Solution of Finite Cauchy difference equations on Free Abelian Group

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Abstract: Let (X, .) be a semigroup, (Y, +) an abelian group and $g{:}X\to Y$. The first and second order Cauchy differences of g are $D^1g(a,b) = g(ab) - g(a) - g(b)$, $D^2g(a,b,c) = g(abc) - g(ab) - g(bc) - g(ac) + g(a) + g(b) + g(c)$. Finite order Cauchy differences D^nf are defined recursively. In the case of $Y = Z$, a ring where multiplication is distributive over addition, we show that functions g : $X\to Z$ with finite Cauchy differences are closed under multiplication. The equation $D^ng = 0$ is considered for finite abelian groups.

Keywords: Semigroups, rings, Cauchy difference equation, finite abelian groups.

1. Introduction

Let $(X,.)$ be a semigroup, $(Y, +)$ an abelian groups and $0 \in Y$ denote the identity element. For a function g : X → Y, its Cauchy differences, D^ng , are defined by

$$D^0g = g,$$
$$D^1g(a_1,a_2) = g(a_1a_2) - g(a_1) - g(a_2),$$
$$D^{n+1}g(a_1,a_2,\dots,a_{n+2}) = D^ng(a_1,a_2,a_3,\dots,a_{n+2})$$
$$-D^ng(a_1,a_3,\dots,a_{n+2}) - D^ng(a_2,a_3,\dots,a_{n+2}). \text{------------------------------------}(1)$$

Thefirst order Cauchy difference D^1g will often be abbreviated as Dg.[5]
For an n-place function S : $X^n \to Y$ its first order difference in the j-th variable is defined by

$$D_j^1G(a_1,\dots,a_{j-1},x,y,a_{j+1},\dots,a_m) = G(a_1,\dots,a_{j-1},xy,a_{j+1},\dots,a_m)$$
$$-G(a_1,\dots,a_{j-1},x,a_{j+1},\dots,a_m) - G(a_1,\dots,a_{j-1},y,a_{j+1},\dots,a_m).$$

Taking $G = D^1g : X^2 \to Y$ in particular it follows from the associativity of the operation on X that $D_1{}^1G = D_2{}^1G = D^2g$.

Hence we may see D^2g as the first order difference of D^1g in either the first or the second variable. This observation can be extended to higher order differnces.
In (1) we have defined $D^{n+1}g$ as the first order difference of D^ng in the first variable, and the result is the same as the first order difference of D^ng in any of its $n+1$ variables.

A function g is said to have vanishing D^n, or to be in the kernel of D^n, if $D^ng = 0$
Let Ker $D^n(X,Y) = \{g : X \to Y \mid D^ng = 0\}$.--------------------------------------*(2)*

It is clear that kernels are abelian groups under the usual addition of functions and we have an ascending chain
$\mathrm{Hom}(X,Y) = \mathrm{Ker}\ D(X,Y) \le \mathrm{Ker}\ D^2(X,Y) \le \mathrm{Ker}\ D^3(X,Y) \le \dots$

Abelian groups under addition will be treated as modules over R. In particular Y and Ker $D^n(X,Y)$ are modules over R.
While we consider functions with vanishing Cauchy differences, there are interesting questions and results dealing with weaker assumptions, e.g., that the range of Dg is confined differently. The paper [1] is in such a direction on topological vector spaces.

In [4], the general solution of (2) for $n = 2$ is obtained on free groups, cyclic groups and permutation groups. In [2], the general solution of (2) for $n = 3$ is obtained when S is the free group with a single letter. In [3] results are given for higher n, including the general solution of (2) for $n = 3$ on free groups.

The result is extended to show that functions from X to a ring Z with finite Cauchy differences are closed under multiplication. We exhibit the general solution of the equation $D^3g = 0$ on free abelian groups using the product.

Preliminaries :
Definition 1

Let $r \geq 0$ be an integer and consider the finite sequence $< 1, 2, \ldots, r+1>$. Subsequences $n = <n_1, n_2, \ldots, n_{l+1}>$ and $k = <m_1, m_2, \ldots, m_{x+1}>$ of $< 1, 2, \ldots, r+1 >$ is a *covering pair* if the union of their range sets, $\{n_1, \ldots, n_{l+1}\} \cup \{m_1, \ldots, m_{x+1}\}$, equals $\{1, \ldots, r+1\}$. The family of all such covering pairs will be denoted by $P_{(r,l,x)}$.

