, C_s denotes heat capacity of the solid phase, ρ_s is the density of the solid phase, $\vec{M}$ is magnetization, T is temperature and K_1 is thermal conductivity.

When there is no displacement current in a non-conducting fluid, Maxwell equations are

$$\nabla\times\vec{H}=0, \tag{4a}$$

$$\nabla\cdot\vec{B}=0, \tag{4b}$$

where $\vec{B}$ is magnetic induction and is given by

$$\vec{B}=\mu_0\left(\vec{M}+\vec{H}\right). \tag{5}$$

Considering that magnetic field and magnetization are aligned parallelly, and magnetization varies with temperature and magnitude of the magnetic field, and is given by

$$\vec{M}=\left(\frac{\vec{H}}{H}\right)M(H,T). \tag{6}$$

The equation representing the linearized magnetization is given by

$$M=M_0+K_2(T_0-T)-\chi(H_0-H), \tag{7}$$

where M_0 denotes magnetization at a certain temperature T_0 and magnetic field intensity H_0, $K_2=-\left(\frac{\partial\vec{M}}{\partial T}\right)_{T_0,H_0}$ is the pyromagnetic coefficient and $\chi=\left(\frac{\partial\vec{M}}{\partial H}\right)_{T_0,H_0}$ is the magnetic susceptibility.

The basic state solutions are presented in the following manner

$$\vec{q}=\overrightarrow{q_b}=0,\ T=T_b(z)=T_0-\beta z,\rho=\rho_b(z),p=p_b(z),\beta=\frac{T_0-T_1}{d},$$
$$\overrightarrow{M_b}=\left(M_0+\frac{K_2\beta z}{1+\chi}\right)\hat{k},\overrightarrow{H_b}=\left(H_0-\frac{K_2\beta z}{1+\chi}\right)\hat{k},H_0+M_0=H_0^{ext} \tag{8}$$

where $\hat{k}$ represents the unit vector along z-axis. In this analysis, only the spatially varying components of M_0 and H_0 are considered, and convection is not dependent on the orientation of the external magnetic field.

By disturbing the stationary state, the perturbed equations are expressed as:

$$T=T_b(z)+\theta',\quad p=p_b(z)+p',\quad \vec{q}=\overrightarrow{q_b}+\overrightarrow{q'},$$
$$\rho=\rho_b(z)+\rho',\quad \vec{M}=\overrightarrow{M_b}(z)+\vec{M}',\quad \vec{H}=\overrightarrow{H_b}(z)+\overrightarrow{H'}, \tag{9}$$

where $\theta',p',\overrightarrow{q'}=(u',v',w'),\rho',\overrightarrow{M'}$ and $\overrightarrow{H'}$ are very small disturbances in temperature, pressure, velocity, density, magnetization and magnetic field intensity respectively.

Utilizing Eqs. (8)-(9) in Eqs. (1)-(7), we obtain the subsequent linearized perturbation equations:

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0, \tag{10}$$

$$\frac{\rho_0}{\epsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u'}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left(-\frac{\partial p'}{\partial x}+\mu_0(H_0+M_0)\frac{\partial H_1'}{\partial z}\right)-\frac{\mu}{k_1}u', \tag{11}$$

$$\frac{\rho_0}{\epsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial v'}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left(-\frac{\partial p'}{\partial y}+\mu_0(H_0+M_0)\frac{\partial H_2'}{\partial z}\right)-\frac{\mu}{k_1}v', \tag{12}$$

$$\frac{\rho_0}{\epsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial w'}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\frac{\partial p'}{\partial z}+\rho_0\alpha\theta' g+\mu_0(H_0+M_0)\frac{\partial H_3'}{\partial z}-\mu_0 K_2\beta H_3'+\frac{\mu_0 K_2^2\beta\theta'}{1+\chi}\right]-\frac{\mu}{k_1}w' \tag{13}$$

$$\rho C_1\frac{\partial\theta'}{\partial t}-\mu_0T_0K_2\frac{\partial}{\partial t}\left(\frac{\partial\phi'}{\partial z}\right)=K_1\nabla^2\theta'+\left(\rho C_2\beta-\frac{\mu_0T_0K_2^2\beta}{1+\chi}\right)w', \tag{14}$$

where $\rho C_1=\epsilon\left(\rho_0C_{H,V}-\rho_sC_s+\mu_0K_2H_0\right)+\rho_sC_s$ and $\rho C_2=\epsilon\left(\rho_0C_{H,V}+\mu_0H_0K_2\right)$.

