Polynomial-Exponential Mixture of Poisson distribution

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Abstract:

This is a compound probability distribution based on the theoretical concept of continuous mixtures of Poisson distribution. It has a single parameter and two irrational numbers. It is named as 'Polynomial-Exponential Mixture of Poisson Distribution (PEMPD) which will play a very important role in the field of statistical modeling of over-dispersed count data. Statistical characteristics have been defined and derived according to the need of the proposed paper. By applying goodness of fit to some over-dispersed count data of secondary in nature, it has been observed that it is a better alternative of Poisson-Lindley distribution of (PLD) Sankaran, Poisson-Mishra distribution (PMD) of Sah and Poisson-Modified Mishra (MMD) distribution of Sah and Sahani,

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Introduction:

The proposed distribution is special because it is based on a single parameter and two irrational numbers. In fact, these two irrational numbers have a big contribution in making the normal distribution so special. What we find from the study and past research that continuous distributions based on two irrational numbers are more powerful for statistical modeling of statistical data than which is based on a single irrational number having a single parameter. In this process, Sah has obtained many continuous distributions having a single parameter and two irrational numbers such as New Linear-Exponential distribution [see,13], Premium Linear-Exponential distribution [see, 12], Modified Mishra distribution [see, 14]. New Quadratic-Exponential distribution [see, 16] and Polynomial-Exponential distribution [see, 11] have been obtained by Sah and Sahani. If these distributions are so special, then the distributions obtained by compounding Poisson distribution with these distributions will also be special for modeling of over-dispersed count data sets.
The proposed distribution has been obtained by mixing Poisson distribution with Polynomial-Exponential distribution (PED) of Sah and Sahani [see,11]. Probability density function (pdf) of PED was given by

$$f_1(y;\alpha)=\frac{\alpha^4}{(6+\pi\alpha^3)}(\pi+y^3)e^{-\alpha y}\ ;y>0;\alpha>0 \tag{1}$$

The work related to Lindley mixture of Poisson distribution was obtained by Sankaran in 1970. It was named as Poisson- Lindley distribution [see, 6] and its probability mass function (pmf) was given as

$$P_2(Y;\alpha)=\frac{\alpha^2(y+\alpha+2)}{(\alpha+1)^{y+3}}\quad ;y=0,1,2,...\quad ;\alpha>0 \tag{2}$$

Where Lindley distribution was obtained by Lindley [see,5] and its pdf was given by

$$f_3(y;\alpha)=\frac{\alpha^2}{(1+\alpha)}(1+y)e^{-\alpha y}\ ;y>0,\alpha>0 \tag{3}$$

Sah published Poisson-Mishra distribution (PMD) [see, 9] and has been obtained by compounding Poisson distribution with Mishra distribution [see, 8]. He showed that PMD is a better alternative to PLD. He also showed that Generalised Poisson-Mishra distribution [see, 10] is a better alternative to PMD for statistical modeling for count data having the variance greater than the mean. Recently, Sah and Sahani have obtained Poisson- Modified Mishra distribution (PMMD) [see, 15] and it has been obtained by compounding Poisson distribution with Modified Mishra distribution (MMD) [see, 14]. PMMD is special one because its pmf contains two irrational numbers.

The probability mass function of PMD and PMMD have been given as in the expression (4) and (5) respectively.

$$P_4(Y;\alpha)=\frac{\alpha^3[(1+\alpha)(y+\alpha+2)+(1+y)(2+y)]}{(\alpha^2+\alpha+1)(1+\alpha)^{y+3}} \quad ;y=0,1,2,... \quad ;\alpha>0 \qquad (4)$$

$$P_5(Y;\alpha)=\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\left[\frac{(1+\alpha)(1+\pi+\pi\alpha+y)+(1+y)(2+y)]}{(1+\alpha)^{y+3}}\right] \quad ;y=0,1,2,... \quad ;\alpha>0 \qquad (5)$$

To give a better and systematic shape of this paper, the various headings and sub-headings studied and analysed for the proposed distribution have been presented sequentially. The work of this paper has been classified under the following headings and sub-headings. The materials and methods adopted have been placed in the second section. The third section contains results obtained which may also be called heart of the paper and this section has been divided into different sub-headings as

- Polynomial-Exponential Mixture of Poisson Distribution (PEMPD) and Its Important Characteristics
- Statistical Moments and Essential Descriptive Measures of Statistics required for PEMPD
- Methods of Estimation, and
- Applications and Goodness of Fit of PEMPD.

The conclusion of this paper has been presented in the last section through scientific and statistical methods. It has been observed that PEMPD is more powerful compound distribution for statistical modeling of over-dispersed count data than PLD, PMD and MMD.

Material and Methods:

The work of this paper is theoretical to experimental oriented. It is based on the concept of continuous mixture of Poisson distribution. At first, all the essential theories required for this paper have been defined and derived in systematic manner and then, to test the validity of the theoretical work, Goodness of fit has been used in some over-dispersed secondary count data.

