Dynamics Of the Polynomial Differential Systems

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Abstract: In this paper, we study the bifurcation of limit cycles for the following Lienard systems

$$\dot{x} = y \ \ , \dot{y} = -f_m(x)y - g_n(x),$$

where, $f_m(x)$ and $g_n(x)$ respectively are polynomials of degree m and n, $g_n(0) = 0$. We prove that, if $m = 5$ and $g_n(x) = x$, there are always Lienard systems of the above form as they have a limit cycle.

Keywords: Limit cycles , The bifurcation set.

1. Introduction

In general, there are several free parameters. By using a method introduced in a previous paper, we obtain a sequence of algebraic a proximations to the bifurcationsets,in the parameter space.

Each algebraic approximation represents an exact lower bound to the bifurcation set.

The method is perturbative. So, it is not necessary to have a small or a large parameter in order to obtain these results.

We consider the following problem

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = \varepsilon(1 - x^2)y - x. \end{cases}$$

(See [1]).

Liénard system

In 1926, German Van Der Pol proposed the differential equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0, \qquad \varepsilon > 0. \quad (1)$$

Can the Lienard system with m > 5 and m + 1 < n < 2m have an algebric limit cycle ?

In this part, by developing the main ideas we prove the next results which give a positive solution to the problem opened above.

2. MAIN RESULT

The first step We choose m = 5 and n = 1 then $f_m(x) = a_0x^5 + a_1$, $g_n(x) = x$.

The system becomes.

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$$\begin{cases} \dot{x} = y \\ \dot{y} = -(a_0 x^5 + a_1)y - x. \end{cases}$$

First approximation the general solution is

$$x = C\cos(t - \alpha):$$

(See [2])

Where C and α are arbitrary, we study how the presence of the term with ϵ affects on the solution

$$x = C\cos t,$$

Being given what should be a first reasonable approximation, we try

$$x = A\cos(\omega t),$$

Where ω is a constant which is close to 1 (See [2]) .

$$\begin{cases} \dot{x} = -A\omega\sin(\omega t) \\ \ddot{x} = -A\omega^2\cos(\omega t). \end{cases}$$

by replacing this in the coming differential equation

$$\ddot{x} + (a_0 x^5 + a_1)\dot{x} + x = 0, \qquad (2)$$

We find that $-Aw^2\cos(\omega t) - (a_0 A^5\cos(\omega t)^5 + a_1)(Aw\sin(\omega t)) + A\cos(\omega t) = 0$

$$A(-w^2)\cos(\omega t) = a_0 A^6 w\cos(\omega t)^5\sin(\omega t) + a_1 Aw\sin(\omega t),$$

$$= Aw(a_0 A^5\cos(\omega t)^5\sin(\omega t) + a_1 A\sin(\omega t)) \quad .$$

According to the simplification, we find the second member equal at :

$$= Aw\left(a_0\frac{A^5}{32}\sin(6\omega t) + \frac{3}{16}a_0 A^5\sin(4\omega t) + \frac{15}{32}a_0 A^5\sin(2\omega t) + a_1\sin(\omega t)\right).$$

This equation can be satisfactory for all t only if the coefficients of the different terms sinusoidal disappear The term cos(wt) diappear if w=1 and the coefficient of sin(wt) is zero if we take

$$w=1, \quad a_1=0.$$

We see that the choice of A is arbitrary that signifies that the system doesn't admit an isolated closed curve as limited cycle (See [9]). So, we search for another more efficient method :

Passage in coordinated polar We take :

$$\begin{cases} x = r\cos(\omega t) \\ y = r\sin(\omega t). \end{cases}$$

According to the study,was :

$$\begin{cases} \dot{r} = (x\dot{x} + y\dot{y})/r \\ \dot{\theta} = (x\dot{y} - y\dot{x})/r^2. \end{cases}$$

