Cauchy-Schwartz's inequality and its application in telecommunication systems

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Abstract: In this article, we first tried to prove Schwartz's inequality and then discuss its application in the field of telecommunication systems, especially in the filtering of discrete and continuous white noise signals**.** Schwartz's inequality is an extremely useful inequality that is used in various engineering optimization problems. For example, it is used in order to obtain the adapted filter as well as the maximum ratio combining technique that occurs in digital telecommunication systems.

Keywords: telecommunication systems, filtering, white noise, maximum ratio, discrete

1. Introduction

First we will prove the theorem and then we will use it in the telecommunication system.

Theorem (Schwartz's inequality).(Steele, 2004) For each inner multiplicative space V, we have:

$$|<x.y>| \leq <x.x>^{\frac{1}{2}}<y.y>^{\frac{1}{2}} \qquad I$$

Equality holds if and only if $x = \alpha y$ (linear independent).

Proof. For each scalar β and t we have:

$$<tx+\beta y.tx+\beta y> \geq 0 \qquad II$$

Expanding the inner multiplication above gives:

$$t\bar{t}<x.x>+t\bar{\beta}<x.y>+\beta\bar{t}<y.x>+\beta\bar{\beta}<y.y> \geq 0\cdot \qquad III$$

If we assume that t is real, then:

$$t^2<x.x>+t\bar{\beta}<x.y>+t\beta<y.x>+\beta\bar{\beta}<y.y> \geq 0\cdot \qquad IV$$

Suppose we choose β such that:

$$\beta = \begin{cases} \dfrac{<x.y>}{|<x.y>|} & if <x.y> \neq 0 \\ 1 & if <x.y> = 0 \end{cases} \qquad V$$

Then $\bar{\beta}<x.y> = <y.x>\beta = |<x.y>|$ and $\beta\bar{\beta} = 1\cdot$ Therefore, equation IV reduces to:

$$t^2<x.x>+2t|<x.y>|+<y.y> \geq 0\cdot \qquad VI$$

For every real number t.

Now, if we assume $t = -\frac{|<x.y>|}{<x.x>}$ then:

$$\frac{|<x.y>|^2}{<x.x>} - 2\frac{|<x.y>|^2}{<x.x>} + <y.y> \geq 0\cdot \qquad VII$$

So:

$$-|<x.y>|^2+<x.x><y.y> \geq 0\cdot \qquad VIII$$

Therefore:

$$|< x.y >| \leq < x.x >^{\frac{1}{2}} < y.y >^{\frac{1}{2}}$$

The following inequalities are the result of applying Schwartz's inequality on C^n and $L(a.b)$.

$$\left|\sum_{i=1}^{n} x_i \bar{y}_i\right| \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$$

And

$$\left|\int_a^b |x(t)dt|\overline{y(t)}dt\right| \leq \left[\int_a^b |x(t)|^2 dt\right]^{\frac{1}{2}} \left[\int_a^b |y(t)|^2 dt\right]^{\frac{1}{2}}$$

Norm generated from inner multiplication

Theorem. If V is an inner product space with inner product $< x.y >$, then; V is a normed vector space with the following induced norm:

$$\|x\| = \sqrt{< x.y >}$$

2. Significance of The Study

The study we did is about Schwartz's inequality and its applications. As a result of the study, we came to the conclusion that Schwartz's inequality theorem is one of the most famous theorems in mathematical sciences. In this research, an attempt is made to find out how white noise signals are propagated in telecommunication systems. In this article, we tried to analyse discrete and continuous noise signals in telecommunication systems separately.

3. Review Of Related Studies

Jiangbo Si and Zan Li, (2021) conducted study on Covert Transmission Assisted by Intelligent Reflecting Surface**.** Theu studied about Covert transmission is studied for an intelligent reflecting surface (IRS) aided communication system. **Yash Vasavada, (2017)** conducted study on Amplitude and Phase Calibration of Antenna Arrays and he studied about Array of antenna elements interfaced with a digital signal processor (DSP) allows powerful and flexible (adaptive) beamforming algorithms to be implemented digitally at DSP. **Björnson. Emil and Wymeersch. Henk**, **(2021)** conducted study on Reconfigurable Intelligent Surfaces, they used Schwartz's inequality to describe a Signal Processing Perspective with Wireless Applications. **Mohammad Sababheh and Hamid Reza Moradi, (2022)** conducted study on BUZANO, KREˇIN AND CAUCHY-SCHWARZ INEQUALITIES they compared, Cauchy-Schwarz, Buzano and Kreˇın inequalities are three inequalities about inner product. **Tiago Roux Oliveira, (2021)** conducted a study on Extremum Seeking Feedback with Wave Partial Differential Equation Compensation. They studied the addresses the compensation of wave actuator dynamics in scalar extremum seeking (ES) for static maps. Infinite-dimensional systems described by partial differential equations (PDEs) of wave type have not been considered so far in the literature of ES. **Chengye Lu and Sheng Wu. (2019)** conducted a study on Weak harmonic signal detection method in chaotic interference based on extended Kalman filter. They discussed about the better performance of the neural network method, the performance with the optimal filtering method, but with lower computational complexity. Salim **Bouzebda and Sultana Didi, (2017)** conducted a study on Multivariate wavelet density and regression estimators for stationary and ergodic discrete time processes: Asymptotic results