Results:

This section is considered as the main part of paper. Work of this section has been classified under the following different sub-headings

- Polynomial-Exponential Mixture of Poisson Distribution (PEMPD) and Its Important Characteristics
- Statistical Moments and Essential Descriptive Measures of Statistics required for PEMPD
- Methods of Estimation, and
- Applications and Goodness of Fit of PEMPD

Polynomial-exponential Mixture of Poisson Distribution (PEMPD) and Its Important Characteristics:

In this sub-heading, Probability mass function (pmf) of PEMPD has been constructed by mixing Poisson distribution with PED such that the Parameter (λ) of Poisson distribution act as continuous variable and follows Polynomial-Exponential distribution, and pmf of this distribution can be obtained as

$$P(Y;\alpha)=\int_0^\infty\left[\left(\frac{e^{-\lambda}\lambda^y}{y!}\right)\left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)(\pi+\lambda^3)e^{-\alpha\lambda}\right]d\lambda \ ;y=0,1,2,...;\lambda>0;\alpha>0$$

$$=\left(\frac{\alpha^4}{(6+\pi\alpha^3)}\frac{1}{y!}\right)\left[\pi\int_0^\infty\lambda^y e^{-(1+\alpha)\lambda}d\lambda+\int_0^\infty\lambda^{y+3}e^{-(1+\alpha)\lambda}d\lambda\right]$$

$$=\left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\left[\frac{\pi(1+\alpha)^3+(1+y)(2+y)(3+y)}{(1+\alpha)^{y+4}}\right];y=0,1,2,...;\alpha>0 \qquad (6)$$

The expression (6) is the formula for obtaining probability of $Y=y$ for $y=0,1,2,...$ and it is the pmf of PEMPD.

Graphical representations of pmf of PEMPD at varying values of parameter (α) are given as follows