(See [11,13]) We have :

$$x\dot{x} + y\dot{y} = xy - (a_0 x^5 + a_1)y^2 - xy,$$

$$= -(a_0 x^5 + a_1)y^2,$$

$$= -r^2(a_0 r^5\cos(\theta)^5 + a_1)\sin(\theta)^2.$$

Then

$\dot{r} = -r(a_0 r^5 \cos\theta^5 + a_1)\ \sin\theta^2,$

$$=-r(a_0 r^5 \cos\theta^5 \sin\theta^2 + a_1 \sin\theta^2)\ ,$$

$$= -\ \mathrm{r}(a_0 r^5\ \left(\frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta\right)\quad \sin\theta^2 + a_1 \frac{1-\cos 2\theta}{2}),$$

$$\dot{r} = \frac{3}{64} a_0 r^6 \cos 5\theta + \frac{1}{64} a_0 r^6 \cos 7\theta + a_1 \frac{r^2}{2}\cos 2\theta - \frac{5}{64} a_0 r^6 \cos\theta - a_1 \frac{r^2}{2} + \frac{1}{64} a_0 r^6 \cos 3\theta.$$

We have :

$x\dot{y} - y\dot{x} = -(a_0 x^5 + a_1)xy - x^2 - y^2,$

$$= -(a_0 x^5 + a_1)xy - (x^2 + y^2),$$

$$= -\ r^2((a_0 r^5 \cos 5\theta + a_1)\cos\theta \quad \sin\theta + 1)$$

We have :

$$\dot{\theta} = -\ ((a_0 r^5 \cos 5\theta + a_1)\cos\theta \quad \sin\theta + 1),$$

$$\dot{\theta}\ = -\ \ (a_0 r^5\ \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta + a_1)\frac{\sin 2\theta}{2} + 1),$$

$$\dot{\theta} = \ a_0 r^5 (\frac{1}{32}\ \cos 5\theta\ \ \sin 2\theta - \frac{5}{32}\ \cos 3\theta\ \ \sin 2\theta\ - \frac{5}{16}\ \cos\theta\ \ \sin 2\theta - a_1 \frac{\sin 2\theta}{2} - 1),$$

$$\dot{\theta} = -\ \frac{1}{64} a_0 r^5\ \ \sin 7\theta - \frac{5}{64} a_0 r^5\ \ \sin 5\theta\ - \frac{5}{64} a_0 r^5\ \ \sin 3\theta + \frac{15}{64} a_0 r^5\ \ \sin\theta - \frac{a_0\, r^5 a_1}{2}\ \sin 2\theta - 1\ \ .$$

So the system becomes :

$$\begin{cases} \dot{r} = \dfrac{3}{64} a_0 r^6 \cos 5\theta\ + \dfrac{1}{64} a_0 r^6 \cos 7\theta + a_1 \dfrac{r^2}{2}\cos 2\theta - \dfrac{5}{64} a_0 r^6 \cos\theta - a_1 \dfrac{r^2}{2} + \dfrac{1}{64} a_0 r^6 \cos 3\theta. \\ \dot{\theta} = -\dfrac{1}{64} a_0 r^5\ \ \sin 7\theta - \dfrac{5}{64} a_0 r^5\ \ \sin 5\theta\ - \dfrac{5}{64} a_0 r^5\ \ \sin 3\theta + \dfrac{15}{64} a_0 r^5\ \ \sin\theta - \dfrac{a_0\, r^5 a_1}{2}\ \sin 2\theta - 1\ . \end{cases}$$

We estimate the conditions starting from the existence of the limited cycle (th Bedixon)(See [9,6,3,10])

, so that the hypothesis doesn't change $a_0;a_1\ \neq 0$.

Perturbation of the system We take the equation:

$$\ddot{x} + (a_0 x^5 + a_1)\dot{x} + x = 0,$$

Development in the neighbourhood limit (0, 0). (See [9])

For the value of $\varepsilon = 0$, we find the exact solution.