Note: necessarily $(l+1)+(x+1) \geq r+1$ for $\{n_1, \ldots, n_{l+1}\} \cup \{m_1, \ldots, m_{x+1}\} = \{1, \ldots, r+1\}$.

Theorem 2. *Let X be a semigroup and Z be a ring in which multiplication is distributive over addition from both sides. For any $g, q : X \to Z$ we have*

$$D^r(gq)(a_1, a_2, \ldots, a_{r+1}) = \sum_{0\leq l,x\leq r (n,m)\in P_{(r,l,x)}} \sum D^r g(a_{n1}, \ldots, an_{l+1}) D^x q(a_{m1}, \ldots, am_{x+1}) \quad \text{--------------(3)}$$

Proof

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We shall establish the identity by induction on r.

If $r = 0$ then $l = 0$, $x = 0$, and $n =< 1 >$, $m =< 1 >$ is the only covering pair.

The left side of (3) gives $D^0(gq)(a_1) = (gq)(a_1) = g(a_1)q(a_1)$.

The right side of (3) gives $D^0(g)(a_1)D^0(q)(a_1) = g(a_1)q(a_1)$.

So (3) holds for $r = 0$.

By the definition of the first order Cauchy difference and the distributivity we have

$$\begin{aligned} D(gq)(a_1, a_2) &= g(a_1a_2)q(a_1a_2) - g(a_1)q(a_1) - g(a_2)q(a_2) \\ &= [Dg(a_1, a_2) + g(a_1) + g(a_2)][Dq(a_1,a_2) + q(a_1) + q(a_2)] \\ &\quad - g(a_1)q(a_1) - g(a_2)q(a_2) \\ &= Dg(a_1, a_2)\, Dq(a_1, a_2) + Dg(a_1, a_2)\, q(a_1) + Dg(a_1, a_2)q(a_2) \\ &\quad + g(a_1)Dq(a_1,a_2) + g(a_2)Dq(a_1,a_2) + g(a_1)q(a_2) + g(a_2)q(a_1). \quad (4) \end{aligned}$$

This proves (3) for $r = 1$.

Suppose as an induction hypothesis that for some $r = n \geq 1$ (3) holds, i.e.,

$$D^r(gq)(a_1, a_2, \ldots, a_{r+2}) = \sum_{0\leq l,x\leq r (n,m)\in P_{(r,l,x)}} \sum difference\ of\ D^r g(a_{n1}, \ldots, an_{l+1}) D^x q(a_{m1}, \ldots, am_{x+1})$$

--(5)

Then, taking the difference with respect to the variable a_{n+1} we obtain

$$D^r(gq)(a_1, a_2, \ldots, a_{r+1}) = \sum_{0\leq l,x\leq r (n,m)\in P_{(r,l,x)}} \sum D^r g(a_{n1}, \ldots, an_{l+1}) D^x q(a_{m1}, \ldots, am_{x+1})$$

With respect to a_{n+1}

$$= \sum_{\substack{0\leq l,x\leq r (n,m)\in P_{(r,l,x)} \\ x\, l+1=n+1, m\, x+1=n+1}} \sum difference\ \ D^r g(a_{n1}, \ldots, an_{l+1}) D^x q(a_{m1}, \ldots, am_{x+1})$$

With respect to a_{n+1}

$$+ \sum_{0\leq l,x\leq r (n,m)\in P} \sum_{\substack{(r,l,x) \\ x\, l+1=n+1, m\, x+1=n+1}} D^{r+1} g(a_{n1}, \ldots, an_{l+1}, an_{l+2}) D^x q(a_{m1}, \ldots, am_{x+1})$$

$$\sum_{\substack{0\leq l,x\leq r (n,m)\in P_{(r,l,x)} \\ x\, l+1=n+1, m\, x+1=n+1}} \sum D^r g(a_{n1}, \ldots, an_{l+1}) D^{x+1} q(a_{m1}, \ldots, am_{x+1}, am_{x+2})$$

---(6)

Substituting (6) into (5) we arrive at (3) with $r = n + 1$. This completes the induction.

