Rational Type Contraction in Partially Ordered Metric Space and Classical Results of Common Coupled Fixed Points.

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Abstract- We construct theorems that are widely used to satisfy rational type contractive requirements in partially ordered metric spaces in the current work using a few coupled fixed point theorems. In numerous earlier investigations on this topic, I have attempted to show common related fixed locations. To emphasize our conclusions, we have also included instances.

Keywords- Coupled fixed point, common coupled fixed point, and rational type contraction, partially ordered metric space

I. INTRODUCTION

In 1886, H. Poincare [1] was given the idea of a theory of fixed points. Eventually, M. Frechet [2] introduced the fixed point theorem in 1906, which takes into account both the length among areas and the associated perception by traders at points in metric spaces. Earlier the next year, in 1922, S. Banach [3] established the theorem of a fixed point for contractive mapping in the entire metric space. It's a crucial tool for solving problems caused by nonlinear analysis's excessive number of outcomes. This time, some researchers have further developed the BCMT (Banach contraction principle theorem) for certain functions on different metric spaces. Wolk [4] and Monjardet [5] expanded their BCP (Banach contraction theory) and introduced the partially ordered metric space. The theory of fixed points was researched by Ran and Reuring [6] and used to solve matrix equations on partially ordered metric spaces. Alternately, Nieto et al. [7-9] adapted this concept to the equation of an ordinary differential of the first order by extending the resolution of [6] to partially ordered sets with periodic conditions. Multifarious creators have discussed, expanded, and generalized unique fixed points in partially ordered metric space (see [10–12, 13–18]).
The results of the unique coupled fixed point were proved and demonstrated by T. G. Bhaskar and V. Lakshmikantham [19] in 2006 and linked mixed monotone functions with periodic boundary value problems. Likewise, Likewise, common coupled fixed points and coupled coincidences were connected or drafted by V. Lakshmikantham and L.B. Ciric [20] in ordered metric spaces for nonlinear contraction. Many conclusions have been generalized about coupled fixed points and common coupled fixed points in partially ordered metric spaces (for instance, see [21–35]). Lately, Tiwari and Ganvir [38] extended and generalized the outcomes of coupled fixed points in partially ordered metric spaces for the dynamic technique of rational type contraction. This result was deduced from some prominent results in the literature. Someone had also given him an illustration and an operation to support the result.
The motive of this work is to generalize and meliorate common coupled fixed point theorems for rational type contraction in partially ordered metric spaces. Our conferred results generalize and extend many well-known results [31], [36], and [38] in the literature.

II. BASIC CONCEPT OF MATHEMATICAL NOTES

The terminology and notations that are important to the primary outcomes have been given below.
Definition 2.1 [19]: An element $(\alpha,\beta)\in\mathcal{H}\times\mathcal{H}$ is said to be the coupled fixed point of the function $\xi:\mathcal{H}\times\mathcal{H}\rightarrow\mathcal{H}$, on a partially ordered set $(\mathcal{H},\leq)$, if $\xi(\alpha,\beta)=\alpha$ and $\xi(\beta,\alpha)=\beta$.
Definition 2.2 [39, 40]: An element $(\alpha,\beta)\in\mathcal{H}\times\mathcal{H}$ is said to be the common coupled fixed point of the two functions $\xi_1,\xi_2:\mathcal{H}\times\mathcal{H}\rightarrow\mathcal{H}$ on partially ordered sets $(\mathcal{H},\leq)$, if $\xi_1(\alpha,\beta)=\xi_2(\alpha,\beta)=\alpha$ and $\xi_1(\beta,\alpha)=\xi_2(\beta,\alpha)=\beta$.

Definition 2.3[36, 38]: A function $\xi: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ on partially ordered set $(\mathcal{H}, \leq)$ is said to be increasing if $\xi(\alpha) \leq \xi(\beta)$ for all $\alpha. \beta, \in \mathcal{H}$, when $\alpha \leq \beta$ and is also known as decreasing if $\xi(\alpha) \geq \xi(\beta)$, when $\alpha \leq \beta$.