The wanted idea is to extend the solution in power serial ε:

$$x(t) = x_0(t) + x_1(t)\varepsilon + x_2(t)\varepsilon^2 + 0(\varepsilon^3). \quad (3)$$

So :

$$\dot{x}(t)=\dot{x}_0(t)+\dot{x}_1(t)\varepsilon+\dot{x}_2(t)\varepsilon^2+0(\varepsilon^3).$$

$$\ddot{x}(t)=\ddot{x}_0(t)+\ddot{x}_1(t)\varepsilon+\ddot{x}_2(t)\varepsilon^2+0(\varepsilon^3).$$

$$(a_0x^5+a_1)=a_0\,(x_0(t)+x_1(t)\varepsilon+x_2(t)\varepsilon^2+0(\varepsilon^3))^5+a_1,$$

$$= a_0((x_0(t)+x_1(t)\varepsilon)^5 +5\ (x_0(t)+x_1(t)\varepsilon)^4x_2\,\varepsilon^2 +10\ (x_0(t)+x_1(t)\varepsilon)^3{x_2}^2\varepsilon^4 +10(x_0(t)+x_1(t)\varepsilon)^4{x_2}^3\varepsilon^6+5(x_0(t)+x_1(t)\varepsilon)^4{x_2}^4\varepsilon^8 +{x_2}^5\varepsilon^{10} +a_1\ ,$$

$$= a_0({x_0}^5(t)+5{x_0}^4(t)x_1(t)\varepsilon+10\,{x_0}^4(t){x_1}^2(t)\varepsilon^2+10\,{x_0}^2(t){x_1}^3(t)\varepsilon^3+ \ +5\,x_0(t){x_1}^4(t)\varepsilon^4+{x_1}^5(t)\varepsilon^5 \ +5\,x_2(t){x_0}^4(t)\varepsilon^2)+a_1,$$

$$=a_0({x_0}^5(t)+5{x_0}^4(t)x_1(t)\varepsilon+10\,{x_0}^3(t){x_1}^2(t)\varepsilon^2+5\,x_2(t){x_0}^4(t)\varepsilon^2)+a_1.$$

Is replaced in equation:

$$x\ddot{(t)}+x(t)=\big(\ddot{x}_0(t)+x_0(t)\big)+\big(\ddot{x}_1(t)+x_1(t)\big)\varepsilon+\big(\ddot{x}_2(t)+x_2(t)\big)\varepsilon^2+0(\varepsilon^3).$$

$$a_1\dot{x}(t)=a_1\dot{x}_0(t)+a_1\dot{x}_1(t)\varepsilon+a_2\dot{x}_2(t)\varepsilon^2+0(\varepsilon^3).$$

$$a_0\,x^5(t)\dot{x}(t)=a_0\,{x_0}^5(t)\dot{x}_0(t)+5a_0\,{x_0}^4(t)x_1(t)\dot{x}_0(t)\,\varepsilon+10\,a_0\,{x_0}^3(t){x_1}^2(t)\dot{x}_0(t)\,\varepsilon^2 +5\,a_0\,{x_0}^4(t)x_2(t)\dot{x}_0(t)\,\varepsilon^2+$$

$$a_0\,{x_0}^5(t)\,\dot{x}_1(t)\,\varepsilon+5\,a_0\,{x_0}^4(t)x_1(t)\dot{x}_1(t)\,\varepsilon^2+\ a_0\,{x_0}^5(t)\,\dot{x}_2(t)\,\varepsilon^2.$$

We find

$$\big(x_0+\ddot{x_0}+a_1\dot{x_0}+a_0x_0{}^5\dot{x_0}\big)+\big(x_1+\ddot{x_1}+a_1\dot{x_1}+a_0x_0{}^5\dot{x_1}+5a_0x_0{}^4\,x_1\dot{x_0}\big)\varepsilon +\big(x_2+\ddot{x_2}+a_1\dot{x_2}+10a_0x_0{}^3x_1{}^2\ \dot{x}_0+5a_0x_0{}^4\,x_2\dot{x_0}+5a_0x_0{}^4\,x_1\dot{x_1}+a_0x_0{}^5\,\dot{x_2}\big)\varepsilon^2+0(\varepsilon^3) =0$$

We want find a solution that is valid for all little values of ε. We cancel each of the coefficient $of\varepsilon^n$.

