DETOUR CERTIFIED DOMINATION NUMBER OF GRAPHS

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Abstract:

In this paper, we introduce the concept of detour certified domination number of Graphs. Here we characterised graphs with $\gamma_{dcer}(G) = 2$ and studied the relation of $\gamma_{dcer}(G)$ with other parameters. A subset S of V(G) is said to be a certified dominating set if it is a dominating set and every vertex of S has either zero or atleast two neighbours in V(G) – S. The certified domination number is the minimum cardinality of the certified dominating set and is denoted by $\gamma_{cer}(G)$. A set S of vertices of G is called a detour certified dominating set if S is a certified dominating set and every vertex of G lie in some detour of vertices of S. The minimum cardinality of a detour certified dominating set is the detour certified domination number of G and is denoted by $\gamma_{dcer}(G)$.

Keywords: Certified dominating set, certified domination number, detour domination number, detour certified dominating set, detour certified domination number.

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1 Introduction

Let G = (V,E) be a connected graph where V is the set of vertices and E is the set of edges of G. The *open neighborhood* N(v) of the vertex v consists of the set of vertices adjacent to v, that is, N(v) = $\{w\epsilon V : vw\epsilon E\}$ and the *closed neighbourhood* of v is N[v]=N(v) ∪ $\{v\}$. For a set S$\subseteq V$, the open neighbourhood N(S) is defined to be $\cup_{v\in S} N(v)$ and the closed neighbourhood of S is N$[S] = N(S) \cup$ S. The *degree* of a vertex v is the number of edges incident with v and is denoted by deg(v) or d(v). The minimum and maximum degrees of vertices in V(G) are denoted by $\delta(G)$ *and* $\Delta(G)$ respectively. The distance d(v, x) between v and x is the minimum length of a $v - x$ walk. The *eccentricity* of vertex v is ecc(v) = max $\{d(v,w): w \in V\}$. The *diameter* of G is diam (G) = max $\{ecc(v): v \in V\}$.

The middle graph of a graph $G = (V,E)$ is a graph $M(G) = (V \cup E, E')$ where uv$\in E'$ if and only if either u is a vertex of G and v is an edge incident with u, or u and v are adjacent edges of G. A *unicyclic graph* is a connected graph containing exactly one cycle. Let G be a connected graph. The minimum number of vertices whose removal makes G either disconnected or reduces G into a trivial graph is called its *vertex connectivity* and denoted by $\kappa(G)$. The *chromatic number* $\chi(G)$ of a graph G is the minimal number of colours needed to colour the vertices in such a way that no two adjacent vertices have the same colour. The m-*book graph* is defined as the cartesian product of S_{m+1} and P_2 where S_{m+1} is a star graph and P_2 is a path graph. The *friendship graph* F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex, which becomes a universal vertex.

A set S⊆V(G) is said to be a *dominating set* if every vertex v∈V(G) is either an element of S or is adjacent to an element of S. The minimum cardinality taken over all dominating sets is called the *domination number* and is denoted by $\gamma(G)$. For u, v$\in V(G)$, the length of a maximum u-v path is called detour distance D(u,v). A u-v path of length D(u,v) is called u-v detour. A vertex w ∈ V(G) is said to lie on a u-v detour Q if w is a vertex of V(Q) including u and v. A detour set S is a subset of V(G) such that every vertex v of G lies on an x-y detour of some x, y ∈ S. The detour number is the minimum cardinality of a detour set and is denoted by dn(G). A subset S of V(G) is called a *detour dominating set* if S is both a detour set and dominating set of G. The detour domination number $\gamma_d(G)$ is the minimum order of its detour dominating set and the detour dominating set is denoted by γ_d-set. A set S ⊆ V(G) is called a *certified dominating set* of G if S is a dominating set of G and every vertex belonging to S has either zero or atleast two neighbours in V(G) –S. The cardinality of the smallest certified dominating set is called the certified domination number of G and is denoted by γ_{cer}(G)[3,4].