Fig.1 Graph of pmf of PEMPD at $\alpha = 2,4,6,8$

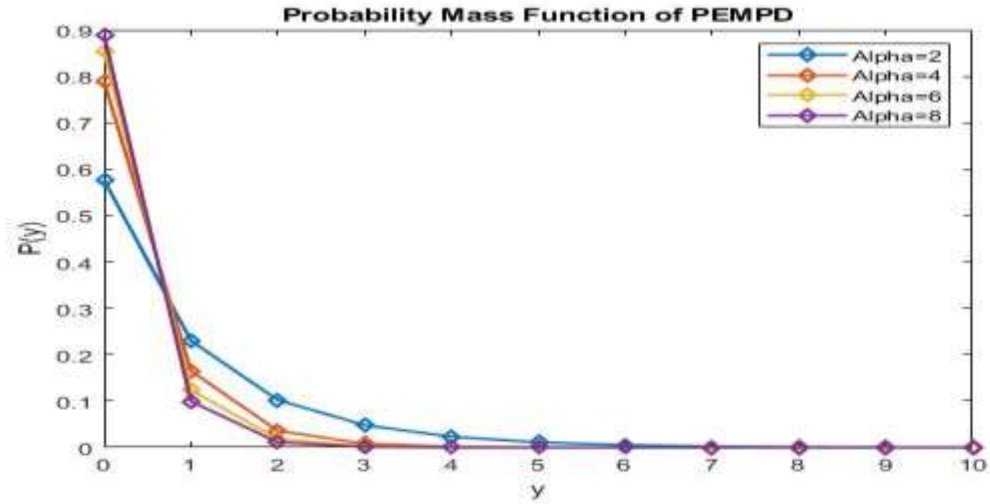


Fig.2 Graph of pmf of PEMPD at $\alpha = 0.5,1.0,2.0,2.5$

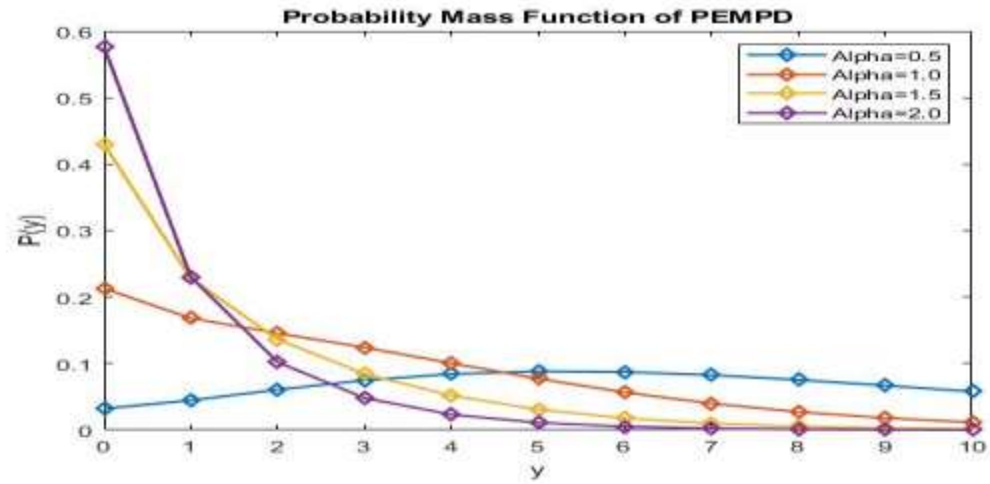
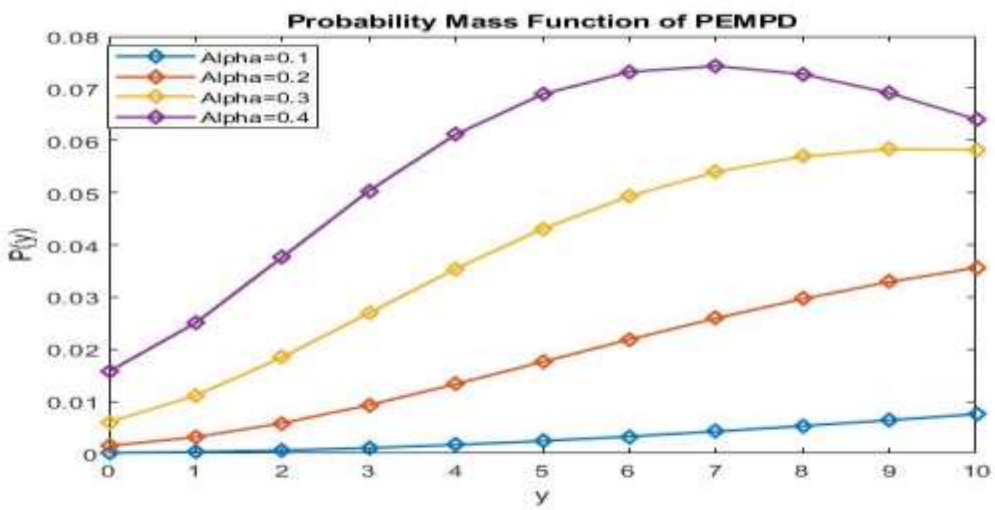


Fig.3 Graph of pmf of PEMPDD a $\alpha = 0.1,0.2,0.3,0.4$



From the above figures, it has been observed that at $y=0$, value of $f(y)$ also increases, as the value of α increases. It can also be observed that the $f(y)$ curve touches X-axis first having the greater value of α.

Probability Generating Function(pgf.) of PEMPD: It is derived as

$$P_Y^{(t)} = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty e^{-(1-t)\lambda}(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty (\pi+\lambda^3)e^{-(1+\alpha-t)\lambda}d\lambda$$

$$= \frac{\alpha^4}{(6+\pi\alpha^3)}\left[\pi\int_0^\infty e^{-\lambda(1+\alpha-t)}d\lambda + \int_0^\infty \lambda^3 e^{-\lambda(1+\alpha-t)}d\lambda\right] = \left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\left[\frac{\pi(1+\alpha-t)^3+6}{(1+\alpha-t)^4}\right]; \alpha>0 \quad (7)$$

Moment Generating Function (M.G.F.) of PEMPD: It plays an important role in statistical analysis by generating the statistical moments and it is obtained as

$$M_Y^{(t)} = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty (\pi+\lambda^3)e^{-\lambda(1+\alpha-e^t)}d\lambda = \frac{\alpha^4}{(6+\pi\alpha^3)}\left[\frac{\pi\Gamma 1}{(1+\alpha-e^t)^1} + \frac{\Gamma 4}{(1+\alpha-e^t)^4}\right]$$

$$= \left(\frac{\alpha^4}{(6+\pi\alpha^3}\right)\left[\frac{\pi(1+\alpha-e^t)^3+6}{(1+\alpha-e^t)^4}\right] \quad (8)$$

The equations (7) and (8) are the probability generating function and moment generating function of PEMPD respectively.

- *Statistical Moments and Essential Descriptive Measures of Statistics required for PEMPD:*

It has been observed that statistical moments play a significant role in studying about shape, size and variability of any statistical distribution. Therefore, it is required to obtain the first four moments about the origin as well as the mean of the proposed distribution. The rth moment about the origin of PEMPD can be obtained as

$$\mu_r' = E\left[E(Y^r/\lambda\right] = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\sum_{y=0}^\infty \frac{y^r e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda \quad (9)$$