For n = 0, 1, 2,...

For n = 0; 1 and 2, we obtained :

$$x_0+\ddot{x_0}+a_1\dot{x_0}+a_0x_0{}^5\dot{x_0}=0 \qquad (4)$$

$$x_1+\ddot{x_1}+a_1\dot{x_1}+a_0x_0{}^5\dot{x_1}+5a_0x_0{}^4\,x_1\dot{x_0}=0 \qquad (5)$$

$$x_2+\ddot{x_2}+a_1\dot{x_2}+10a_0x_0{}^3x_1{}^2\ \dot{x}_0+5a_0x_0{}^4\,x_2\dot{x_0}+5a_0x_0{}^4\,x_1\dot{x_1}+a_0x_0{}^5\,\dot{x_2} =0 \qquad (6)$$

As the equation is autonomous we can choose the instant for corresponding t = 0 to any point of the limited cycle.

Thus,we can choose the initial condition $\dot{x}(0)=0$with out losing genralty from developing (5).

we obtain the initial conditions :

$$\dot{x}_0(0)=\ \dot{x}_1(0)=\dot{x}_2(0)=\cdots=0 \quad (7)$$

EQS (6) and (9) given $x_{0h}(t)=\alpha\cos(\theta)$, homogeneous solution. (8)

Where α is yet determined.

we find the particular solution equal at :

$$x_{0p}(t)=A_0\cos(6\,\theta)+B_0\cos(4\,\theta)+R_0\cos(2\theta)+M_0\cos(\theta).$$

We need of all terms (are not periodic) A_0, B_0, R_0, M_0 to be removed .by substituting (10) in (7) and by using trigonometric identities, we obtain :

We need of all terms (are not periodic) A_0; B_0; R_0; M_0 to be removed .
by substituting (10) in (7) and by using trigonometric identities.
We obtain :

$$\ddot{x_1} + x_1 = -(a_1 + a_0\alpha^5 \cos\theta^5)x_1 + 5a_0\alpha^5 \cos\theta^4 \sin\theta\, x_1 \quad (9)$$

$$\ddot{x_1} + (a_1 + a_0\alpha^5 \cos\theta^5)\dot{x_1} + (1 - 5a_0\alpha^5 \cos\theta^4 \sin\theta)x_1 = 0 \quad (10)$$

as a result, this is a differential equation of second order with variable coefficients. This equation is ,relatively, easy to solve in the general case. Be the equation

Supposing that the function $x_1$that satisfies the differential equation x_1=exp(k t) where k can be a complex number.

So we have

$$k^2 exp^{kt} + (a_1 + a_0\alpha^5 \cos\theta^5)k exp^{kt} + (1 - 5\alpha^5 \cos\theta^4 \sin\theta)\, exp^{kt} = 0,$$

or

$$k^2 + (a_1 + a_0\alpha^5 \cos\theta^5)k + (1 - 5\alpha^5 \cos\theta^4 \sin\theta) = 0,$$

this last relation is the quadratic equation auxiliary of the differential equation (polynomial charac-teristic)(See [9]).

It has to be solutions / roots that we will notice in the general case : k_1,k_2.

Immediately comes that :

$$k_{1,2} = \frac{-(a_1 + a_0\alpha^5 \cos\theta^5) \pm \sqrt{(a_1 + a_0\alpha^5 \cos\theta^5)^2 - 4(1 - 5\alpha^5 \cos\theta^4 \sin\theta)}}{2}$$

So

$$x_1 = s_1 exp^{k_1 t} + s_2 exp^{k_2 t}$$

If we take a_0= 0 or $\alpha = 0$ we find that x_1 is a periodic solution, but it's contradictory to the hypothesis or to the trivial solution x = 0.