Definition 1.1. A dominating set S $\subseteq V(G)$ is said to be a detour certified dominating set of a connected graph G if every vertex of S has either zero or atleast two neighbours in $V - S$ and every vertex of G lie on some detour path of vertices of S. The minimum cardinality of a detour certified dominating set is called the detour certified domination number and is denoted by $\gamma_{dcer}(G)$.

2 Detour Certified Domination Number of Standard Graphs:

1. If G is a complete graph on n vertices then $\gamma_{dcer}(G) = \begin{cases} 3 & if\ n = 3 \\ 2 & otherwise \end{cases}$
2. Let G be the path graph on n vertices. Then $\gamma_{dcer}(G) = n$.
3. If G is a cycle graph on n vertices where n≠ 3,5 , then $\gamma_{dcer}(G) = \left\lceil \frac{n}{3} \right\rceil$.
4. For any star graph G on $n \geq 2$ vertices, $\gamma_{dcer}(G) = n$.
5. If G is a complete bipartite graph on $m + n$ vertices where $m, n \geq 2$, then $\gamma_{dcer}(G) = 2$.
6. If G is a wheel graph on n vertices then $\gamma_{dcer}(G) = 2$.
7. Let G be a friendship graph of order $2n + 1$. Then $\gamma_{dcer}(G) = n$.
8. Let G be a book graph. Then $\gamma_{dcer}(G) = 2$.
9. If G is a pan graph, then $\gamma_{dcer}(G) = \left\lceil \frac{n}{3} \right\rceil$+1.
10. For the Peterson graph G, $\gamma_{dcer}(G)$ = 3.

Observations 2.1.

1. Every pendant vertex of a graph G belongs to every detour certified dominating set of G.
2. Every support vertex of a graph G belongs to every detour certified dominating set of G.

Remark 2.2.

For any connected graph G, $2 \leq \gamma_{dcer}$(G) $\leq n$. The bound $2 \leq \gamma_{dcer}(G) \leq n$ is sharp. The lower bound is attained for K_n $n \neq 3$, $n \geq 2$, $K_{m,n}$ and W_n and it attains the upper bound for $P_n, n \geq 2$ and $S_n, n \geq 2$.

Limitation 2.3. $\gamma_{dcer}(G)$ never be $n - 1$.

Proof: Suppose $\gamma_{dcer}(G) = n - 1$. Let S be the γ_{dcer} - set. Then there exists a vertex v in G such that $v \notin$ S. Since G is a connected graph, v contains atleast one neighbour in S. Let $A \subseteq S$ be the set of all neighbours of v. Let $u_i \in S$. If $u_i \notin A$ then $|N(u_i) \cap (V - S)| = 0$. If $u_i \in A$ then $|N(u_i) \cap (V - S)| = 1$ which is a contradiction to S is a γ_{dcer} -set. Thus γ_{dcer} never attains the value $n - 1$.

Theorem 2.4. Let T be a connected acyclic graph of order n. Then γ_{dcer}(T) = n.

Proof : Let T be a tree with l leaves and s support. We prove this theorem by considering the following cases.

$Case\ (i)$: T has no internal vertices.

Let S be the set of all leaves and support vertices of T. Then by observations, S is the minimum detour certified dominating set. Hence $\gamma_{dcer}(T) = |S| = l + s = n$.

$Case\ (ii)$: T has internal vertices.

Let S be the γ_{dcer}-set. Then by observations, leaves and support vertices belongs to S. Suppose no internal vertex belongs to S. Since T is a tree, there exists support vertices, say $u \in S$, for which the number of neighbourhood of u in V- S is 1, ie., $|N(u) \cap (V - S)| = 1$ which is a contradiction to S is a γ_{dcer} -set. Hence all the internal vertices belongs to S. Thus γ_{dcer}(T) $= n$.