Definition 2.4[5]: A function $\xi: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be strict mixed monotone property on partially ordered set $(\mathcal{H}, \leq)$ if $\xi(\alpha, \beta)$ is increasing in α and decreasing β as well, $i.e\ \alpha_1, \alpha_2 \in \mathcal{H}$ with $\alpha_1 \leq \alpha_2 \Rightarrow \xi(\alpha_1, \beta) \leq \xi(\alpha_2, \beta)$ for all $\beta \in \mathcal{H}$ and $\beta_1, \beta_2 \in \mathcal{H}$ with $\beta_1 \leq \beta_2 \Rightarrow \xi(\alpha, \beta_1) \geq \xi(\alpha, \beta_2)$ for all $\alpha \in \mathcal{H}$.

Definition 2.5[19]: If **(**$\mathcal{H}, \leq)$ is a partially ordered set in addition **t**o $(\mathcal{H}, d)$ being a metric space, then triple $(\mathcal{H}, d, \leq)$ is referred to as a partially ordered metric space.

III. MAIN RESULTS

In this section, we display key outcomes and deliver their proof.

Theorem 3.1: Let $\xi_1, \xi_2: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be ant two functions having the mixed monotone property on $(\mathcal{H}, d, \leq)$, where $(\mathcal{H}, d, \leq)$ is a partially ordered complete metric space, satisfying the conditions:

$$d(\xi_1(\alpha,\beta)\xi_2(u,v)) \leq l_1 d(\alpha,u) + l_2 d(\beta,v) + l_3\left[\frac{d(\alpha,\xi_1(\alpha,\beta)),\ d(u,\xi_2(u,v))}{d(\alpha,u)}\right] + l_4\left[\frac{d(\beta,\xi_1(\beta,\alpha)),\ d(v,\xi_2(v,u))}{d(\beta,v)}\right] \ldots \quad (3.1.1)$$

For all $\alpha, \beta, u, v \in \mathcal{H}$ with $\alpha \geq u$ and $\beta \geq v$, where $l_i \in [0,1]$ for$i = 1,2, \ldots 4$ such that $\sum_{i=0}^{4} l_i < 1$ and there exist $\alpha_0, \beta_0 \in \mathcal{H}$ where $\alpha_0 \leq \xi_1(\alpha_0, \beta_0) \leq \xi_2(\alpha_0, \beta_0)$ and $\beta_0 \geq \xi_1(\beta_0, \alpha_0) \geq \xi_2(\beta_0, \alpha_0)$. Then all $\xi_1$and ξ_2 have a common coupled fixed point in $\mathcal{H} \times \mathcal{H}$.

Proof: Take $\alpha_0, \beta_0 \in \mathcal{H}$ where $\alpha_0 \leq \xi_1(\alpha_0, \beta_0) \leq \xi_2(\alpha_0, \beta_0)$and $\beta_0 \geq \xi_1(\beta_0, \alpha_0) \geq \xi_2(\beta_0, \alpha_0)$. Now we define two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ in $\mathcal{H}$ such that

$$\alpha_{2k+1} = \xi_1(\alpha_{2k}, \beta_{2k}),\quad \beta_{2k+1} = \xi_1(\beta_{2k}, \alpha_{2k})$$
and
$$\alpha_{2k+2} = \xi_1(\alpha_{2k+1}, \beta_{2k+1}),\quad \beta_{2k+2} = \xi_1(\beta_{2k+1}, \alpha_{2k+1}) \quad (3.1.2)$$

Now, we have show that

$$\alpha_{2k} \leq \alpha_{2k+1} \leq \alpha_{2k+2} \text{ and } \beta_{2k} \geq \beta_{2k+1} \geq \alpha_{2k+2}, \text{ for all } k \geq 0. \quad (3.1.3)$$