Bifurcation system We have the following system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(a_0 x^5 + a_1)y - x. \end{cases}$$

We search equilibrium points for this system :

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$$

The system accepts one equilibrium point which is : (0; 0).

After that, we look for the Jacobean:

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -(5a_0x^5y)-1 & -(a_0x^5+a_1) \end{pmatrix}$$

So, at the equilibrium point:

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -a_1 \end{pmatrix}$$

We search for the polynomial characteristic :

$$\lambda^2 + a_1\lambda + 1 = 0. \quad \delta. = a_1{}^2 - 4$$

We find three cases δ :

If *$\delta=0$ then* $a_1 = \pm 2$ *thus the equilibrium point is a node.*

If *$\delta>0$ then* $a_1 > 2$, thus λ it accepts two values of different signs , and the equilibrium point is a saddle point..

The interesting case is $\delta<0$, then the value of λ is complex and in this case , we have :

$$\lambda = -a_1 \pm \sqrt{a_1{}^2 - 4}.$$

If $a_1 = 0$ the equilibrium points is central in the linear case for the non linear part it's a passage of polar coordinated because linear is topologically equivalent with the non linear but the centre can be a limited cycle.

If $0 < a_1 < 2$ the equilibrium point is an unstable focus (positive real part).

If $0 < a_1$ the equilibrium point is a steady focus (negative real part).

In the non linear case, we can use lyaponov's criterium :

The system linear predicts centres when the setting is equal to 0.

In the non linear system, we can see that those centres aren't in fact conserved. To determine the steadiness of the origin , we consider positive defined function $v(x, y)$.I take some examples of $v(x, y)$but they aren't efficient.

The second step

We choose :

$$f_m(x) = a_0x^5 + a_1x^4 + a_2, \qquad g_n(x) = x.$$

The system becomes :

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(a_0x^5 + a_1x^4 + a_2)y - x \end{cases}$$

In the first approximation, the general solution is :

$$x = C\cos(t - \alpha).$$

Where C and α are arbitrary.

We study how the presence of the term with ε affects on the solution :

$$x = C\, cost.$$

By giving what can be a first reasonable approximation , we try

$$x = A\, cos(\omega t).$$

Where w is a constant close to 1.

$$\begin{cases} \dot{x} = -A\, w \sin wt \\ \ddot{x} = -Aw^2 \cos wt \end{cases}$$

We replace in the following differential equation

$$\ddot{x} + (a_0\, x^5 + a_1\, x^4 + a_2\,)\dot{x} + x = 0. \qquad (12)$$

We find that :

$$-Aw^2 \cos wt - (a_0 A^5\ \cos wt^5 + a_1 A^4\ \cos wt^4 + a_2)\,(Aw \sin wt + A\cos wt\,) = 0,$$

$$A(1-w^2)\cos wt = a_0 A^6 w\ \cos wt^5 \sin wt + a_1 A^5 w\ \cos wt^4 \sin wt + a_2 Aw\ \sin wt,$$

$$= Aw\,(a_0 A^5\ \cos wt^5 + a_1 A^4\ \cos wt^4 + a_2)\sin wt.$$

After the simplification, we find the second member equal at :

$$= a_0 \frac{A^6 w}{32} \sin 6wt$$
$$+ a_1 \frac{A^5 w}{16} \sin 5wt + a_0 \frac{A^6 w}{8} \sin 4wt$$
$$+ 3a_1 \frac{A^5 w}{16} \sin 3wt + a_0 \frac{A^6 w}{32} \sin 2wt + \left(a_1 \frac{A^5 w}{8} + a_2 Aw\right) \sin wt.$$

This equation can be satisfied for all t only if the coefficients of the different sinusoidal terms disappear .

The term cos(wt) disappears if w = 1, and the coefficient of $A = \sqrt[4]{-8\frac{a_2}{a_1}}$ is zero as if we take $a_1, a_2 < 0, a_1 \neq 0$.