Now,

$$\frac{\partial}{\partial x}(H_1'+M_1')+\frac{\partial}{\partial y}(H_2'+M_2')\frac{\partial}{\partial z}+(H'_3+M_3')=0, \tag{15}$$

where $\overrightarrow{H'}=\nabla\phi'$ and ϕ' represents the perturbed magnetic potential.

$$M_3'+H_3'=H_3'(1+\chi)-K_2\theta',\ H_i'+M_i'=\left(1+\frac{M_0}{H_0}\right)H_i', \tag{16}$$

where $i=1{,}2$ and it is assumed that $\beta K_2d\ll H_0(\chi+1)$.

Eliminating $u',v'$and p' among Eqs. (11)-(13) by utilizing Eq. (10), we obtain

$$\frac{\rho_0}{\epsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\nabla^2w'=\left(1+\lambda\frac{\partial}{\partial t}\right)\left(\rho_0\alpha g\nabla_1^2\theta'-\mu_0K_2\beta\frac{\partial}{\partial z}\nabla_1^2\phi'+\frac{\mu_0K_2^2\beta\nabla_1^2\theta'}{1+\chi}\right)-\frac{\mu}{k_1}\nabla^2w'. \tag{17}$$

Combining Eqs. (15) and (16), we obtain

$$(\chi+1)\frac{\partial^2\phi'}{\partial z^2}+\left(\frac{M_0}{H_0}+1\right)\nabla_1^2\phi'-K_2\frac{\partial\theta'}{\partial z}=0. \tag{18}$$

Now, in normal mode analysis, let the perturbed quantities be expressed in the following manner

$$(w',\phi',\theta')(x,y,z,t)=[w''(z),\phi''(z),\theta''(z)]e^{i(xk_x+yk_y)+nt}, \tag{19}$$

where $k=\left[(k_x)^2+\left(k_y\right)^2\right]^{1/2}$ is the resulting wave number, k_x and k_y are the respective wave numbers along the $x-$ and $y-$directions.

By utilizing Eq. (19) into Eqs. (14), (17)-(18) and subsequently making the variables dimensionless by specific setting

$$w'=\frac{d}{\nu}w'',\quad z'=\frac{z}{d},\quad a=kd,\quad D'=Dd,\quad \theta=\frac{aR^{1/2}K_1}{\beta\nu d\rho C_2}\theta'',\quad \omega=\frac{nd^2}{\nu},\quad \phi'=\frac{(\chi+1)K_1aR^{1/2}}{\rho C_2\beta\nu d^2K_2}\phi'',$$

$$P_r'=\frac{\nu\rho C_2}{K_1},\quad G=\frac{\lambda\nu}{d^2},\quad R=\frac{\rho C_2g\alpha\beta d^4}{\nu K_1},\quad M_1=\frac{\mu_0K_2^2\beta}{(\chi+1)\alpha\rho_0g},\quad M_2=\frac{\mu_0K_2^2T_0}{(\chi+1)\rho C_2},\quad M_3=\frac{1+\frac{M_0}{H_0}}{(1+\chi)},\quad P_r=\frac{\nu\rho C_1}{K_1}. \tag{20}$$