4. Objectives of The Study

- To find out N_0 and N_1 normal random variables with zero mean of uniform distribution.
- To find out the variance of random variables of N_0 and N_1.
- To find out that SNR is maximized.

Applications of Schwartz inequality in telecommunication systems

As an important application of Schwartz's inequality, we consider the matched filter detector used in telecommunications and radar systems. First, we will examine the discrete matched filter detector and then the continuous one.(Alomari, 2022; Kim et al., 2009; Vasavada & Reed, 2017)

1. Detecting a discrete signal "buried" in white noise

Suppose the received signal with noise is given as follows:

$$r_i = s_i + n_i \quad . \quad i = 1.2.3. \dots . m$$

It is assumed that the values of sample $s_i(i = 1.2. . \dots \ . m)$ signals are known. Also, it is assumed that the samples of noise n_i are independent and with zero mean, and the variance of each noise sample is σ_n^2. The goal is to process r_i samples for presence or absence of s_i signal. The general approach is to calculate a weighted average of the received signals and compare them to a threshold (Xu et al., 2019). The weighted mean is obtained as follows:

$$y = \sum_{i=1}^{m} h_i r_i = \underbrace{\sum_{i=1}^{m} h_i s_i}_{Signal} + \underbrace{\sum_{i=1}^{m} h_i n_i}_{Noise}$$

This is equivalent to filtering the received signal using a filter with h_i coefficients. This object is shown in Figure 1. The filter coefficients are calculated by maximizing the signal-to-noise ratio (SNR)[1] in the filter output. This ratio is defined as follows:

$$SNR = \frac{Signal\ power}{Noise\ power} = \frac{(\sum_{i=1}^{m} h_i s_i)^2}{\sigma_n^2 \sum_{i=1}^{m} {h_i}^2}$$

SNR can be written in vector representation as:

$$SNR = \frac{Signal\ power}{Noise\ power} = \frac{(< h.s >)^2}{\sigma_n^2 \|h\|^2}$$

in which, $h = [h_1 \quad h_2 \quad \dots \quad h_m]^T$ and $s = [s_1 \quad s_2 \quad \dots \quad s_m]^T$. The filter coefficients are represented by the vector h and the maximum SNR is obtained using the Schwartz inequality:

$$SNR = \frac{(< h.s >)^2}{\sigma_n^2 \|h\|^2} \leq \frac{\|s\|^2 \ \|h\|^2}{\sigma_n^2 \|h\|^2} = \frac{\|s\|^2}{\sigma_n^2 \|h\|^2}$$

The above inequality holds when:

$$h = \alpha s$$

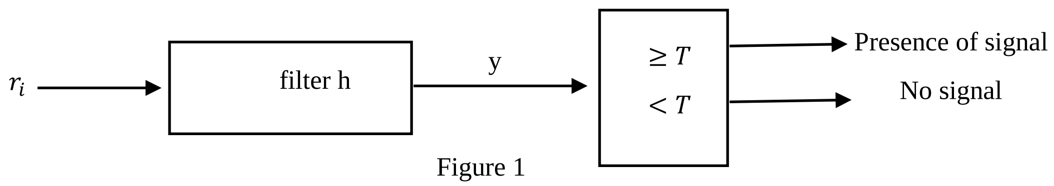

Figure 1

Matched filter detector

Therefore, when the filter coefficients are proportional to the signal samples, SNR is maximized. The proportionality constant α is chosen as 1, because it has no effect on SNR. (Si et al., 2021). Therefore:

$$h_i = s_i \quad i = 1.2. \dots . m \quad . \quad SNR_{max} = \frac{1}{\sigma_n^2} \sum_{i=1}^{m} s_i^2$$

Threshold T is chosen as follows in order to minimize the possibility of error:

$$T = \frac{\|s\|^2}{2}$$

[1] . Signal-to-noise Ratio

The detector above is known as the matched filter detector.