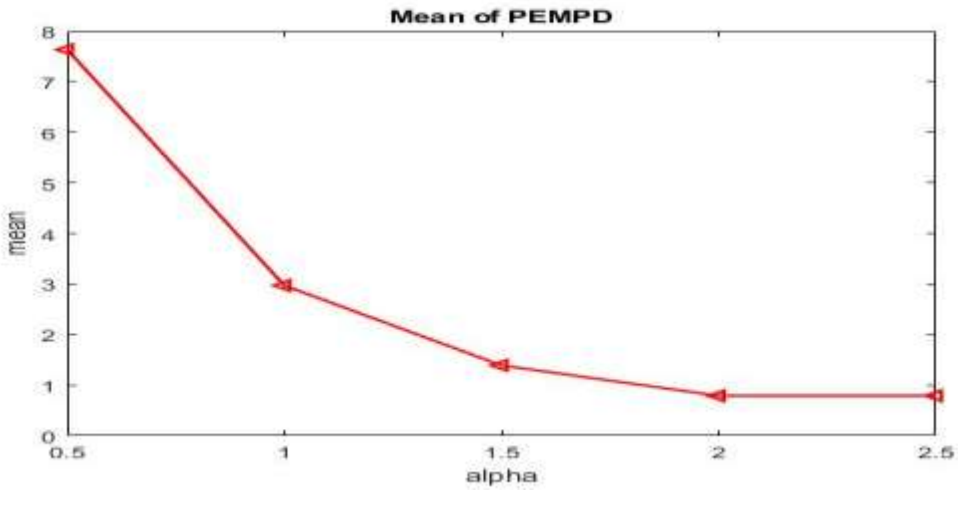
Substituting $r=1,2,3,4$ in the equation (9), the first four moments about origin of PEMPD can be obtained as follows. The mean of PNQED has been obtained as

$$\mu_1' = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\sum_{y=0}^\infty \frac{y^1 e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty(\lambda)(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^4}{(6+\pi\alpha^3)}\left[\int_0^\infty \pi\lambda e^{-\alpha\lambda}d\lambda + \int_0^\infty \lambda^4 e^{-\alpha\lambda}d\lambda\right] = \frac{\alpha^4}{(6+\pi\alpha^3)}\left[\frac{\pi}{\alpha^2}+\frac{24}{\alpha^5}\right] = \frac{(24+\pi\alpha^3)}{\alpha(6+\pi\alpha^3)} \quad (10)$$

Pictorial form of the mean of PEMPD for different values of alpha is given below.

Fig.4 The mean of PEMPD for $\alpha = 0.5, 1.0, 1.5, 2.0, 2.5$



It can be observed, from the figure (4), that the mean of PEMPD is the inversely proportional to the parametric value. Substituting $r=2$ in the equation (9), the second moment about the origin of PEMPD can be obtained as follows

$$\mu_2' = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\sum_{y=0}^{\infty}\frac{y^2e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\lambda+\lambda^2\right)(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^4}{(6+\pi\alpha^3)}\left[\left(\frac{\pi}{\alpha^2}++\frac{24}{\alpha^5}\right)+\left(\frac{2\pi}{\alpha^3}++\frac{120}{\alpha^6}\right)\right] = \frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}+\frac{(2\pi\alpha^3+24)}{\alpha^2(6+\pi\alpha^3)} \quad (11)$$

Putting $r=3$ in the equation (9), μ_3' can be obtained as

$$\mu_3' = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\sum_{y=0}^{\infty}\frac{y^3e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\lambda^3+3\lambda^2+\lambda\right)(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\pi\lambda+3\pi\lambda^2+\pi\lambda^3+\lambda^4+3\lambda^5+\lambda^6\right)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^4}{(6+\pi\alpha^3)}\left[\left(\frac{\pi}{\alpha^2}+\frac{24}{\alpha^5}\right)+\left(\frac{6\pi}{\alpha^3}+\frac{360}{\alpha^6}\right)+\left(\frac{6\pi}{\alpha^4}+\frac{720}{\alpha^7}\right)\right]$$

$$= \frac{(24+\pi\alpha^3)}{\alpha(6+\pi\alpha^3)}+\frac{6(60+\pi\alpha^3)}{\alpha^2(6+\pi\alpha^3)}+\frac{6(120+\pi\alpha^3)}{\alpha^3(6+\pi\alpha^3)} \quad (12)$$

Putting $r=4$ in the equation (9), the fourth moment about the origin can be obtained as

$$\mu_4' = \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\sum_{y=0}^{\infty}\frac{y^4e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^3)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\lambda+7\lambda^2+6\lambda^3+\lambda^4\right)(\pi+\lambda^4)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^4}{(6+\pi\alpha^3)}\int_0^\infty\left(\pi\lambda+7\pi\lambda^2+6\pi\lambda^3+\pi\lambda^4+\lambda^4+7\lambda^5+6\lambda^6+\lambda^7\right)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha 4}{(6+\pi\alpha^3)}\left[\left(\frac{\pi}{\alpha^2}+\frac{24}{\alpha^5}\right)+\left(\frac{14\pi}{\alpha^3}+\frac{840}{\alpha^6}\right)+\left(\frac{36\pi}{\alpha^4}+\frac{4320}{\alpha^7}\right)+\left(\frac{24\pi}{\alpha^5}+\frac{5040}{\alpha^8}\right)\right]$$

$$= \frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}+\frac{14(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)}+\frac{36(\pi\alpha^3+120)}{\alpha^3(6+\pi\alpha^3)}+\frac{24(\pi\alpha^3+210)}{\alpha^4(6+\pi\alpha^3)} \quad (13)$$

Central Moments of PEMPD:

It is essential to study about the nature proposed distribution according to variability, shape and size. So, the first four central moments of PEMPD have been obtained as

$\mu_1 = 0$

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2 = \frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}+\frac{(2\pi\alpha^3+120)}{\alpha^2(6+\pi\alpha^3)}-\left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}\right)^2$$

$$= \frac{[\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^3+16\pi\alpha^4+84\pi\alpha^3+144\alpha+144]}{[\alpha(\pi\alpha^3+6)]^2} \quad (14)$$

Theorem (1): *PEMPD is an over-dispersed compound distribution.*

Proof: if $\mu_2 > \mu_1'$, it is said to be over-dispersed.