Removing the asterisks for the sake of simplicity, we obtain

$$\left[\frac{\omega}{\epsilon}+\frac{1}{k_1}(1+G\omega)^{-1}\right](D^2-a^2)w=aR^{1/2}[D\phi M_1-(M_1+1)\theta], \tag{21}$$

$$D^2\theta-(P_r\omega+a^2)\theta+P_rM_2\omega D\phi=(M_2-1)R^{1/2}aw, \tag{22}$$

$$(D^2-a^2M_3)\phi-D\theta=0, \tag{23}$$

where a is wave number, $P_r(>0)$ is the Prandtl number, z is an independent variable which is real and lies in the closed interval [0,1], $R(>0)$ is the Rayleigh number, G is stress relaxation parameter, $M_1(>0)$ is magnetic number, $M_2(>0)$ is dimensionless parameter, $M_3(>0)$ measures the non-linearity of magnetization, D represents the derivative with respect to z and ω is complex growth rate constant which can be expressed as $\omega=\omega_r+i\omega_i$, where ω_i and ω_r are real constants.

Also the dependent variables $\theta=(\theta_r,\theta_i),w=(w_r,w_i)$ and $\phi=(\phi_r,\phi_i)$ in the complex plane are functions of real variable z, where functions $\theta_r,\theta_i,w_r,w_i,\phi_r$ and ϕ_i are real-valued.

As M_2 is infinitesimal quantity so it is ignored in the later analysis [3] and therefore Eq. (22) can be rewritten as:

$$D^2\theta-(P_r\omega+a^2)\theta=-R^{1/2}aw. \tag{24}$$

The ferromagnetic boundaries are given as:

Free-Free boundaries

$$w = \theta = D^2 w = D\phi = 0 \quad \text{at} \quad z = 0,1. \tag{25}$$

Rigid-Rigid boundaries

$$w = \theta = Dw = \phi = 0 \quad \text{at} \quad z = 0,1. \tag{26}$$

It is noteworthy that Eqs. (21) and (23)–(26) can be interpreted as an eigenvalue problem for ω, which regulates the occurrence of Maxwell ferromagnetic convection within a densely packed porous medium.

3. Mathematical Analysis

Theorem 1. If $R > 0, M_1 > 0, G > 0, \omega_r \geq 0$ and $\omega_i \neq 0$, then the essential condition for the occurrence of non-trivial solutions $(w, \theta, \phi, \omega)$ of Eqs. (21), (23)-(24) along with the boundary conditions (25) is that

$$|\omega|^2 < max\left\{\epsilon\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right), \frac{\epsilon}{k_1 G}\right\}.$$

Proof: Multiplying Eq. (21) by w^* (where w^* is the complex conjugate of w) and integrating the resultant equation from $z = 0$ to $z = 1$, we obtain

$$\left[\frac{\omega}{\epsilon} + \frac{1}{k_1}(1 + G\omega)^{-1}\right] \int_0^1 w^*(D^2 - a^2)w \, \mathrm{dz} = aR^{1/2} M_1 \int_0^1 w^* D\,\phi \mathrm{dz} - aR^{1/2}(M_1 + 1) \int_0^1 w^* \theta \mathrm{dz}. \tag{27}$$

The terms on the R.H.S. of Eq. (27) by utilizing Eqs. (23)-(24) and boundary conditions (25), can be written as:

$$\begin{aligned} aR^{1/2}(M_1 + 1) \int_0^1 w^* \theta \mathrm{dz} &= aR^{1/2}(M_1 + 1) \int_0^1 \theta w^* \mathrm{dz} \\ &= -(M_1 + 1)\left[\int_0^1 \theta D^2 \theta^* \mathrm{dz} - (a^2 + P_r \omega^*) \int_0^1 \theta \theta^* \mathrm{dz}\right] \end{aligned} \tag{28}$$