Result 2.5. For any connected graph G with l leaves and s support, $l + s \leq \gamma_{dcer}(G) \leq n$.

Proof: By observations, γ_{dcer}(G) $\geq l + s$. Since the set of all vertices of G is a detour certified dominating set, we have γ_{dcer}(G) $\leq n$. Hence $l + s \leq \gamma_{dcer}(G) \leq n$.

Theorem 2.6. Let G be a connected graph of order $n \geq 2$ and let H be a spanning subgraph of G. Then $\gamma_{dcer}(G) \leq \gamma_{dcer}(H)$.

Proof: Let G be a connected graph of order $n \geq 2$ and let S be the minimum detour certified dominating set of G. Let H be the spanning subgraph of G obtained by removal of certain edges of G and $|V(G)| = |V(H)|$. Then H may be a cyclic graph or acyclic graph or null graph. Consequently, there exists either a set K with cardinality same as S or a set L with cardinality greater than S, which forms a minimum detour certified dominating set of H. If K is a minimum detour certified dominating set of H then $\gamma_{dcer}(G) = |S| = |K| = \gamma_{dcer}(H)$. Otherwise L is a minimum detour certified dominating set of H such that $\gamma_{dcer}(G) = |S| < |L| = \gamma_{dcer}(H)$. Thus we get $\gamma_{dcer}(G) \leq \gamma_{dcer}(H)$.

3 Characterisation

Theorem 3.1. Let G be a connected graph of order n $\geq$ 6 with atmost one universal vertex and G contains a cycle of length n. Then γ_{dcer} (G) = 2 or 3.

Proof : Let $V(G) = \{v_1, v_2, , \dots, v_n\}$ be the vertices of G.

$Case\ (i)$: n is even

Let v_1 be the universal vertex. Let v_2 and v_n be the adjacent vertices of v_1 each of degree 2. Since G contains a universal vertex $\gamma_{cer}\ (G) = 1$. Now consider the detour path of two vertices. n is even implies the detour path of v_1 - v_i contains all the vertices of G where d(v_i)≥ 3. Therefore S = $\{v_1, v_i\}$ is a minimum detour certified dominating set. Hence γ_{dcer} (G) = 2.

$Case\ (ii)$: n is odd

In this case, the value of γ_{dcer} (G) depends on the degree of the vertices v_2 and v_n.

$Subcase\ (i):\ v_2\ \ and\ \ v_n\ \ are\ of\ degree\ 2.$

Let S = $\{v_1\}$ be the minimum certified dominating set. If v_2 or $v_n \in S$ then $|N(v_2) \cap (V - S)|$ = $|N(v_n) \cap (V - S)|$ =1. Thus $v_2,\ v_n \notin S$. Now, G contains a cycle of length n, n is odd and S is a certified dominating set implies atmost three vertex is needed for minimum detour certified dominating set. Hence γ_{dcer} (G) =3.

$Subcase\ (ii):\ Atmost\ one\ of\ \ v_2\ \ or\ v_n\ \ is\ of\ degree\ 2.$

Assume d(v_2)> 2. Clearly S = $\{v_1, v_2\}$ forms a minimum detour certified dominating set because $|N(v_2) \cap (V - S)| \geq 2$. Therefore γ_{dcer} (G) =2.

$Subcase\ (iii):$ d(v_2) and d(v_n) are atleast 3.

Then $\{v_1, v_2\}$ will form a minimum detour certified dominating set. Therefore γ_{dcer} (G) = 2.

Corollary 3.2.

Let G be a connected graph of order n≥ 6 with more than two universal vertex and G contains a cycle of length n. Then γ_{dcer} (G) = $\gamma_{cer}(G) + 1 = 2$.

4 Relations with other parameters

Theorem 4.1. Let G be a connected graph of order atleast 6. Then γ(G) = γ_{dcer} (G) if G has γ - set S such that every vertex in S has atleast two neighbours in V-S and the detour path of vertices of S contain all the vertices of G.