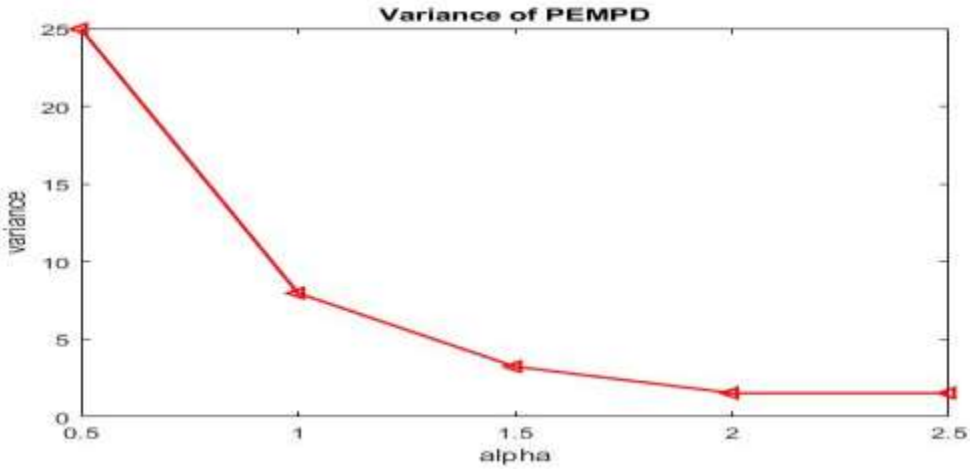
Or, $\dfrac{[\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^3+16\pi\alpha^4+84\pi\alpha^3+144\alpha+144]}{[\alpha(\pi\alpha^3+6)]^2} > \dfrac{(24+\pi\alpha^3)}{\alpha(6+\pi\alpha^3)}$

Or, $[\pi^2\alpha^7 + \pi^2\alpha^6 + 30\pi\alpha^3 + 16\pi\alpha^4 + 84\pi\alpha^3 + 144\alpha + 144] - \alpha(24+\pi\alpha^3)(6+\pi\alpha^3) > 0$

Or, $[\pi^2\alpha^6 + 80\pi\alpha^3 + 144] > 0$ (15)

The value of expression (15) is mathematically true. Hence, PEMPD is an over-dispersed. Pictorial form of μ_2 with varying values of α is presented as

Fig.5 The Variance of PEMPD for $\alpha = 0.5, 1.0, 1.5, 2.0, 2.5$



From the figure (5), we can observe that the variance of PEMPD decreases as the value of α increases.

The third moment about the mean of PEMPD can be obtained as

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= \left[\frac{(24+\pi\alpha^3)}{\alpha(6+\pi\alpha^3)} + \frac{6(60+\pi\alpha^3)}{\alpha^2(6+\pi\alpha^3)} + \frac{6(120+\pi\alpha^3)}{\alpha^3(6+\pi\alpha^3)}\right] - 3\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)} + \frac{(2\pi\alpha^3+24)}{\alpha^2(6+\pi\alpha^3)}\right]\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}\right] + 2\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}\right]^3$$

$$= \frac{[(\pi\alpha^3)^3(\alpha^2+3\alpha+2) + (\pi\alpha^3)^2(36\alpha^2+270\alpha+396) + (\pi\alpha^3)(324\alpha^2 + 1944\alpha+648) + (864\alpha^2+2592\alpha+1728)]}{[\alpha(6+\pi\alpha^3)]^3} \quad (16)$$

The fourth moment about the mean (μ_4) of PEMPD can be obtained as

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 + 3(\mu_1')^4$$

$$= \left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)} + \frac{14(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)} + \frac{36(\pi\alpha^3+120)}{\alpha^3(6+\pi\alpha^3)} + \frac{24(\pi\alpha^3+210)}{\alpha^4(6+\pi\alpha^3)}\right] - 4\left[\frac{(24+\pi\alpha^3)}{\alpha(6+\pi\alpha^3)} + \frac{6(60+\pi\alpha^3)}{\alpha^2(6+\pi\alpha^3)} + \frac{6(120+\pi\alpha^3)}{\alpha^3(6+\pi\alpha^3)}\right]\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}\right] + 6\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)} + \frac{(2\pi\alpha^3+24)}{\alpha^2(6+\pi\alpha^3)}\right]\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}\right]^2 - 3\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)}\right]^4$$

$$= \frac{[(\pi\alpha^3)^4(\alpha^3+10\alpha^2+18\alpha+9) + (\pi\alpha^3)^3(42\alpha^3+852\alpha^2+5206\alpha+2796) + (\pi\alpha^3)^2(108\alpha^3 + 11880\alpha^2+69552\alpha+89768) + (\pi\alpha^3)(2808\alpha^3+59184\alpha^2+132192\alpha+93312) + (5184\alpha^3+98496\alpha^2+186624\alpha+93312))}{[\alpha(6+\pi\alpha^3)]^4} \quad (17)$$