and

$$\begin{aligned} aR^{1/2} M_1 \int_0^1 w^* D\,\phi \mathrm{dz} &= -M_1 \int_0^1 D\,\phi (D^2 - a^2 - P_r \omega^*) \theta^* \mathrm{dz} \\ &= -M_1 \int_0^1 D\,\phi D^2 \theta^* \mathrm{dz} + M_1 (a^2 + P_r \omega^*) \int_0^1 (D\phi) \theta^* \mathrm{dz} \\ &= M_1 \int_0^1 D^2 \phi D\,\theta^* \mathrm{dz} + M_1 (a^2 + P_r \omega^*) \int_0^1 \phi D\,\theta^* \mathrm{dz} \\ &= M_1 \int_0^1 D^2 \phi (D^2 - a^2 M_3) \phi^* \mathrm{dz} - M_1 (a^2 + P_r \omega^*) \int_0^1 \phi (D^2 - a^2 M_3) \phi^* \mathrm{dz}. \end{aligned} \tag{29}$$

Combining Eqs. (27)-(29), which yield

$$\begin{aligned} \left[\frac{\omega}{\epsilon} + \frac{1}{k_1}(1 + G\omega)^{-1}\right] \int_0^1 w^*(D^2 - a^2)w \, \mathrm{dz} &= (M_1 + 1) \int_0^1 \theta (D^2 - a^2 - P_r \omega^*) \theta^* \mathrm{dz} - M_1 (a^2 + P_r \omega^*) \\ &\int_0^1 \phi (D^2 - a^2 M_3) \phi^* \mathrm{dz} + M_1 \int_0^1 D^2 \phi (D^2 - a^2 M_3) \phi^* \mathrm{dz}. \end{aligned} \tag{30}$$

Utilizing the boundary conditions (25) to integrate the each integral of Eq. (30).

The integral on the L.H.S. of (30) gives

$$\int_0^1 w^*(D^2 - a^2)w \, \mathrm{dz} = -\int_0^1 (a^2 |w|^2 + |Dw|^2) \mathrm{dz}. \tag{31}$$

The first integral on the R.H.S. of (30) gives

$$\int_0^1 \theta (D^2 - a^2 - P_r \omega^*) \theta^* \mathrm{dz} = -\int_0^1 (a^2 |\theta|^2 + |D\theta|^2 + P_r \omega^* |\theta|^2) \mathrm{dz}. \tag{32}$$

The second integral on the R.H.S. of (30) gives

$$\int_0^1 \phi (D^2 - a^2 M_3) \phi^* \mathrm{dz} = -\int_0^1 (a^2 M_3 |\phi|^2 + |D\phi|^2) \mathrm{dz}. \tag{33}$$

The third integral on the R.H.S. of (30) gives

$$\int_0^1 D^2 \phi (D^2 - a^2 M_3) \phi^* \mathrm{dz} = \int_0^1 (a^2 M_3 |D\phi|^2 + |D^2 \phi|^2) \mathrm{dz}. \tag{34}$$

Utilizing Eqs. (31)-(34) in Eq. (30), we get

$$\begin{aligned} -\left[\frac{\omega}{\epsilon} + \frac{1}{k_1}(1 + G\omega)^{-1}\right] \int_0^1 (a^2 |w|^2 + |Dw|^2) \mathrm{dz} &= -(M_1 + 1) \int_0^1 (a^2 |\theta|^2 + |D\theta|^2 + P_r \omega^* |\theta|^2) \mathrm{dz} \\ &+ M_1 (P_r \omega^* + a^2) \int_0^1 (a^2 M_3 |\phi|^2 + |D\phi|^2) \mathrm{dz} + M_1 \int_0^1 (a^2 M_3 |D\phi|^2 + |D^2 \phi|^2) \mathrm{dz}. \end{aligned} \tag{35}$$

In Eq. (35), substituting $\omega = \omega_r + i\omega_i$ and $\omega^* = \omega_r - i\omega_i$ and comparing the imaginary coefficient on either side of the resultant equation and cancelling ω_i, we obtain

$$-\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1}\right)\int_0^1 (a^2|w|^2 + |Dw|^2)\mathrm{dz} = (M_1+1)P_r\int_0^1 |\theta|^2\mathrm{dz} - P_r M_1\int_0^1 (a^2 M_3|\phi|^2 + |D\phi|^2)\mathrm{dz}. \tag{36}$$

or

$$\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1}\right)\int_0^1 (a^2|w|^2 + |Dw|^2)\mathrm{dz} = -(M_1+1)P_r\int_0^1 |\theta|^2\mathrm{dz} + P_r M_1\int_0^1 (a^2 M_3|\phi|^2 + |D\phi|^2)\mathrm{dz}. \tag{37}$$