2. Detecting a continuous signal "buried" in noise

Consider the bitwise digital data communication system of Figure 2, where the transmitted signals are binary bits "1" and "0" indicating the presence of s(t) and the absence of $s(t)$, respectively.(Mozzon et al., 2020). The duration of the signal is assumed to be in seconds. The communication channel is an ideal additive [2]Gaussian white noise channel with zero mean and power spectrum density:

$$S_n(f) = \frac{N}{2}$$

Modelled.

The receiver consists of a linear filter with impulse response $h(t)$ and duration T_b called the matched filter, an ideal sampler that samples the outputs of the matched filter at bit rate $f_s = R_b = \frac{1}{T_b}$, and a one-bit quantized. As in Figure 3. As a result, the quantized output is a result of the transferred binary bit n-k. A one-bit quantized $Q[0]$ is defined as follows:(Başaran & Gürdal, 2023)[16]

$$Q[x] = \begin{bmatrix} 1 & x \geq T \\ 1 & x < T \end{bmatrix}$$

where T is a threshold constant. The purpose of filtering the received signal is to reduce the effect of white noise, as a result of which SNR and signal recognisability increases. Hence, the optimal filter, known as the matched filter, is the one that maximizes SNR. Threshold T is chosen to minimize bit error rate $(BER)^2$.(Altwaijry et al., 2023).

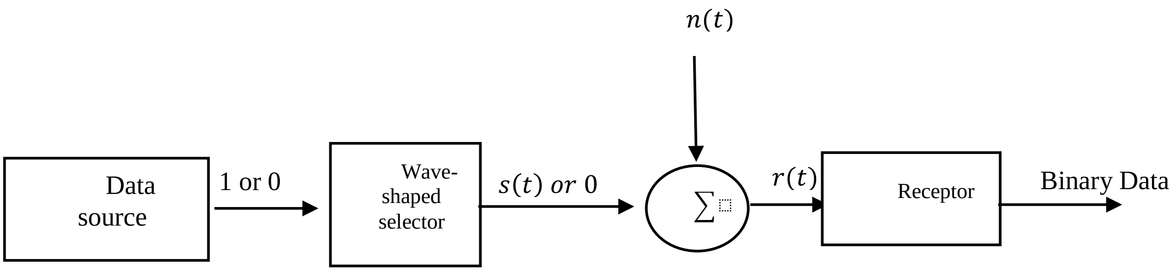

Figure 2

A digital data telecommunication system

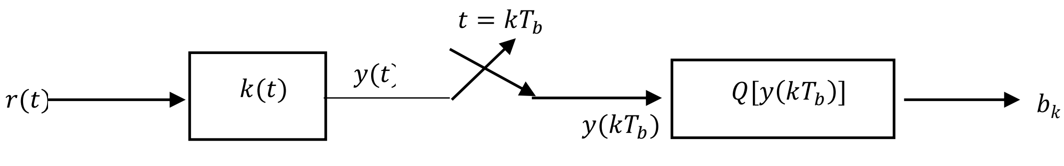

Figure 3

Recipient Matched filter

We want to determine a filter $h(t)$ that maximizes SNR at the input of the quantizer. The output of the filter $y(t)$ is the combination of the input with the impulse response of the filter and is defined as:

$$y(t) = r(t) * h(t) = \int_0^r h(\tau) r(t-\tau) d\tau$$

[2] . Additive noise means: the received signal is the transmission signal plus some noise so that the noise is statistically independent of a signal. The word additive indicates that noise has an additive effect on a signal.

Assuming that the binary bit "1" is transmitted, the sampler is equal to:

$$y(T_b) = \int_0^{T_b} h(T_b - \tau) r(\tau) d\tau = \int_0^{T_b} h(T_b - \tau) s(\tau) d\tau + N_1$$

And if the binary bit "0" is transmitted, the output of the sampler is equal to:

$$y(T_b) = \int_0^{T_b} h(T_b - \tau) r(\tau) d\tau = \int_0^{T_b} h(T_b - \tau) n(\tau) d\tau + N_0$$

where N_0 and N_1 are normal random variables with zero mean of uniform distribution which are defined as follows:

$$N = \int_0^{T_b} h(T_b - \tau) n(\tau) d\tau$$

The variance of random variables N_0 is the same as N_1 and can be calculated as follows:

$$\sigma^2 = E\left(\int_0^{T_b} h(T_b - \tau) n(\tau) d\tau\right)^2 = E\left(\int_0^{T_b}\int_0^{T_b} h(T_b - \tau_1) h(T_b - \tau_2) n(\tau_1) n(\tau_2) d\tau_1 d\tau_2\right)$$