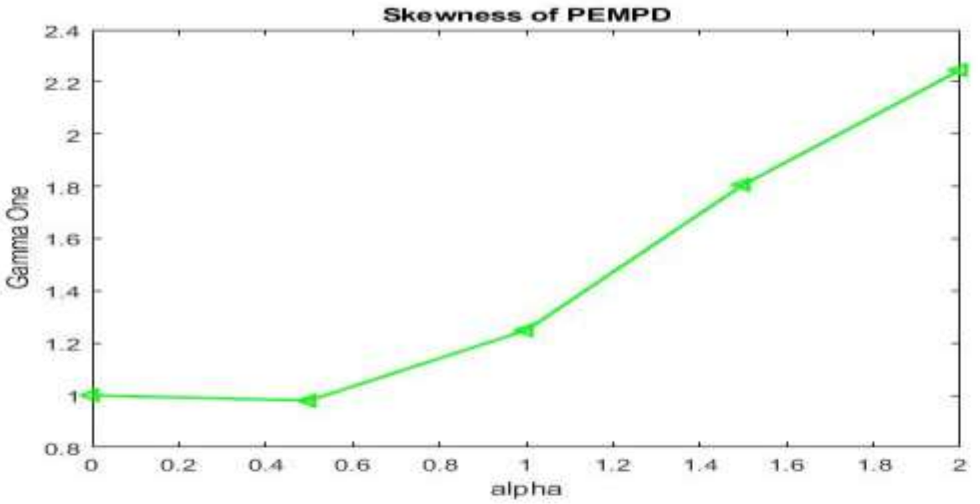
To study about shape and size of this distribution, the co-efficient of skewness and kurtosis based on moments have been obtained as

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{[(\pi\alpha^3)^3(\alpha^2+3\alpha+2) + (\pi\alpha^3)^2(36\alpha^2+270\alpha+396) + (\pi\alpha^3)(324\alpha^2 + 1944\alpha+648) + (864\alpha^2+2592\alpha+1728)]}{[\pi^2\alpha^7 + \pi^2\alpha^6 + 30\pi\alpha^3 + 16\pi\alpha^4 + 84\pi\alpha^3 + 144\alpha + 144]^{3/2}} \quad (18)$$

The expression (18) represents co-efficient of skewness of PEMPD based on moments. From this expression, it has been found that $1 < \gamma_1 < \infty$.

Pictorial form of γ_1 with different values of the parameter α is presented as.

Fig.6: Skewness of PEMPD for $\alpha = 0.0, 0.5, 1.0, 1.5, 2.0$

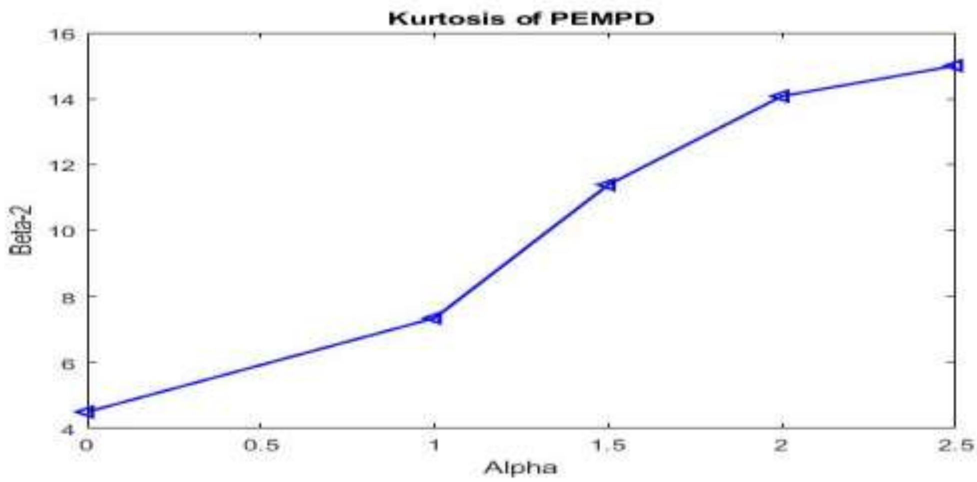


$$\beta_2 = \frac{\mu_4}{(\mu_2)^2}$$

$$= \frac{\begin{array}{l}[(\pi\alpha^3)^4(\alpha^3+10\alpha^2+18\alpha+9)+(\pi\alpha^3)^3(42\alpha^3+852\alpha^2+5206\alpha+2796)+(\pi\alpha^3)^2(108\alpha^3 \\ +11880\alpha^2+69552\alpha+89768)+(\pi\alpha^3)(2808\alpha^3+59184\alpha^2+132192\alpha+93312) \\ +(5184\alpha^3+98496\alpha^2+186624\alpha+93312)]\end{array}}{[\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^3+16\pi\alpha^4+84\pi\alpha^3+144\alpha+144]^2} \quad (19)$$