By multiplying Eq. (24) with its complex conjugate and integrating it within the limits from $z = 0$ to $z = 1$ and utilizing the boundary conditions (25), we get

$$\int_0^1 (a^4|\theta|^2 + 2a^2|D\theta|^2 + |D^2\theta|^2)\,dz + P_r^2|\omega|^2\int_0^1 |\theta|^2\mathrm{dz} + 2P_r\omega_r\int_0^1 (a^2|\theta|^2 + |D\theta|^2)\mathrm{dz} = Ra^2\int_0^1 |w|^2\mathrm{dz}, \tag{38}$$

since, $\omega_r \geq 0$, we attain from Eq. (38) that

$$\int_0^1 |\theta|^2\mathrm{dz} \leq \frac{Ra^2}{P_r^2|\omega|^2}\int_0^1 |w|^2\mathrm{dz}. \tag{39}$$

Multiplying Eq. (23) by ϕ^* and integrating it within the limits from $z = 0$ to $z = 1$, we get

$$\int_0^1 \phi^*(D^2 - a^2M_3)\phi\mathrm{dz} = \int_0^1 \phi^* D\,\theta\mathrm{dz}. \tag{40}$$

or

$$\begin{aligned}\int_0^1 (a^2M_3|\phi|^2 + |D\phi|^2)\mathrm{dz} &= -\int_0^1 \phi^* D\,\theta\mathrm{dz}\\ &= \int_0^1 D\phi^*\theta\,\mathrm{dz}\\ &\leq \left|\int_0^1 D\phi^*\theta\,\mathrm{dz}\right|\\ &\leq \int_0^1 |D\phi^*||\theta|\mathrm{dz}\\ &\leq \int_0^1 |D\,\phi|\;|\theta|\mathrm{dz}\\ &\leq \left[\int_0^1 |D\,\phi|^2\,\mathrm{dz}\right]^{1/2}\left[\int_0^1 |\theta|^2\mathrm{dz}\right]^{1/2}. \end{aligned}\tag{41}$$

(utilizing Schwartz inequality)

It follows that

$$\int_0^1 |D\phi|^2\mathrm{dz} \leq \left[\int_0^1 |D\phi|^2\mathrm{dz}\right]^{1/2}\left[\int_0^1 |\theta|^2\mathrm{dz}\right]^{1/2}. \tag{42}$$

Thus, we get

$$\left[\int_0^1 |D\phi|^2\mathrm{dz}\right]^{1/2} \leq \left[\int_0^1 |\theta|^2\mathrm{dz}\right]^{1/2}. \tag{43}$$

Combining inequalities (41) and (43), we get

$$\int_0^1 (a^2M_3|\phi|^2 + |D\phi|^2)\mathrm{dz} \leq \int_0^1 |\theta|^2\mathrm{dz}. \tag{44}$$

Now, utilizing inequalities (39) and (44) in Eq. (37), which yield

$$\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1}\right)\int_0^1 (a^2|w|^2 + |Dw|^2)\mathrm{dz} + (M_1+1)P_r\int_0^1 |\theta|^2\mathrm{dz} \leq \frac{RM_1a^2}{P_r|\omega|^2}\int_0^1 |w|^2\mathrm{dz}, \tag{45}$$

which can be rewritten as

$$\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1}\right)\int_0^1 |Dw|^2\mathrm{dz} + a^2\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1} - \frac{RM_1}{P_r|\omega|^2}\right)\int_0^1 |w|^2\mathrm{dz} + (M_1+1)P_r\int_0^1 |\theta|^2\mathrm{dz} \leq 0. \tag{46}$$

Case 1: If $\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1}\right) < 0$ then $\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1} - \frac{RM_1}{P_r|\omega|^2}\right) < 0$, the above inequalities imply that

$|\omega|^2 < \frac{\epsilon}{k_1 G}$ and $|\omega|^2 < \epsilon\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right)$.