By reducing this relation, we have:

$$\sigma^2 = \int_0^{T_b}\int_0^{T_b} h(T_b - \tau_1) h(T_b - \tau_2) E(n(\tau_1) n(\tau_2)) d\tau_1 d\tau_2$$

$$= \int_0^{T_b}\int_0^{T_b} h(T_b - \tau_1) h(T_b - \tau_2) \frac{N_0}{2} \delta(\tau_1 - \tau_2) d\tau_1 d\tau_2$$

$$= \frac{N_0}{2} \int_0^{T_b} h^2(T_b - \tau_2) \int_0^{T_b} \delta(\tau_1 - \tau_2) d\tau_1 d\tau_2$$

$$= \frac{N_0}{2} \int_0^{T_b} h^2(T_b - \tau_2)(u(T_b - \tau_2) - u(-\tau_2)) d\tau_2$$

$$= \frac{N_0}{2} \int_0^{T_b} h^2(T_b - \tau_2) d\tau_2 = \frac{N_0}{2} \int_0^{T_b} h^2(t) dt$$

The SNR at the quantizer input is defined as:

$$SNR = \frac{\left[\int_0^{T_b} h(T_b - \tau) s(\tau) d\tau\right]^2}{\sigma^2} = \frac{2\left[\int_0^{T_b} h(T_b - \tau) s(\tau) d\tau\right]^2}{N_0 \int_0^{T_b} h^2(\tau) d\tau}$$

Applying Schwartz's inequality to the above equation, we have:

$$SNR \leq \frac{2\int_0^{T_b} s^2(\tau) d\tau \int_0^{T_b} h^2(T_b - \tau) d\tau}{N_0 \int_0^{T_b} h^2(\tau) d\tau} = \frac{2\int_0^{T_b} s^2(\tau) d\tau \int_0^{T_b} h^2(\tau) d\tau}{N_0 \int_0^{T_b} h^2(\tau) d\tau} = \frac{2\int_0^{T_b} s^2(\tau) d\tau}{N_0}$$

Equality holds if:

$$s(\tau) = h(T_b - \tau) \quad or \quad h(t) = s(T_b - t)$$

The noise variance is obtained as follows:

$$\sigma^2 = \frac{N_0}{2}\int_0^{T_b} h^2(t)dt = \frac{N_0}{2}\int_0^{T_b} s^2(t)dt = \frac{N_0}{2}E$$

And the maximum SNR is equal to: $SNR_{max} = \frac{2E}{N_0}$, where $E = \int_0^{T_b} s^2(t)dt$ is the energy in signal $s(t)$. As a result, the input of the quantizer is:

$$y(kT_b) = \begin{cases} E + N_1 & if \quad b_k = 1 \\ N_0 & if \quad b_k = 1 \end{cases}$$

The threshold constant T is chosen to minimize BER. For a symmetric binary data source where $p(0) = p(1) = 0.5$ is the optimal choice for the threshold of T, $E/2$ and the resulting BER is:

$$BER = p(e|o)p(0) + p(e|1)p(1) = 0.5p(N_0 > T) + 0.5p(N_1 +< T) \qquad I$$

where in:

$$p(N_0 > T) = \frac{1}{\sqrt{2\pi}\sigma}\int_T^{\infty} e^{-\frac{x^2}{2\sigma^2}}dx = Q\left(\frac{\alpha}{\sigma}\right) \qquad II$$

And

$$p(N_1 < T) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{T} e^{-\frac{x^2}{2\sigma^2}}dx = Q\left(\frac{\alpha}{\sigma}\right) \qquad III$$

The function b is defined as follows:$Q(x) = (\frac{1}{\sqrt{2\pi}})\int_x^{\infty} e^{-\frac{u^2}{2}}du$ · By substituting equations II and III in equation I, we have:

$$BER = 0.5Q\left(\frac{\alpha}{\sigma}\right) + 0.5Q\left(\frac{\alpha}{\sigma}\right) = Q\left(\frac{\alpha}{\sigma}\right) = Q\left(\frac{\frac{E}{2}}{\sqrt{\frac{N\cdot E}{2}}}\right) = Q\left(\sqrt{\frac{2E}{N_0}}\right) \cdot$$

5. Recommendation

- We can calculate the maximum value of SNR by using of Cauchy Schwartz's inequality.
- We can calculate BER with Cauchy Schwartz's inequality.

6. Conclusion

We know that the detector of telecommunication systems is very important in our daily life, and about this system, we should research what is the role of mathematics in the structure of such detectors of telecommunication systems. Therefore, Schwartz's inequality is one of the most important inequalities and has many applications, one of its applications in telecommunication systems, especially in detecting discrete and continuous signals in noise and white noise.

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