The expression (19) is the co-efficient of kurtosis based on moments and it has been found that $(4.5 < \beta_2 < \infty)$ .Hence, it is leptokurtic by size. Pictorial form of β_2 for different values of α is presented as

Fig.7: Kurtosis of PEMPD for $\alpha = 0.0, 1.0, 1.5, 2.0, 2.5$



From figure (6), It has been observed that the values of γ_1 are directly proportional to the values of α such that $1 < \gamma_1 < \infty$. From figure (7) also, It has been observed that the values of β_2 are directly proportional to the values of α , while the values of mean and variance are inversely proportional to the value of α .

- *Remarks:*
  - PEMPD is always over-dispersed.
  - It is always positively skewed by shape, and
  - It is always leptokurtic by size.

- *Methods of Estimation of the Parameter of PEMPD:*

These two methods of estimation, namely (a) Method of moments and (b) Maximum likelihood method, have been used to derive the point estimator of the parameter of this distribution.
(a)*Method of moments*:

Since this distribution depends on only one parameter, we can use the expression (10) of the mean of this distribution to get a sufficient estimator of the parameter of this distribution as follows

$$\mu_1' = \frac{(24+\pi\alpha^3)}{\alpha(6+\pi\alpha^3)}$$

Or, $\mu_1'(6\alpha+\pi\alpha^4)-(24+\pi\alpha^3)=0$ (20)

To get an estimator of the parameter of this distribution, population mean is replaced by the sample mean and solving the expression (20), the polynomial equation of α in third degree, by Regula- Falsi or Newton-Rapson method which gives a point estimator of α .

(b) *Method of maximum likelihood:*
A sample of size n is taken from the PEMPD population to get the estimator of the parameter (α) from this method as follows

y_i : y_1 y_2 y_3 ... y_k

f_i : f_1 f_2 f_3 ... f_k

The maximum likelihood equation has been obtained as

$$L = \left(\frac{\alpha^4}{6+\pi\alpha^3}\right)^n (1+\alpha)^{-\sum_{i=1}^{k}(y_i+4)f_i} \prod_{i=1}^{k}[\pi(1+\alpha)^3+(1+y_i)(2+y_i)(3+y_i)]^{f_i} \quad (21)$$

Or, $log(L) = 4n\log a - n\log(6+\pi\alpha^3) - \left(\sum_{i=1}^{k}(y_i+4)f_i\right)\left(\log(1+\alpha)\right) + \sum_{i=1}^{k} f_i \log\left(\pi(1+\alpha)^3+(1+y_i)(2+y_i)(3+y_i)\right)$

Or, $$\frac{\partial(log(L))}{\partial\alpha} = \frac{4n}{\alpha} - \frac{3\pi n\alpha^2}{(6+\pi\alpha^3)} - \frac{(\bar{y}n+4n)}{(1+\alpha)} + \sum_{i=1}^{k}\frac{f_i[3\pi(1+\alpha)^2]}{[\pi(1+\alpha)^3+(1+y_i)(2+y_i)(3+y_i)]} = 0 \quad (22)$$

An estimate of α can also be obtained by solving the expression (22).

- *Goodness of Fit and Applications of PEMPD:*

Due to the fact that this distribution is over-dispersed in nature, its use can be seen in Biology, ecology, errors theory, accident proneness and etc. related fields. The following data have been used to test the validity of this distribution, for which the goodness of test has been applied.

Example (1): Distribution of mistakes in copying groups of random digits, Kemp and Kemp (1965), [see.4].

Number of errors per group	0	1	2	3	4^+
Observed Frequency	35	11	8	4	2

Example (2): Distribution of Pyrausta nablilalis in 1937, Beall (1940), [see, 1].

Number of insects per leaf	0	1	2	3	4	5
Observed Frequency	33	12	6	3	1	1

Example (3): Distribution of mammalian cytogenic dosimetry lesions in rabbit lymphoblast included by Streptonigrin [NSC-45383], Exposure-70($\mu g / kg$), [see,2].

Class / Exposure ($\mu g / kg$)	0	1	2	3	4	5	6
Observed Frequency	200	57	30	7	4	0	2

Example (4): Distribution of number of red mites on apple leaves, reported by Garman (1923) [see,3].