Hence, we get

$|\omega|^2 < max\left\{\epsilon\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right), \frac{\epsilon}{k_1 G}\right\}$.

Case 2: If $\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1}\right) > 0$ then, we must have

$\left(\frac{1}{\epsilon} - \frac{G}{|1+G\omega|^2 k_1} - \frac{RM_1}{P_r|\omega|^2}\right) < 0$.

This implies that

$|\omega|^2 < \epsilon\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right)$.

Combining both the cases it can be written as

$$|\omega|^2 < max\left\{\epsilon\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right), \frac{\epsilon}{k_1 G}\right\}. \tag{47}$$

Hence, the proof is established.

Special Cases: The following outcomes can be deduced as special cases from Theorem 1:

- For classical ferromagnetic convection (Taking $\epsilon = 1$ and $G \to \infty$)
  $|\omega| < \sqrt{\frac{RM_1}{P_r}}$.

This result aligned well with the complex growth rate observed in ferromagnetic convection as presented by Jyoti Prakash [24].

- For classical ferromagnetic convection in a densely packed porous medium (Taking $G \to \infty$)
  $|\omega| < \sqrt{\epsilon\left(\frac{RM_1}{P_r}\right)}$.

Theorem 2: If $R > 0, M_1 > 0, G > 0$, and $\omega_r \geq 0$, then the essential condition for the occurrence of non-trivial solutions $(w, \theta, \phi, \omega)$ of Eqs. (21), (23), and (24) along with the boundary conditions (26) is that

$$|\omega_i|^2|\omega|^2 < max\left\{\epsilon^2\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right)^2, \left(\frac{\epsilon}{k_1 G}\right)^2\right\}.$$

Proof: Multiplying Eq. (21) by the complex conjugate of w, denoted as w^*, and integrating the resultant equation within the limits from $z = 0$ to $z = 1$, which yield

$$\left[\frac{\omega}{\epsilon} + \frac{1}{k_1}(1 + G\omega)^{-1}\right]\int_0^1 w^*(D^2 - a^2)w\text{dz} = aR^{1/2}M_1\int_0^1 w^* D\phi\text{dz} - aR^{1/2}(M_1 + 1)\int_0^1 w^*\theta\text{dz}. \tag{48}$$

Integrating each term of Eq. (48) and utilizing the boundary conditions (26), we have

The integral on the L.H.S. of Eq. (48) gives,

$$\int_0^1 w^*(D^2 - a^2)w\,\text{dz} = -\int_0^1(a^2|w|^2 + |Dw|^2)\text{dz}. \tag{49}$$

The second integral on the R.H.S. of Eq. (48) gives,

$$\int_0^1 w^*\theta\text{dz} = -\frac{1}{aR^{1/2}}\int_0^1 \theta(D^2 - a^2 - P_r\omega^*)\theta^*\text{dz}. \tag{50}$$

or

$$\int_0^1 w^*\theta\text{dz} = \frac{1}{aR^{1/2}}\int_0^1(a^2|\theta|^2 + |D\theta|^2 + P_r\omega^*|\theta|^2)\text{dz}. \tag{51}$$

Utilizing Eqs. (49), (51) in Eq. (48), we get

$$-\left[\frac{\omega}{\epsilon} + \frac{1}{k_1}(1 + G\omega)^{-1}\right]\int_0^1(a^2|w|^2 + |Dw|^2)\text{dz} = -(M_1 + 1)\int_0^1(a^2|\theta|^2 + |D\theta|^2 + P_r\omega^*|\theta|^2)\text{dz}.$$

$$+aR^{1/2}M_1\int_0^1 w^* D\,\phi\text{dz}. \tag{52}$$

or

$$\left[\frac{\omega}{\epsilon}+\frac{1}{k_1}(1+G\omega)^{-1}\right]\int_0^1(a^2|w|^2+|Dw|^2)\mathrm{dz}=(M_1+1)\int_0^1(a^2|\theta|^2+|D\theta|^2+P_r\omega^*|\theta|^2)\mathrm{dz}$$
$$-aR^{1/2}M_1\int_0^1 w^*D\,\phi\mathrm{dz}. \quad (53)$$