Proof: Assume γ(G) = γ_{dcer} (G). Let S be a γ_{dcer} - set. Then $\gamma(G) = \gamma_{dcer}(G)$ implies that S is a γ - set of G. G is a connected graph and so there will be no isolated vertices. Therefore, for any vertex $v \in S$ the set $N(v) \cap (V - S)$ is non-empty. Since S is a minimum detour certified dominating set, we have $|N(v) \cap (V - S)| \geq 2$ for all $v \in S$ and all the vertices of G lies in the detour path of vertices of S.

Conversely, suppose that S is a γ - set of G with $|N(v) \cap (V-S)| \geq 2$ for all $v \in S$ and the detour path of vertices of S contains all the vertices of G. Then S is a minimum detour certified dominating set of G. Hence γ_{dcer} (G) $\leq |S| = \gamma$(G) $\leq \gamma_{dcer}$ (G). Thus $\gamma(G) = \gamma_{dcer}(G)$.

Theorem 4.2. Let G be a connected graph of order n. Then $\gamma_{dcer}(G) \geq \left\lfloor \frac{\gamma(G)+\gamma_d(G)+\gamma_{cer}(G)}{3} \right\rfloor$.

Proof: To prove this theorem, we consider the following cases:

$Case(i)$: Let S be the minimum dominating set of G.

For every $v \in S$ if $|N(v) \cap (V-S)|$ is either zero or atleast two and all the vertices of G lie in some detour path of S then S itself is a minimum detour certified dominating set of G. Otherwise we can find a set $A \subseteq V(G)$ such that $S \cup A$ forms a γ_{dcer} - set of G.

Thus $\gamma(G) \leq \gamma_{dcer}(G)$. ---------------------(1)

$Case(ii)$: Let S be the minimum detour dominating set of G.

If every vertex of S has either zero or atleast two neighbours in V-S then S itself is a minimum detour certified dominating set. Else, there exists a set $B \subseteq V(G)$ such that $S \cup B$ forms a minimum detour certified dominating set.

Therefore, $\gamma_d(G) \leq \gamma_{dcer}(G)$. ---------------------(2)

$Case(iii)$: Let S be the $\gamma_{cer}(G)$ - set of G.

If all the vertices of G lies in the some detour path of S, then S is the minimum detour certified dominating set. Or else, the set S together with some other vertices of G forms a minimum detour certified dominating set of G.

Hence $\gamma_{cer}(G) \leq \gamma_{dcer}(G)$. ---------------------(3)

Now on addition of (1), (2) and (3), we get

$$\gamma_{dcer}(G) \geq \left\lfloor \frac{\gamma(G)+\gamma_d(G)+\gamma_{cer}(G)}{3} \right\rfloor.$$

Theorem 4.3. Let G be a connected graph of order n and let d be the diameter of G. Then $\gamma_{dcer}(G) \geq \frac{d+4}{3}$

Proof : We have $\left\lceil\frac{d+1}{3}\right\rceil \leq \gamma(G)$ which implies $\frac{d+1}{3} + 1 \leq \gamma(G)$.

$$\Rightarrow \quad d + 1 \leq 3\gamma(G) - 3$$

$$\Rightarrow \quad d + 1 \leq 3(\gamma(G) - 1)$$

$$\Rightarrow \quad d + 1 \leq 3(\gamma_{dcer}(G) - 1)$$

$$\Rightarrow \quad \gamma_{dcer}(\mathrm{G}) \geq \frac{\mathrm{d}+4}{3}$$

Theorem 4.4. Let G be a connected graph having no pendant vertices. If γ_d(G) = r then γ_{dcer}(G) = r for any positive integer r.