Now from (3.1.2) with $k = 0$, we have

$$\alpha_0 \leq \xi_1(\alpha_0, \beta_0) = \alpha_1 \text{ and } \beta_0 \geq \xi_1(\beta_0, \alpha_0) = \beta_1 \text{ Such that}$$
$$\alpha_1 = \xi_1(\alpha_0, \beta_0) \leq \xi_2(\alpha_0, \beta_0) \leq \xi_2(\alpha_1, \beta_1) = \alpha_2 \text{ and } \beta_1 = \xi_1(\beta_0, \alpha_0) \geq \xi_2(\beta_0, \alpha_0) \geq \xi_2(\beta_1, \alpha_1) = \beta_2.$$

Thus, the inequality in (3.1.3) holds for $k = 0$. Let (3.1.3) hold for k and according to the mixed monotone property of ξ_1 and ξ_2. We have

$$\alpha_{2k+1} = \xi_1(\alpha_{2k}, \beta_{2k}) \leq \xi_1(\alpha_{2k+1}, \beta_{2k+1}) = \alpha_{2k+2},$$
$$\beta_{2k+1} = \xi_1(\beta_{2k}, \alpha_{2k}) \geq \xi_1(\beta_{2k+1,}\alpha_{2k+1}) = \beta_{2k+2}. \quad (3.1.4)$$
and
$$\alpha_{2k+2} = \xi_1(\alpha_{2k+1}, \beta_{2k+1}) \leq \xi_1(\alpha_{2k+2}, \beta_{2k+2}) = \alpha_{2k+3},$$
$$\beta_{2k+2} = \xi_1(\beta_{2k+1}, \alpha_{2k+1}) \geq \xi_1(\beta_{2k+2,}\alpha_{2k+2}) = \beta_{2k+3}. \quad (3.1.5)$$

Now, from (3.1.4) and (3.1.5) $\alpha_{2k+1} \leq \alpha_{2k+2}$, $\beta_{2k+1} \leq \beta_{2k+2}$ and $\alpha_{2k+2} \geq \alpha_{2k+3}, \beta_{2k+2} \geq \beta_{2k+3}$.Thus inequality (3.1.3) hold for $k + 1$. and we get

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{2k} \leq \alpha_{2k+1} \leq \cdots \text{ and } \beta_0 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{2k} \geq \beta_{2k+1} \geq \cdots.$$

Now, using (3.1.1), we get

$$\begin{aligned} d(\alpha_{2k+1}, \alpha_{2k+2}) &= d(\xi_1(\alpha_{2k}, \beta_{2k}), \xi_2 d(\alpha_{2k+1}, \beta_{2k+1})) \\ &\leq l_1 d(\alpha_{2k}, \alpha_{2k+1}) + l_2 d(\beta_{2k}, \beta_{2k+1}) + l_3\left[\frac{d(\alpha_{2k,}\xi_1(\alpha_{2k,}\beta_{2k}))d(\alpha_{2k+1,}\xi_2(\alpha_{2k+1,}\beta_{2k+1}))}{d(\alpha_{2k,}\alpha_{2k+1})}\right] \\ &\quad + l_4\left[\frac{d(\beta_{2k,}\xi_1(\beta_{2k,}\alpha_{2k}))d(\beta_{2k+1,}\xi_2(\beta_{2k+1,}\alpha_{2k+1}))}{d(\beta_{2k,}\beta_{2k+1})}\right] \\ &\leq l_1 d(\alpha_{2k}, \alpha_{2k+1}) + l_2 d(\beta_{2k}, \beta_{2k+1}) + l_3 d(\alpha_{2k+1}, \alpha_{2k+2}) + l_4 d(\beta_{2k+1}, \beta_{2k+2}).. \quad (3.1.6) \end{aligned}$$

Similarly we can prove that

$$d(\beta_{2k+1}, \beta_{2k+2}) \leq l_1 d(\beta_{2k}, \beta_{2k+1}) + l_2 d(\alpha_{2k}, \alpha_{2k+1}) + l_3 d(\beta_{2k+1}, \beta_{2k+2}) + l_4 d(\alpha_{2k+1}, \alpha_{2k+2}).. \quad (3.1.7)$$