By substituting $\omega=\omega_r+i\omega_i$ and $\omega^*=\omega_r-i\omega_i$ into Eq. (53), setting the imaginary parts equal in the resulting equation, and dividing it by ω_i (where $\omega_i\neq 0$), which yield

$$\left(\frac{1}{\epsilon}-\frac{G}{|1+G\omega|^2k_1}\right)\int_0^1(a^2|w|^2+|Dw|^2)\mathrm{dz}=-(M_1+1)P_r\int_0^1|\theta|^2\mathrm{dz}-\frac{aR^{1/2}M_1}{\omega_i}imag\int_0^1 w^*D\,\phi\mathrm{dz}. \quad (54)$$

Now,

$$-\frac{aR^{1/2}M_1}{\omega_i}imag\int_0^1 w^*D\phi dz\leq\left|\frac{aR^{1/2}M_1}{\omega_i}imag\int_0^1 w^*D\phi\mathrm{dz}\right|$$
$$\leq aR^{1/2}M_1\left|\frac{1}{\omega_i}\int_0^1 w^*D\phi\mathrm{dz}\right|$$
$$\leq\frac{aR^{1/2}M_1}{|\omega_i|}\left|\int_0^1 w^*D\phi\mathrm{dz}\right|$$
$$\leq\frac{aR^{1/2}M_1}{|\omega_i|}\int_0^1|D\phi||w|\mathrm{dz}$$
$$\leq\frac{aR^{1/2}M_1}{|\omega_i|}\left[\int_0^1|D\phi|\mathrm{dz}\right]^{1/2}\left[\int_0^1|w|^2\mathrm{dz}\right]^{1/2}. \quad (55)$$

(by using Schwarz inequality)

By employing both inequalities (39) and (44), we get

$$\left[\int_0^1|D\phi|^2\mathrm{dz}\right]^{1/2}\leq\frac{aR^{1/2}}{P_r|\omega|}\left[\int_0^1|w|^2\mathrm{dz}\right]^{1/2}. \quad (56)$$

Making use of inequalities (55) and (56), we get

$$-\frac{aR^{1/2}M_1}{\omega_i}imag\int_0^1 w^*D\,\phi\mathrm{dz}\leq\frac{a^2RM_1}{P_r|\omega||\omega_i|}\left[\int_0^1|w|^2\mathrm{dz}\right]. \quad (57)$$

By utilizing inequality (57) in (54), which yields

$$\left(\frac{1}{\epsilon}-\frac{G}{|1+G\omega|^2k_1}\right)\int_0^1(a^2|w|^2+|Dw|^2)\mathrm{dz}\leq-(M_1+1)P_r\int_0^1|\theta|^2\mathrm{dz}+\frac{RM_1a^2}{P_r|\omega||\omega_i|}\int_0^1|w|^2\mathrm{dz}, \quad (58)$$

which can be rewritten as:

$$\left(\frac{1}{\epsilon}-\frac{G}{|1+G\omega|^2k_1}\right)\int_0^1|Dw|^2\mathrm{dz}+a^2\left(\frac{1}{\epsilon}-\frac{G}{|1+G\omega|^2k_1}-\frac{RM_1}{P_r|\omega||\omega_i|}\right)\int_0^1|w^2|\mathrm{dz}+(M_1+1)P_r\int_0^1|\theta|^2\mathrm{dz}\leq 0. \quad (59)$$

Case 1: If $\left(\frac{1}{\epsilon}-\frac{G}{|1+G\omega|^2k_1}\right)<0$ then $\left(\frac{1}{\epsilon}-\frac{G}{|1+G\omega|^2k_1}-\frac{RM_1}{P_r|\omega||\omega_i|}\right)<0$.