Proof: Let G be a connected graph having no pendant vertices such that γ_d(G) = r. Let S be the minimum detour certified dominating set. To prove γ_{dcer}(G) = r. Suppose that assume γ_{dcer}(G) $\neq$ r. Then either γ_{dcer}(G) $<$ r or γ_{dcer}(G) $>$ r. Since every detour certified dominating set is a detour dominating set and γ_d(G) = r, we have γ_{dcer}(G) $\nless$ r. Therefore, assume γ_{dcer}(G) $>$ r. Without loss of generality, we assume that γ_{dcer}(G) = r+k where 1 $\leq$ k $\leq$ n-r-1. At any one of this particular stage definitely γ_{dcer} (G) = r+k contradicts the minimality of S. Hence γ_{dcer} (G) never be greater than r. Therefore γ_{dcer} (G) = r.

Remark: The above theorem does not hold for C_3 and C_5.

Theorem 4.5. For any connected graph of order n$\geq$ 2, γ_{dcer}(G)+κ(G) $\leq$ 2n-1 with equality if G is isomorphic to K_2, K_3.

Proof: Let G be a connected graph of order n$\geq$ 2. Obviously κ(G) $\leq$ n-1. We have γ_{dcer}(G) $\leq$ n. Hence we get γ_{dcer}(G)+κ(G)$\leq$n+n-1=2n-1. For equality consider, $G \cong K_2$. We know $\kappa(K_n)$=n-1. Therefore, $\kappa(K_2)$ = 1. Also $\gamma_{dcer}(K_2)$ = 2.

Thus γ_{dcer}(G)+κ(G) $\leq$2n-1.

Similarly we can prove the result if $G \cong K_3$.

Theorem 4.6. Let G be a connected graph of order n$\geq$ 2 .Then γ_{dcer}(G)+χ(G) $\leq$ 2n. Equality holds if G is isomorphic to K_2, K_3.

Proof: Clearly for any connected graph G, χ(G) $\leq$ n. Hence γ_{dcer}(G)+χ(G) $\leq$ 2n. For K_2, the chromatic number is n. Also $\gamma_{dcer}(K_2)$ is the number of vertices of K_2. Hence γ_{dcer}(G)+χ(G) = 2n. Similarly for K_3, γ_{dcer}(G)+χ(G) = 2n.

Theorem 4.7. For any connected graph of order n≥ 2, γ_{dcer}(G)+Δ(G) ≤ 2n-1. The bound is sharp if G is isomorphic to K_2, K_3.

Proof: Since G is a connected graph of order n≥ 2, the maximum degree of G is at most n-1. Also we have γ_{dcer}(G) ≤ n. Thus we find that, γ_{dcer}(G)+ Δ(G) ≤ 2n-1.

Sharpness: $\Delta(K_2) = 1$ and $\Delta(K_3) = 2$. Also $\gamma_{dcer}(K_2) = 2$ and $\gamma_{dcer}(K_3) = 3$.

Thus γ_{dcer}(G)+ Δ(G) = 2n -1 for K_2 and K_3.

5 Middle Graphs of Path, Cycle and Star:

Theorem 5.1. Let M(P_n), n≥ 2 denote the middle graph of P_n and m denote the number of edges of P_n . Then γ_{dcer} (M(P_n)) = $\begin{cases} n+m & if \quad n = 2,3 \\ \left\lceil \frac{n}{2} \right\rceil + 2 & if \quad n \geq 4 \end{cases}$

Proof: We prove this theorem by considering the following cases:

$Case\ (i)$: $n = 2$

When n=2, the middle graph of P_n is nothing but the star graph S_3. Therefore

γ_{dcer} (M(P_n)) = $\gamma_{dcer}(S_3)$ = 3 $= 2 + 1 = n + m$.