Thus, we choose : $w = 1,\ A = \sqrt[4]{-8\frac{a_2}{a_1}}$.

This lets yet the term that containing sin(6wt); sin(5wt); sin(4wt); sin(3wt); sin(2wt) to disappear because we have fixed w and A.

Thus, we find that the system admits a limited cycle of radius $A = \sqrt[4]{-8\frac{a_2}{a_1}}$ as $a_1, a_2 < 0, a_1 \neq 0$.

3rdStep

We choose $f_m(x) = a_0x^5 + a_1x^4 + a_2x^3 + a_3$, $g_n(x) = x$.

The system becomes :

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(a_0x^5 + a_1x^4 + a_2x^3 + a_3)y - x \end{cases}$$

In first approximation, the general solution is

$$x = C\cos(t - \alpha).$$

Where C and α are arbitrary.

We study how the presence of the term with ε affects on the solution

$$x = C\cos t.$$

By giving what can be a first reasonable approximation , we try

$$x = A\cos wt..$$

Where ω is a constant close to 1.

$$\begin{cases} \dot{x} = -A\,w\sin wt \\ \ddot{x} = -Aw^2\cos wt \end{cases}$$

-

We replace in the following differential equation

$$\ddot{x} + (a_0\,x^5 + a_1\,x^4 + a_2x^3 + a_3\,)\dot{x} + x = 0. \qquad (13)$$

We find that :

$$-Aw^2\cos wt - (a_0A^5\;\cos wt^5 + a_1A^4\;\cos wt^4 + a_2A^3\cos wt^3 + a_3)\,(Aw\sin wt + A\cos wt\,) = 0,$$

$$A(1-w^2)\cos wt = a_0A^6w\;\cos wt^5\sin wt + a_1A^5w\;\cos wt^4\sin wt$$
$$+ a_2Aw\;\sin wt + a_2A^4w\;\cos wt^3\sin wt + a_3Aw\;\sin wt,$$

$$= Aw\,(a_0A^5\;\cos wt^5 + a_1A^4\;\cos wt^4 + a_2A^3\cos wt^3 + a_3)\sin wt.$$

After the simplification, we find the second member equal at :

$$= a_0\frac{A^6w}{32}\sin 6wt + a_1\frac{A^5w}{16}\sin 5wt + (a_0\frac{A^6w}{8}$$
$$+ a_2\frac{A^4w}{8})\sin 4wt$$
$$+ 3a_1\frac{A^5w}{16}\sin 3wt + (a_0\frac{A^6w}{32} + a_2\frac{A^4w}{4})\sin 2wt + (a_1\frac{A^5w}{8} + a_3Aw)\sin wt.$$

This equation can be satisfied for all t only if the coefficients of the different sinusoidal terms disappear .

The term cos(wt) disappears if w = 1, and the coefficient of sin wt is zero as if we take A $=\sqrt[4]{-8\frac{a_3}{a_1}}$ $a_1, a_3 < 0, a_1 \neq 0.$

This lets yet the term that containing sin(6wt); sin(5wt); sin(4wt); sin(3wt); sin(2wt) to disappear because we have fixed w and A.

Thus, we find that the system admits a limited cycle of radius A $=\sqrt[4]{-8\frac{a_3}{a_1}}$ as $a_1, a_3 < 0, a_1 \neq 0.$

4^th^step

We choose $f_m(x)=a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4,$ $g_n(x)=x.$

The system becomes :

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4)y - x \end{cases}$$

In first approximation, the general solution is

$$x = C\,cos(t - \alpha).$$

Where C and α are arbitrary.