The above inequalities imply that

$$|\omega_i||\omega|<\left(\frac{\epsilon}{k_1G}\right)\quad\text{and}\quad|\omega_i||\omega|<\epsilon\left(\frac{1}{k_1G}+\frac{RM_1}{P_r}\right).$$

or

$$|\omega_i|^2|\omega|^2<\left(\frac{\epsilon}{k_1G}\right)^2\quad\text{and}\quad|\omega_i|^2|\omega|^2<\epsilon^2\left(\frac{1}{k_1G}+\frac{RM_1}{P_r}\right)^2.$$

Hence, we get

$$|\omega_i|^2|\omega|^2<max\left\{\epsilon^2\left(\frac{1}{k_1G}+\frac{RM_1}{P_r}\right)^2,\left(\frac{\epsilon}{k_1G}\right)^2\right\}.$$

Case 2: If $\left(\frac{1}{\epsilon}-\frac{G}{|1+G\omega|^2k_1}\right)>0$,

then, we must have

$$\left(\frac{1}{\epsilon}-\frac{G}{|1+G\omega|^2k_1}-\frac{RM_1}{P_r|\omega||\omega_i|}\right)<0.$$

It implies that

$$|\omega_i||\omega| < \epsilon\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right)$$

or

$$|\omega_i|^2|\omega|^2 < \epsilon^2\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right)^2.$$

Combining both the cases it can be written as:

$$|\omega_i|^2|\omega|^2 < max\left\{\epsilon^2\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right)^2, \left(\frac{\epsilon}{k_1 G}\right)^2\right\}. \quad (60)$$

Hence, the proof of the theorem is established.

In terms of the physical aspects, the above theorem (in view of inequality (60)) can be stated as `in the context of rigid-rigid boundaries, the complex growth of a random oscillating movement of increasing amplitude in Maxwell ferroconvection, lies in the right half plane $\omega_r\omega_i$ of ω'.

Special Cases: From Theorem 2, one can derive the subsequent results as specific outcomes:

- For classical ferromagnetic convection (Taking $\epsilon = 1$ and $G \to \infty$)
  $|\omega_i|^2|\omega|^2 < \left(\frac{RM_1}{P_r}\right)^2.$
  This result aligned well with the complex growth rate observed in ferromagnetic convection as presented by Jyoti Prakash [24].
- For classical ferromagnetic convection within a densely packed porous medium (Taking $G \to \infty$)
  $|\omega_i|^2|\omega|^2 < \epsilon^2\left(\frac{RM_1}{P_r}\right)^2.$

4. Conclusion

The normal mode analysis and linear stability analysis are utilized to explore the growth rate of disturbances in a Maxwell ferromagnetic fluid layer that is heated from underside and subjected to a vertically acting magnetic field. The fundamental essence of the Theorem 1 indicates that when Maxwell ferromagnetic fluid flows through a densely packed porous medium bounded by both free-free boundaries, the complex growth rate of oscillating movement with increasing amplitude is located within a semi-circle positioned in the right half of the $\omega_r\omega_i$ −plane. This semi-circle has its radius $\sqrt{\max\left\{\epsilon\left(\frac{1}{k_1 G} + \frac{RM_1}{P_r}\right), \frac{\epsilon}{k_1 G}\right\}}$ and centre at origin. Also, the case of estimating the growth rate of perturbations in Maxwell ferromagnetic fluid for rigid-rigid boundaries is derived in Theorem 2. For higher value of stress relaxation parameter G and standard value of medium porosity, the general results can be drawn for convection in ferrofluid layer for the cases of rigid-rigid and free-free boundaries. The outcomes obtained in this paper concern only unitless parameters and are not containing wave number, so they are uniformly valid and applicable.

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