$Case\ (ii)$: $n = 3$

Let V(M(P_n)) be the vertex set of M(P_n) with V(M(P_n)) = V(P_n) ∪ E(P_n) =5. Then the detour dominating set S of M(P_n) contains three vertices say $\{v_1, v_2, v_3\}$. But for u, v $\in V - S$, we have $|N(v_i) \cap (V - S)| = 1$. Therefore S ∪ {u,v} forms a minimum detour certified dominating set of M(P_n). Hence γ_{dcer} (M(P_n)) = |S ∪ {u, v}| = $|S| + 2 = n + m$.

$Case\ (iii)$: $n \geq 4$

Let V(M(P_n)) be the set of all vertices of M(P_n) such that $|V(M(P_n))| = m + n$ and M(P_n) has exactly two pendant vertices. Let the pendant vertices of M(P_n) be v_1 and v_{n+m}.

When $n \geq 4$, we consider the following sub cases:

$Subcase\ (i)$: $n\ is\ even$

Let S = $\{v_2, v_6, v_{10}, v_{14}, \dots, v_{n+m-1}\}$ be the minimum dominating set of M(P_n). Then $|S| = \left\lceil \frac{n}{2} \right\rceil$. Consider the $v_2 - v_{n+m-1}$ detour path. All the vertices other than v_1 and v_{n+m}

lies in the $v_2 - v_{n+m-1}$ detour path where $v_2, v_{n+m-1} \in S$. Therefore $S \cup \{v_1, v_{n+m}\}$ forms a minimum detour dominating set. Now, since $|N(v) \cap (V-S)| \geq 2$ for every $V \in S$ and $|N(v_1) \cap (V-S)| = 0$, $|N(v_{n+m}) \cap (V-S)| = 0$, by the definition of certified dominating set, $S \cup \{v_1, v_{n+m}\}$ forms a minimum detour certified dominating set. Hence γ_{dcer} (M(P_n)) = $|S \cup \{v_1, v_{n+m}\}| = |S| + 2 = \left\lceil \frac{n}{2} \right\rceil + 2$.

Subcase (ii): n is odd

Let $S = \{ v_2, v_6, v_{10}, v_{14}, \dots, v_{n+m-3}, v_{n+m}\}$ be the minimum dominating set of M(P_n). Then $|S| = \left\lceil \frac{n}{2} \right\rceil$. Consider the detour path $v_2 - v_{n+m}$. Clearly all the vertices of M(P_n) lies in this detour path except v_1. Therefore $S \cup \{v_1\}$ forms a minimum detour dominating set of M(P_n). This implies that γ_d (M(P_n)) = $|S \cup \{v_1\}| = |S| + 1 = \left\lceil \frac{n}{2} \right\rceil + 1$. Now consider v_{n+m}. $|N(v_{n+m}) \cap (V-S)| = 1$ because v_{n+m} is a pendant vertex. Hence we need to choose v_{n+m-1} for the certified dominating set. But if we choose v_{n+m-1}, we get $|N(v_{n+m-1}) \cap (V-S)| = 1$ which again contradicts the definition of certified dominating set.

Thus $K=\{v_1\} \cup \{ v_2, v_6, v_{10}, v_{14}, \dots, v_{n+m-3}, v_{n+m}\} - \{v_{n+m-3}\} \cup \{v_{n+m-4}\} \cup \{v_{n+m-1}\}$ forms a minimum detour certified dominating set. Therefore, γ_{dcer} (M(P_n)) = |K| = $1 + \left\lceil \frac{n}{2} \right\rceil - 1 + 1 + 1 = \left\lceil \frac{n}{2} \right\rceil + 2$.

From both the cases we get, γ_{dcer} (M(P_n)) = $\left\lceil \frac{n}{2} \right\rceil + 2$ when $n \geq 4$.

Theorem 5.2. Let M($K_{1,n-1}$) denote the middle graph of $K_{1,n-1}$. Then γ_{dcer} (M($K_{1,n-1}$)) = 2n-1.