Adding (3.1.6) and (3.1.7) we have

$$d(\alpha_{2k+1}, \alpha_{2k+2}) + d(\beta_{2k+1}, \beta_{2k+2}) \leq \lambda[d(\alpha_{2k}, \alpha_{2k+1}) + d(\beta_{2k}, \beta_{2k+1})] \text{ , where } \lambda = \frac{l_1 + l_2}{1\_(l_3 + l_4)} < 1.$$

Now from above equation, we get

$$\begin{aligned} d(\alpha_{2k+1}, \alpha_{2k+2}) + d(\beta_{2k+1}, \beta_{2k+2}) &\leq \lambda[d(\alpha_{2k}, \alpha_{2k+1}) + d(\beta_{2k}, \beta_{2k+1})] \\ &\leq \lambda^2[d(\alpha_{2k-1}, \alpha_{2k}) + d(\beta_{2k-1}, \beta_{2k})] \\ &\leq \lambda^3[d(\alpha_{2k-2}, \alpha_{2k-1}) + d(\beta_{2k-2}, \beta_{2k-1})] \end{aligned}$$

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$$\leq \lambda^k[d(\alpha_0,\alpha_1)+(\beta_0,\beta_1)].$$

Thus it follows that

$$d(\alpha_{k+1},\alpha_k)+\ d(\beta_{k+1},\beta_k)\leq \lambda^k[d(\alpha_0,\alpha_1)+(\beta_0,\beta_1)]\ldots \tag{3.1.8}$$

This implies that

$\lim_{k\to\infty}[d(\alpha_{k+1},\alpha_k)+\ d(\beta_{k+1},\beta_k)]=0$. Consequently, we have

$\lim_{n\to\infty}\ d(\alpha_{k+1},\alpha_k)=0$, and $\lim_{k\to\infty}\ d(\beta_{k+1},\beta_k)=0$. Therefore, for $k,r\in N$, using (3.1.8) by the triangle inequality, we have

$$\begin{aligned}d(\alpha_k,\alpha_{k+r})+\ d(\beta_k,\beta_{k+r}) &\leq [\ d(\alpha_k,\alpha_{k+1})+\ d(\alpha_{k+1},\alpha_{k+2})+\cdots+\ d(\alpha_{k+r-1},\alpha_{k+r})]\\ &\quad +[d(\beta_k,\beta_{k+1})+d(\beta_{k+1},\beta_{k+2})+\cdots+\ d(\beta_{k+r-1},\beta_{k+r})]\\ &= \begin{bmatrix}\{d(\alpha_k,\alpha_{k+1})+d(\beta_k,\beta_{k+1})\}+\{d(\alpha_{k+1},\alpha_{k+2})+d(\beta_{k+1},\beta_{k+2})\}+\cdots+\\ \{d(\alpha_{k+r-1},\alpha_{k+r})+d(\beta_{k+r-1},\beta_{k+r})\}\end{bmatrix}\\ &= \begin{bmatrix}\lambda^k\{d(\alpha_0,\alpha_1)+d(\beta_0,\beta_1)\}+\lambda^{k+1}\{d(\alpha_0,\alpha_1)+d(\beta_0,\beta_1)\}+\cdots+\\ \lambda^{k+r-1}\{d(\alpha_0,\alpha_1)+d(\beta_0,\beta_1)\}\end{bmatrix}\\ &\leq [d(\alpha_0,\alpha_1)+d(\beta_0,\beta_1)][\lambda^k+\lambda^{k+1}+\cdots+\lambda^{k+r-1}]\\ &\leq \lambda^k(1+\lambda+\lambda^2+\cdots+\lambda^{r-1})[d(\alpha_0,\alpha_1)+d(\beta_0,\beta_1)]\\ &\leq \frac{\lambda^k}{1-\lambda}\ [d(\alpha_0,\alpha_1)+d(\beta_0,\beta_1)] \text{ as } k\to 0.\end{aligned}$$

This implies that

$\lim_{k\to 0}[d(\alpha_k,\alpha_{k+r})+\ d(\beta_k,\beta_{k+r})\ ]=0$. Hence $\{\alpha_k\}$ and $\{\beta_k\}$ are Cauchy sequence in $\mathcal{H}$.