We study how the presence of the term with ε affects on the solution

$$x = C\,cost.$$

By giving what can be a first reasonable approximation , we try

$$x = A\,cos\,wt..$$

Where ω is a constant close to 1.

$$\begin{cases} \dot{x} = -A\,w \sin wt \\ \ddot{x} = -Aw^2 \cos wt \end{cases}$$

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We replace in the following differential equation

$$\ddot{x} + (a_0\,x^5 + a_1\,x^4 + a_2x^3 + a_3x^2 + a_4\,)\dot{x} + x = 0. \qquad (14)$$

We find that :

$$-Aw^2 \cos wt - (a_0A^5\ \cos wt^5 + a_1A^4\ \cos wt^4 + a_2A^3 \cos wt^3 + a_3A^2 \cos wt^2 + a_4\,)\,(Aw \sin wt + A\cos wt\,) = 0,$$

$$A(1 - w^2)\cos wt = a_0A^6w\ \cos wt^5 \sin wt + a_1A^2w\ \cos wt^4 \sin wt + a_2A^4w \cos wt^3\ \sin wt + a_3A^3w\ \cos wt^3 \sin wt + a_4Aw\ \sin wt,$$

$$= Aw\,(a_0A^5\ \cos wt^5 + a_1A^4\ \cos wt^4 + a_2A^3 \cos wt^3 + a_3A^2 \cos wt^2 + a_4) \sin wt.$$

After the simplification, we find the second member equal at :

$$= a_0\frac{A^6w}{32}\sin 6wt + a_1\frac{A^5w}{16}\sin 5wt + (a_0\frac{A^6w}{8} + a_2\frac{A^4w}{8}) \sin 4wt + (3a_1\frac{A^5w}{16} + a_3\frac{A^5w}{4}) \sin 3wt + (a_0\frac{A^6w}{32} + a_2\frac{A^4w}{4}) \sin 2wt + (a_1\frac{A^5w}{8} + a_3\frac{A^3w}{4} + a_4Aw) \sin wt.$$

This equation can be satisfied for all t only if the coefficients of the different sinusoidal terms disappear .

The term cos(wt) disappears if w = 1 and the coefficient of sin wt is zero as if we take

$$Aw\left(a_1\frac{A^4}{8} + a_3\frac{A^2}{4} + a_4\right) = 0 \qquad Aw \neq 0 \quad in\ order\ not\ to\ fall\ in\ trivial\ case.$$

$$a_1A^4 + 2a_3A^2 + 8a_4 = 0. We\ put\ y = A^2, so\ \delta = 4\,{a_3}^2 - 32\,a_1a_4.$$

If $\delta = 0 \rightarrow {a_3}^2 = 8a_1a_4$, then $y = \frac{-a_3}{a_1}$ so the system accepts a limited cycle of radius $A = \sqrt{\frac{-a_3}{a_1}}$ as $a_1, a_3 < 0, a_1 \neq 0$.

If $\delta > 0 \rightarrow {a_3}^2 > 8a_1a_4$, we find two solutions : $y_{1,2} = \frac{-4a_3 \pm \sqrt{\delta}}{2a_{11}}$.

So, the system accepts a limited cycle only if $A_{1,2} = \sqrt{y_{1,2}}$ provided that $y_{1,2} > 0$.

If $\delta < 0$, it is not accepts any limited cycle.

5^{th}step

We choose $f_m(x)=a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5$, $g_n(x) = x$.

The system becomes :

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5)y - x \end{cases}$$

In first approximation, the general solution is

$$x = C\ cos(t - \alpha).$$

Where C and α are arbitrary. We study how the presence of the term with ε affects on the solution

$$x = C\ cost.$$

By giving what can be a first reasonable approximation , we try

$$x = A\ cos\ wt..$$

Where ω is a constant close to 1.

$$\begin{cases} \dot{x} = -A\ w \sin wt \\ \ddot{x} = -Aw^2 \cos wt \end{cases}$$

-

We replace in the following differential equation

$$\ddot{x} + (a_0\ x^5 + a_1\ x^4 + a_2x^3 + a_3x^2 + a_4\ x + a_5)\dot{x} + x = 0. \qquad (15)$$