Proof: It is clear from the construction of M($K_{1,n-1}$), M($K_{1,n-1}$) has n-1 pendant vertices, one vertex of degree n-1 and n-1 vertices of degree n. Let S be the detour certified dominating set of M($K_{1,n-1}$) so that γ_{dcer} (M($K_{1,n-1}$)) = $|S|$. Now by the observations that pendant vertices and support vertices belongs to the detour certified dominating set implies $|S| = n-1+n-1 = 2n-2$. Again by the observation $\gamma_{dcer}(G)$ never be n-1, $|S| = 2n-2+1 = 2n-1$. Therefore, γ_{dcer} (M($K_{1,n-1}$)) = 2n-1.

Theorem 5.3. Let M(C_n), $n \geq 3$ denote the middle graph of cycle C_n. Then γ_{dcer}(M(C_n)) = $\left\lceil \frac{n}{2} \right\rceil$.

Proof: Let V(M(C_n)) = $\{v_1, v_2, \dots, v_{2n}\}$ be the vertices of M(C_n). Let S be the minimum dominating set of M(C_n). In M(C_n), it is noted that n vertices have degree 4 and n vertices have degree 2.

$Case\ (i)$: $n\ is\ even$

In this case, the non consecutive vertices with degree 4 gives the minimum dominating set. Clearly S itself is a minimum detour certified dominating set. Therefore $|S| = \frac{n}{2}$.

$Case\ (ii)$: $n\ is\ odd$

Let the vertices $\{v_1, v_3, v_5, \dots, v_{2n-1}\}$ have degree 4 and $\{v_2, v_4, v_6, \dots, v_{2n}\}$ have degree 2. Now S=$\{v_1, v_5, \dots, v_{2n-5}, v_{2n-2}\}$ is a minimum dominating set with $|N(v_i) \cap (V - S)| \geq 2$ and $v_i - v_j$ detour path contains all the vertices of S, where $v_i, v_j \in S$. Thus S is a minimum detour certified dominating set and $|S| = \left\lceil \frac{n}{2} \right\rceil$. From the above cases, we can say that γ_{dcer}(M(C_n)) = $\left\lceil \frac{n}{2} \right\rceil$.

6 Results on Unicyclic Graph

Remark 6.1. In this section n-cycle indicates the cycle with n vertices. The general bound for unicyclic graph is $\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{dcer}(G) \leq n$.

Theorem 6.2. Let G be a connected unicyclic graph with a n-cycle that pendant vertices are at the non consecutive vertices of the cycle where n ≥ 6. Let L be the set of all pendant vertices of G. Then $\left\lceil \frac{n}{3} \right\rceil + |L| \leq \gamma_{dcer}(G) \leq \left\lceil \frac{n}{2} \right\rceil + |L|$.

Proof : Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of the cycle in the unicyclic graph G. By observation, all the pendant vertices and support vertices belong to the detour certified dominating set. Therefore, $\gamma_{dcer}(G) \geq \left\lceil \frac{n}{3} \right\rceil + |L|$, because detour certified domination number of cycle is $\left\lceil \frac{n}{3} \right\rceil$. Now, let S be the minimum detour certified dominating set of G.

Let us find the upper bound. Suppose to the contrary, let us assume that pendant vertices in G lie at more than $\left\lfloor\frac{n}{2}\right\rfloor$ position. ie., $|S| > \left\lfloor\frac{n}{2}\right\rfloor + |L|$**.** Consider $v_i, v_j \in S$ for some i, j . Since pendant vertices lie at more than $\left\lfloor\frac{n}{2}\right\rfloor$ vertices, the vertices v_i and v_j lies in consecutive position for some i, j and $|N(v_i) \cap (V - S)| = 1$, $|N(v_j) \cap (V - S)| = 0$ which is a contradiction to S is a minimum detour certified dominating set.

Hence $\gamma_{dcer}(G) \leq |L| + \left\lfloor\frac{n}{2}\right\rfloor$. Thus the inequality, $\left\lceil\frac{n}{3}\right\rceil + |L| \leq \gamma_{dcer}(G) \leq \left\lfloor\frac{n}{2}\right\rfloor + |L|$.