Since $(\mathcal{H},d,\leq)$ is a partially ordered complete metric space, there exist $\alpha^*,\beta^*\in\mathcal{H}$ such that

$\lim_{k\to\infty}\alpha_k=\alpha^*$ and $\lim_{k\to\infty}\beta_k=\beta^*$. Then we have

$$\alpha^*=\ \lim_{k\to\infty}\alpha_{k+1}=\lim_{k\to\infty}\xi_1(\alpha_{2k},\beta_{2k})=\xi_1(\lim_{k\to\infty}\alpha_{2k},\lim_{k\to\infty}\beta_{2k}\ )=\xi_1(\alpha^*,\beta^*)$$

and

$$\beta^*=\ \lim_{k\to\infty}\beta_{k+1}=\lim_{k\to\infty}\xi_1(\beta_{2k},\alpha_{2k})=\xi_1(\lim_{k\to\infty}\beta_{2k},\lim_{k\to\infty}\alpha_{2k}\ )=\xi_1(\beta^*,\alpha^*).$$

Thus, $\alpha^*=\xi_1(\alpha^*,\beta^*)$ and $\beta^*=\xi_1(\beta^*,\alpha^*)$. Hence (α^*,β^*) is a coupled fixed point of ξ_1 in $\mathcal{H}\ \times\mathcal{H}$.
Similarly, we can show that (α^*,β^*) is a coupled fixed point of ξ_2 in $\mathcal{H}\ \times\mathcal{H}$.
Since $\alpha^*=\xi_1(\alpha^*,\beta^*)=\xi_2(\alpha^*,\beta^*)$ and $\beta^*=\xi_1(\beta^*,\alpha^*).=\xi_2(\beta^*,\alpha^*)$. Therefore (α^*,β^*) is a common coupled fixed point of ξ_1 and ξ_2 in $\mathcal{H}\ \times\mathcal{H}$. This completes the theorem.

Example 3.2: Let $\mathcal{H}=[0,1]$ be endowed with the usual metric $\xi:\mathcal{H}\ \times\mathcal{H}\to\mathcal{H}$ in the usual order. Then $(\mathcal{H},d,\leq)$ be a partially ordered complete metric space. Define the mappings $\xi_1,\xi_2:\mathcal{H}\ \times\mathcal{H}\to\mathcal{H}$ by

$$\xi_1(\alpha,\beta)=3\alpha-2\beta$$

$$\xi_2(\alpha,\beta)=\begin{cases}2\alpha-\beta+1, & \text{if } \alpha\geq\beta\\ \frac{1}{7} & \text{, if } \alpha\leq\beta\end{cases};\quad \text{for } \alpha,\beta\in[0,1].$$

Clearly, $\xi_1$and ξ_2 have the mixed monotone property. Let $\alpha_0=\frac{2}{3},\beta_0=\frac{1}{2}$. So,

$\xi_1(\alpha_0,\beta_0)=\xi_1(\frac{2}{3},\frac{1}{2})=3\left(\frac{2}{3}\right)-2\left(\frac{1}{2}\right)=1.$, $\xi_1(\beta_0,\alpha_0)=\xi_1(\frac{1}{2},\frac{2}{3})=2\left(\frac{1}{2}\right)-2\left(\frac{2}{3}\right)=\frac{1}{6}$. And

$\xi_2(\alpha_0,\beta_0)=\xi_2(\frac{2}{3},\frac{1}{2})=2\left(\frac{2}{3}\right)-\frac{1}{2}+1=\frac{11}{6}$, $\xi_2(\beta_0,\alpha_0)=\frac{1}{7}$.Thus $\frac{2}{3}\leq 1\leq\frac{11}{6}$ and $\frac{1}{2}\geq\frac{1}{6}\geq\frac{1}{7}$.