We find that :

$$-Aw^2 \cos wt - (a_0A^5\ \cos wt^5 + a_1A^4\ \cos wt^4 + a_2A^3 \cos wt^3 + a_3A^2 \cos wt^2 + a_4\ )\ (Aw \sin wt + A \cos wt\ ) = 0,$$

$$A(1 - w^2) \cos wt = a_0A^6w\ \cos wt^5 \sin wt + a_1A^5w\ \cos wt^4 \sin wt + a_2A^4w \cos wt^3\ \sin wt + a_3A^3w\ \cos wt^2 \sin wt + a_4A^2w\ \sin wt + a_5Aw\ \sin wt,$$

$$= Aw\ (a_0A^5\ \cos wt^5 + a_1A^4\ \cos wt^4 + a_2A^3 \cos wt^3 + a_3A^3 \cos wt^2 + a_3A^2\ \cos wt^2 + a_4A \cos wt + a_5) \sin wt.$$

After the simplification, we find the second member equal at :

$$a_0 \frac{A^6 w}{32} \sin 6wt + a_1 \frac{A^5 w}{16} \sin 5wt + (a_0 \frac{A^6 w}{8}$$
$$+ a_2 \frac{A^4 w}{8}) \sin 4wt \ + \ (3a_1 \frac{A^5 w}{16}$$
$$+ a_3 \frac{A^5 w}{4}) \sin 3wt + (a_0 \frac{A^6 w}{32}$$
$$+ a_2 \frac{A^4 w}{4} \ a_4 \frac{A^2 w}{2}) \sin 2wt + (a_1 \frac{A^5 w}{8} + a_3 \frac{A^3 w}{4} + a_5 Aw) \sin wt.$$

This equation can be satisfied for all t only if the coefficients of the different sinusoidal terms disappear .

The term cos(wt) disappears if w = 1 and the coefficient of sin wt is zero as if we take

$$Aw\left(a_1 \frac{A^4}{8} + a_3 \frac{A^2}{4} + a_5\right) = 0 \qquad Aw \neq 0 \quad in\ order\ not\ to\ fall\ in\ trivial\ case.$$

$$a_1 A^4 + 2a_3 A^2 + 8a_5 = 0. We\ put\ y = A^2, so\ \delta = 4\ {a_3}^2 - 32\ a_1 a_5.$$

If $\delta = 0 \to {a_3}^2 = 8a_1 a_5$, then $y = \frac{-a_3}{a_1}$ so the system accepts a limited cycle of radius $A = \sqrt{\frac{-a_3}{a_1}}$ as $a_1, a_3 < 0, a_1 \neq 0$.

If $\delta > 0 \to {a_3}^2 > 8a_1 a_5$, we find two solutions : $y_{1,2} = \frac{-4a_3 \pm \sqrt{\delta}}{2a_{11}}$.

So, the system accepts a limited cycle only if $A_{1,2} = \sqrt{y_{1,2}}$ provided that $y_{1,2} > 0$.

If $\delta < 0$, it accepts any limited cycle.

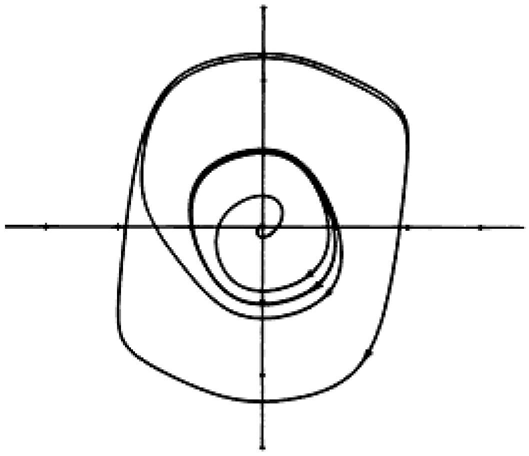

Figure : The two limit cycle of the Lienard system

8.Conclusion

In this paper, we proved the system can have at most 2 limit cycles. If f(x) is an odd polynomial of degree 5 then the probabilities that the Liénard equation for f(x) has at least 2 periodic solutions is greater than 47,23 % and that it has no periodic solution is greater than 34.54%.

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