With the above general expansion for $D^r(gq)$ we arrive at the followingresult:

Corollary 3. *For any $g, q : X \to Z$, $D^j g = D^i q = 0$ implies $D^{j+i}(gq) = 0$.*

According to formula (3) we have $D^{j+i}(gq)(a_1, a_2, \ldots, a_{j+i+1})$

$$= \sum_{0\leq l,x\leq r (n,m)\in P_{(j+i,l,x)}} \sum D^r g(a_{n1}, \ldots, an_{l+1}) D^x q(a_{m1}, \ldots, am_{x+1})$$

As mentioned in a little note following definition 1, (n,m) $\in P_{(j+I,\, l,x)}$ implies $l+x+1 \geq j+i$.

This either $l \geq j$ or $x \geq i$.

Because $D^j g = 0$, $l \geq j$ implies $D^l g = 0$ and because $D^i q = 0$, $x \geq i$ implies $D^x q = 0$.

So every term $D^r g(a_{n1}, \ldots, an_{l+1}) D^x q(a_{m1}, \ldots, am_{x+1})$ in the above expansion is zero.

Main Results:

Let $G =<Đ>$ be the free group on the alphabet$Đ$ and let $C^{x,y}$ denote the abelianization of C. In other words $<Đ>^{xy}$ is the free abelian group on $Đ$.We shall present the general solution of $C^3 f = 0$ on the free abelian group.

To assist the presentation we shall endow $Đ$ with a linear order $<$ and without loss of generality assume that $Đ$ has at least two letters.

Theorem 4.

Suppose that $|Đ| > 1$. *If* $f \in \text{Ker } C^3(<Đ>^{uv}, Y)$ *then it has therepresentation*

$$G(a) = \sum_x \left\{W(u,x)g(u) + \begin{bmatrix} W(u,a) \\ 2 \end{bmatrix} Dg(u,u) + \begin{bmatrix} W(a,u) \\ 3 \end{bmatrix} D^2(u,u,u)\right\} +$$
$$\sum_{u<v} \left\{W(u,x)W(a,v)Dg(u,b) + W(u,x) \begin{bmatrix} W(u,a) \\ 2 \end{bmatrix} D^2 g(u,v,v) +\right.$$
$$\left.\begin{bmatrix} W(a,u) \\ 2 \end{bmatrix} W(u,v)D^2(u,u,v)\right\} + \sum_{u<v<w} W(a,u)W(b,v)W(c,w)\, D^2 g(u,v,v)$$

---(7)

Conversely, let g , Dg , and D^2g *be arbitrary initial functions mapping indicated subsets of Đ, Đ × Đ and Đ × Đ × Đ, respectively, into Y, then g defined by* (7) *indeed belongs to* $\text{Ker } D^3(<ĐĐ, Y)$.

Note: The definitions of the functions used in the above representation are given in [3] and are included here in the appendix. If *Đ* has only two letters then there won't be distinct letters u , v , w and the last sum in (7) disappears by the convention $\sum_\varnothing = 0$.

Proof. Suppose that $g :< Đ >^{uv} \to Y$ and $D^3 g = 0$.
For each element a $\in < Đ >^{uv}$, using the endowed linear order, we can write it as

$$a = \alpha_1^{k1}\alpha_2^{k2} \dots . \alpha_r^{kr}, \alpha_1 \in Đ, \alpha_1 < \alpha_2 < \cdots < \alpha_r. \qquad (8)$$

From [3] we get

$$G(a) = g(\alpha_1^{k1}\alpha_2^{k2} \dots . \alpha_r^{kr})$$
$$= \sum_{1\le j\le r} k_j g(\alpha_j) + \begin{bmatrix} k_j \\ 2 \end{bmatrix} Dg(\alpha_j,\alpha_j) + \begin{bmatrix} k_j \\ 3 \end{bmatrix} D^2 g(\alpha_j,\alpha_j,\alpha_j)$$
$$+ \sum_{1\le j\le i\le r} k_j k_i g(\alpha_j,\alpha_i) + k_j \begin{bmatrix} k_j \\ 2 \end{bmatrix} D^2 g(\alpha_j,\alpha_i,\alpha_i) + \begin{bmatrix} k_j \\ 2 \end{bmatrix} \alpha_i D^2 g(\alpha_j,\alpha_j,\alpha_i)$$
$$+ \sum_{1\le j\le i\le t\le r} k_j k_i k_t D^2 g(\alpha_j,\alpha_i,\alpha_t) \qquad (9)$$

From the writing (8) we observe that $k_j = W(a,\alpha_j)$ for every j.
Hence (9) proves (7).

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