Hence, $\alpha_0\leq\xi_1(\alpha_0,\beta_0)\leq\xi_2(\alpha_0,\beta_0)$and $\beta_0\geq\xi_1(\beta_0,\alpha_0)\geq\xi_2(\beta_0,\alpha_0)$.

Now consider the contraction condition 3.1.1 in theorem 3.1., Let $\alpha=\frac{4}{3},\beta=\frac{1}{3}$, $l_1=1,l_2=2,l_3=3$ and $l_4=4,u=\frac{4}{5},\ v=\frac{1}{5}$. Then $\xi_1(\alpha,\beta)=\frac{10}{3},\xi_1(\beta,\alpha)=\frac{5}{3},\xi_2(u,v)=\frac{12}{5}$, $d\big(\xi_1(\alpha,\beta),\xi_2(u,v)\big)=\frac{14}{15}$;
$d(\alpha,u)=\frac{8}{15}$, $d(\beta,v)=\frac{2}{15}$, $d\big(\alpha,\xi_1(\alpha,\beta)\big)=\frac{2}{3}$, $d\big(u,\xi_2(u,v)\big)=\frac{8}{5}$, $d\big(\beta,\xi_1(\beta,\alpha,)\big)=\frac{4}{3}$ and
$d\big(v,\xi_2(v,u)\big)=\frac{2}{5}$. Hence by (3.1.1), we get

$$\begin{aligned}\frac{14}{15} &\leq \frac{8}{15}+\frac{4}{15}+3\left[\frac{16}{5}\times\frac{15}{8}\right]+4\left[\frac{8}{15}\times\frac{15}{2}\right]\\ &\leq \frac{8}{15}+\frac{4}{15}+18+16\\ &= \frac{522}{15};\\ &= 34.8.\end{aligned}$$

Since all the assumptions of theorem 3.1 are satisfied. Therefore, ξ_1 and ξ_2 have unique fixed points in $\mathcal{H}\times\mathcal{H}$.

Corollary 3.3: Let $\xi_1, \xi_2: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be ant two functions having the mixed monotone property on $(\mathcal{H}, d, \leq)$, where $(\mathcal{H}, d, \leq)$ is a partially ordered complete metric space, satisfying the conditions:

$$d(\xi_1(\alpha,\beta)\xi_2(u,v)) \leq l_1 d(\alpha,u) + l_2 d(\beta,v) \ldots \quad (3.3.1)$$

For all $\alpha, \beta, u, v \in \mathcal{H}$ with $\alpha \geq u$ and $\beta \geq v$, where $l_i \in [0,1]$ for$i = 1,2$ such that $\sum_{i=0}^{2} l_i < 1$ and there exist $\alpha_0, \beta_0 \in \mathcal{H}$ where $\alpha_0 \leq \xi_1(\alpha_0, \beta_0) \leq \xi_2(\alpha_0, \beta_0)$ and $\beta_0 \geq \xi_1(\beta_0, \alpha_0) \geq \xi_2(\beta_0, \alpha_0)$. Then all $\xi_1$and ξ_2 have a common coupled fixed point in $\mathcal{H} \times \mathcal{H}$.

Proof: Putting $l_3 = 0$ and $l_4 = 0$ in Theorem 3.1, we get Corollary 3.3.

Corollary 3.4: Let $\xi_1, \xi_2: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be ant two functions having the mixed monotone property on $(\mathcal{H}, d, \leq)$, where $(\mathcal{H}, d, \leq)$ is a partially ordered complete metric space, satisfying the conditions:

$$d(\xi_1(\alpha,\beta)\xi_2(u,v)) \leq l_1 d(\alpha,u) + l_2 \left[\frac{d(\alpha,\xi_1(\alpha,\beta)),\ d(u,\xi_2(u,v))}{d(\alpha,u)}\right] \ \ldots \quad (3.4.1)$$

For all $\alpha, \beta, u, v \in \mathcal{H}$ with $\alpha \geq u$ and $\beta \geq v$, where $l_i \in [0,1]$ for$i = 1,2$,such that $\sum_{i=0}^{2} l_i < 1$ and there exist $\alpha_0, \beta_0 \in \mathcal{H}$ where $\alpha_0 \leq \xi_1(\alpha_0, \beta_0) \leq \xi_2(\alpha_0, \beta_0)$ and $\beta_0 \geq \xi_1(\beta_0, \alpha_0) \geq \xi_2(\beta_0, \alpha_0)$. Then all $\xi_1$and ξ_2 have a common coupled fixed point in $\mathcal{H} \times \mathcal{H}$.