Number of red mites per leaf	0	1	2	3	4	5	6	7
Observed Frequency	38	17	10	9	3	2	1	0

Table I: Observed Verses Expected Frequency of Example (1)

Number of errors per group	Observed frequency	Expected frequency			
		PLD	PMD	PMMD	PEMPD
0	35	33.0	32.9	33.6	34.7
1	11	15.3	15.3	14.7	13.8
2	8	6.8	6.8	6.5	6.1
3	4	2.9	3.6	2.9	2.9
4	2	2.0	1.4	2.3	2.5
μ'_1	0.78333333				
μ'_2	1.8500				
$\hat{\alpha}$		1.7434	2.1654758	1.804566256	2.0077743
d.f.		2	2	2	2
χ^2		1.78	1.72	1.44	1.1
P-value		0.61	0.625	0.677	0.738
Total	60	60.0	60.0	60.0	60.0

Table II: Observed Verses Expected Frequency of Example (2)

Number of insects per leaf	Observed frequency	Expected frequency			
		PLD	PMD	PMMD	PNQED
0	33	31.5	31.4	32.0	32.9
1	12	14.2	14.2	13.7	12.8
2	6	6.1	6.2	5.9	5.9
3	3	2.5	2.6	2.6	2.7
4	1	1.0	1.0	1.1	1.2
5	1	0.7	0.6	0.7	0.5
μ'_1	0.75				
μ'_2	1.8571				
$\hat{\alpha}$		1.8081	2.234	1.862795711	2.054914
d.f.		2	2	2	2
χ^2		0.53	0.47	0.29	0.098
P-value		0.83	0.85	0.89	0.93
Total	56	56.0	56.0	56.0	56.0

Table III: Observed Verses Expected Frequency of Example (3)

Class per Exposure ($\mu g / kg$)	Observed frequency	Expected frequency			
		PLD	**PMD**	**PMMD**	**PNQED**
0	200	191.8	191.6	193.4	196.8
1	57	70.3	70.2	68.4	64.9
2	30	24.9	25.1	24.5	23.4
3	7	8.6	8.7	8.8	9.0
4	4	2.9	2.9	3.2	3.6
5	0	1.0	1.0	1.1	1.4
6	2	0.5	0.5	0.6	0.9
μ'_1	0.553333333				
μ'_2	1.253333333				
$\hat{\alpha}$		2.353339	2.784976722	2.339317198	2.438340362
d.f.		3	3	3	3
χ^2		3.91	3.81	3.40	3.12
P-value		0.43	0.45	0.50	0.534
Total	300	300.0	300.0	300.0	300.0

Table IV: Observed Verses Expected Frequency of Example (4)

Number of red mites per leaf	Observed frequency	Expected frequency		
		PLD	**PMMD**	**PNQED**
0	38	35.8	36.8	38.5
1	17	20.7	20.0	18.6
2	10	11.4	10.9	10.1
3	9	6.0	5.9	5.7
4	3	3.1	3.1	3.3
5	2	1.6	1.6	1.8
6	1	0.8	0.8	0.7
7^+	0	0.6	0.9	1.3
μ'_1	1.15			
μ'_2	3.4			
$\hat{\alpha}$		1.255891	1.3724416	1.64889605
d.f.		3	3	3
χ^2		2.47	1.15	0.05
P-value		0.61	0.82	0.995
Total	80	80.0	80.0	80.0

Among the four data given above, the first data was given by Kemp and Kemp related to the number of errors per page [see, 4]. The second data was given by Beall which is based on the number of insects per leaf [see, 1]. The third example is due to Catcheside at al which is about class per exposer [See, 2] and the last example, is related to number of red mites per leaf, was given by Garman [See, 3]. In the above-mentioned tables, the expected frequency due to PLD, PMD, PMMD and PEMPD has been calculated which makes it easy to compare these distributions. The first three examples have already been mentioned in the doctoral thesis [see, 7].

Conclusion:

In table-V, $d.f.$ $\chi^2_{d.f.}$ and $P-Value$ of PLD, PMD, PMMD and PEMPD have been included to make comparison easy and simple.

Table- V

PLD. PMD and PMMD Verses PEMPD

Table	PLD			PMD		PMMD		PEMPD	
	$d.f.$	$\chi^2_{d.f.}$	$P-Value$	$\chi^2_{d.f.}$	$P-Value$	$\chi^2_{d.f.}$	$P-Value$	$\chi^2_{d.f.}$	$P-Value$
I	2	1.78	0.61	1.72	0.625	1.44	0.677	1.11	0.738
II	2	0.53	0.83	0.47	0.85	0.29	0.89	0.098	0.93
III	3	3.91	0.43	3.81	0.45	3.40	0.50	3.12	0.534
IV	3	2.47	0.61	-	-	1.15	0.82	0.05	0.995

The following conclusions have been made on the basis of theoretical as well as application results of this distribution

- PEMPD is a better alternative of PLD [see, 7] and PMD [see, 10] PMMD [see, 13] for statistical modeling.
- Since $\mu_2 > 0$, it is always over-dispersed.
- Since $1 < \gamma_1 < \infty$,it is always positively skewed by shape and
- Since $4.5 < \beta_2 < \infty$, it is Leptokurtic by size.

Conflict of Interest

The authors of this paper have written this paper selflessly with the aim of contributing only to continuous mixtures of Poisson distribution. The authors do not intend to prejudice or offend anyone while writing this paper.

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