Corollary 6.3. If at the non consecutive vertices exactly one pendant vertex exists, then $2\left\lceil\frac{n}{3}\right\rceil \leq \gamma_{dcer}(G) \leq 2\left\lfloor\frac{n}{2}\right\rfloor$ where $n \geq 6$.

Proof : Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of the cycle in the unicyclic graph G. Since exactly one pendant vertex lie at non consecutive $v_i's$,by the theorem,

$$2\left\lceil\frac{n}{3}\right\rceil \leq \gamma_{dcer}(G) \leq 2\left\lfloor\frac{n}{2}\right\rfloor.$$

Theorem 6.4. Let G be a connected unicyclic graph with a n- cycle where n denotes such that pendant vertices are at the consecutive vertices of the cycle where $n \geq 6$ is the number of vertices of the cycle. Then $\gamma_{dcer}(G) = n + l$ where l is the number of pendant vertices of G.

Proof : Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of the cycle in G and L be the set of all pendant vertices of G which lie at $v_i's$ such that $|L| = l$ Let v_i, v_{i+1} be the two consecutive vertices of the cycle. By observation, all the pendant vertices and support vertices belongs to detour certified dominating set implies $L \subseteq S$. Let S be the detour certified dominating set. Let us consider v_i to be a support vertex which has the pendant vertex l_i. Since G is unicyclic, $|N(v_i) \cap (V - S)| = 1$**.** Therefore $v_{i-1} \in S$. Again $|N(v_{i-1}) \cap (V - S)| = 1$. which implies $v_{i-2} \in S$. Continuing in this way we can see that $\{v_i, v_{i-1}, \dots., v_1, v_{i+1}, v_{i+2}, \dots, v_n\} \in S$. Thus S = $\{v_1, \dots, v_n\} \cup L$.

$$\Rightarrow \gamma_{dcer}(G) = n + l$$

Theorem 6.5. Let G be a connected unicyclic graph with a n- cycle where $n \geq 6$. Let Q be the set of all vertices of the trees attached to the non-consecutive vertices of the cycle such that order of G is $n + |Q|$. Then $\gamma_{dcer}\ (G) \leq |Q| + \left\lfloor\frac{n}{2}\right\rfloor$.

Proof : Consider a vertex of G. At that vertex, there may be a tree attached or not. If a tree is attached at vertex, the next tree will be attached at a non consecutive vertex. The maximum number of possibility of such non consecutive vertices is $\left\lfloor\frac{n}{2}\right\rfloor$. By theorem, all the vertices of the tree belongs to the detour certified dominating set. Thus we conclude that γ_{dcer} (G)≤ $\left\lfloor\frac{n}{2}\right\rfloor$+|Q|.

Corollary 6.6. For any connected unicyclic graph G, if the trees are at exactly $\left\lfloor\frac{n}{2}\right\rfloor$ vertices of the cycle then $\gamma_{dcer}\ (G) =\ |Q|$.

Theorem 6.7. Let G be a unicyclic graph such that G is a (m,n) tadpole graph. Then $\gamma_{dcer}(G) = \begin{cases} n+m, n \geq 2, m = 3,5 \\ n + \left\lceil\frac{m}{3}\right\rceil, n \geq 2, m = 4, m \geq 6 \end{cases}$

7 Conclusion

In this paper, the concept of detour certified domination number has been studied. The bound was found to be $2 \leq \gamma_{dcer}(G) \leq n$ for any connected graph G. We have characterised graphs for which $\gamma_{dcer}(G) = 2$. Also relation with other graph theoretical and domination parameters have been studied. The concept of detour certified domination number has been applied to middle graphs of path, cycle and star and their values are calculated. It has been found that the γ_{dcer}value of unicyclic graph lies between $\left\lceil\frac{n}{3}\right\rceil$ and n.

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