Proof; Taking $l_2 = 0, l_4 = 0$ and $l_3 = l_2$ in Theorem 3.1, we get the Corollary 3.4.

Corollary 3.5: Let $\xi: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a function having the mixed monotone property on $(\mathcal{H}, d, \leq)$, where $(\mathcal{H}, d, \leq)$ is a partially ordered complete metric space, satisfying the conditions:

$$d(\xi(\alpha,\beta),\xi(u,v)) \leq l_1 d(\alpha,u) + l_2 d(\beta,v) + l_3 \left[\frac{d(\alpha,\ \xi(\alpha,\beta)),\ d(u,\ \xi(u,v))}{d(\alpha,u)}\right]$$
$$+ l_4 \left[\frac{d(\beta,\ \xi(\beta,\alpha)),\ d(v,\ \xi(v,u))}{d(\beta,v)}\right] \ldots \quad (3.5.1)$$

For all $\alpha, \beta, u, v \in \mathcal{H}$ with $\alpha \geq u$ and $\beta \geq v$, where $l_i \in [0,1]$ for$i = 1,2, \ldots 4$ such that $\sum_{i=0}^{4} l_i < 1$ and there exist $\alpha_0, \beta_0 \in \mathcal{H}$ where $\alpha_0 \leq \xi(\alpha_0, \beta_0)$ and $\beta_0 \geq \xi(\beta_0, \alpha_0)$Then all ξ have a coupled fixed point in $\mathcal{H} \times \mathcal{H}$.

Proof: Taking $\xi_1 = \xi_2 = \xi$ in Theorem 3.1, we get Corollary 3.5.

Corollary 3.6: Let $\xi: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a function having the mixed monotone property on $(\mathcal{H}, d, \leq)$, where $(\mathcal{H}, d, \leq)$ is a partially ordered complete metric space, satisfying the conditions:

$$d(\xi(\alpha,\beta),\xi(u,v)) \leq l_1 d(\alpha,u) + l_2 d(\beta,v) \ldots \quad (3.6.1)$$

for all $\alpha, \beta, u, v \in \mathcal{H}$ with $\alpha \geq u$ and $\beta \geq v$, where $l_i \in [0,1]$ for$i = 1,2, \ldots 4$ such that $\sum_{i=0}^{4} l_i < 1$ and there exist $\alpha_0, \beta_0 \in \mathcal{H}$ where $\alpha_0 \leq \xi(\alpha_0, \beta_0)$ and $\beta_0 \geq \xi(\beta_0, \alpha_0)$Then all ξ have a coupled fixed point in $\mathcal{H} \times \mathcal{H}$.

Proof: Taking $l_3 = l_3 = 0$ in Corollary 3.5, we get Corollary 3.6.

Corollary 3.7: Let $\xi: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a function having the mixed monotone property on $(\mathcal{H}, d, \leq)$, where $(\mathcal{H}, d, \leq)$ is a partially ordered complete metric space, satisfying the conditions:

$$d(\xi(\alpha,\beta),\xi(u,v)) \leq d(\alpha,u) + \psi d(\beta,v) \ldots \quad (3.7.1)$$

for all $\alpha, \beta, u, v \in$ □ with $\alpha \geq u$ and $\beta \geq v$, where $l_i \in [0,1]$ for$i = 1,2, \ldots 4$ such that $\sum_{i=0}^{4} l_i < 1$ and there exist $\alpha_0, \beta_0 \in$ □ where $\alpha_0 \leq \xi(\alpha_0, \beta_0)$ and $\beta_0 \geq \xi(\beta_0, \alpha_0)$, and $\psi: [0,\infty) \to [0,\infty]$ is a continuous. Then ξ has a coupled fixed point in $\mathcal{H} \times \mathcal{H}$.

Proof: Taking $l_1 = 1, and\ l_2 = \psi$ in Corollary 3.6, we get the result of Chouhan and Sharma [31].

Example 3.8: We Consider example 3.3 of [31] and presume $\mathcal{H} = [0,\infty)$be endowed with standard metric $d(\alpha,\beta) = |\alpha - \beta|$, for all $\alpha, \beta \in \mathcal{H}$. Then $(\mathcal{H}, d, \leq)$ be a partially ordered complete metric space.

Define the mappings $\xi: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ by$\xi(\alpha,\beta) = \frac{\alpha - 2\beta}{3}$; $\alpha \geq 2\beta$. Now, we take $\psi: [0,\infty) \to [0,\infty]$ such that $\psi(t) = \frac{2t}{3}$. Clearly ξ is continuous and has the mixed monotone property. Also there are $\alpha_0 = 0$; $\beta_0 = 0$ in $\mathcal{H}$ such that

$$\alpha_0 = 0 \leq \xi(\alpha_0, \beta_0) = \xi(\alpha_0, \beta_0) \text{ and } \beta_0 = 0 \geq \xi(0,0) = \xi(\beta_0, \alpha_0).$$

Then it is obvious that (0, 0) is coupled fixed point of $\mathcal{H}$. Now, the following possibility for values of (α, β) and (u, v) such that $\alpha \geq u$ and $\beta \leq v$. Then

$$\begin{aligned} d(\xi(\alpha,\beta),\xi(u,v)) &= d\left(\frac{\alpha-2\beta}{3}, \frac{u-2v}{3}\right) \\ &= \frac{1}{3}|(\alpha - 2\beta) - (u - 2v)| \\ &= \frac{1}{3}|(\alpha - u) - 2(\beta - v)| \end{aligned}$$

$$\leq \frac{1}{3}|(\alpha - u)| + \frac{2}{3}|(\beta - v)|$$
$$\leq |(\alpha - u)| + \frac{2}{3}|(\beta - v)|$$
$$= d(\alpha, u) + \psi d(\beta, v).$$

Thus all the conditions of Corollary 3.7 are satisfied. Hence ξ has a coupled fixed point in $\mathcal{H} \times \mathcal{H}$.

Theorem 3.1: Let $\xi_1, \xi_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be ant two functions having the mixed monotone property on $(\mathcal{H}, d, \leq)$, where $(\mathcal{H}, d, \leq)$ is a partially ordered complete metric space, satisfying the conditions:

$$d(\xi_1(\alpha,\beta)\xi_2(u,v)) \leq l_1 d(\alpha,u) + l_2 d(\beta,v) + l_3\left[\frac{d(\alpha,\xi_2(u,v)), d(u,\xi_2(\alpha,\beta))}{d(\alpha,u)}\right]$$
$$+ l_4\left[\frac{d(\beta,\xi_2(v\ u)),\ d(v,\xi_1(\beta,\alpha))}{d(\beta,v)}\right] \ldots \quad (3.1.1)$$

For all $\alpha, \beta, u, v \in \mathcal{H}$ with $\alpha \geq u$ and $\beta \geq v$, where $l_i \in [0,1]$ for $i = 1,2, \ldots 4$ such that $\sum_{i=0}^{4} l_i < 1$ and there exist $\alpha_0, \beta_0 \in \mathcal{H}$ where $\alpha_0 \leq \xi_1(\alpha_0, \beta_0) \leq \xi_2(\alpha_0, \beta_0)$ and $\beta_0 \geq \xi_1(\beta_0, \alpha_0) \geq \xi_2(\beta_0, \alpha_0)$. Then all ξ_1 and ξ_2 have a common coupled fixed point in $\mathcal{H} \times \mathcal{H}$.

Proof: The proof is similar to Theorem 3.1.

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