ANALYZE THE GROWTH RATE OF A PREY-PREDATOR SYSTEM WITH SIMULATION USING MATLAB.

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ABSTRACT:

Mathematical models and common sense suggest that a connected system of predator and prey should cycle: predation decreases prey populations to low rates, predator numbers rise, and prey populations rise continually. One such model that models predator-prey interactions is the Lotka-Volterra Model. According to our findings, both prey and predator growth are reliant on additional food sources as well as one another. Utilizing this technique, the stability of the linearized equilibrium point is analyzed. The findings indicate that the equilibrium point in the positive quadrant is stable. To demonstrate the behavior of the cohabitation of prey and predator populations, certain instances are offered. Additionally, it illustrates how the prey or predator population behaves while they are absent, what interactions occurred between them, and evaluates the numerical simulation for various parameters. The biological ramifications of our findings are further touched upon in the study's conclusion.

Keywords: Population Model, Prey-Predator system, logistic growth rate, Lotka-Volterra Model and Numerical simulations.

INTRODUCTION

Mathematics has always prospered from working with developing sciences. Mathematical modeling is the process of applying mathematical systems to analyze various social, technological, and scientific situations. These techniques involve the creation of mathematical models based on real-world circumstances, their simulation and mathematical analysis, and the interpretation and validation of data analysis. The most frequently addressed subject in biomathematics is population dynamics.

A mathematical model of a predator and prey is typically presented in published articles as a Cauchy problem for a set of ordinary differential equations [1]– [5]. Starting with populations of a single species and progressing to increasingly realistic models where different species dwell and interact in the same ecosystem, the study of the evolution of different populations has long been of particular interest [6]. The analysis of the dynamic interaction between predator and prey has also benefited greatly from the use of mathematical models [7]. Alfred J. Lotka first suggested the Lotka-Volterra predator-prey system in the theory of autocatalytic chemical reactions in 1910. It has since been offered to represent the population dynamics of two interacting species of a predator and its prey [8], [9]. The model equations, the mathematical analysis, and the subsequent numerical simulations all combine to show both the quantitative and qualitative effects of that logical framework. If the prey population becomes extinct and the predator populations follow, the predator is referred to as a specialized predator. A generalist predator, however, can endure in the absence of a prey population. Predator considered in all prior research on prey-predator interaction is an expert. A predator-prey model that depends on the prey and includes self- and cross-diffusion and in which the predator has a secondary food supply has been taken into account [10]. Together, the prey and the predator evolve. The predator's surroundings include the prey. The predator must therefore evolve all the skills required to catch prey in order to survive. These skills include speed, stealth, camouflage (to conceal while approaching the prey), good senses of smell, sight, and hearing (to find the prey), immunity to the prey's poison, poison (to kill the prey), the appropriate mouthparts or digestive system, and others [11]. The additional food may ease the population's predation pressure. Although they receive less attention than base prey, this supplementary food is a significant part of the diets of most predators [12]. In order to exploit a single victim, two predators compete, and Freedman and Waltman examined the persistence criteria [13]. In a model of two predators vying for the same prey, B. Mukhopadhyay and R. Bhattacharyya investigated the effects of harvesting and predator interference [14]. Because predator-prey interactions have a significant impact on the dynamical system and the predator plays an intriguing role in maintaining the structure of the food web, studying the interactions between the two species using differential equation models is one of the traditional applications of mathematical biology [15]. [16]. In a population system, there are often two different sorts of predators: one is a specialist (such as little mustelids), while the other is a generalist (like foxes, common buzzards, cats, etc.). Specialist predators depend only on the specific food for a healthy and successful life, but generalist predators need to have a variety of food sources and attack aggressively on their favorite food [17]. It is known as apparent competition [18] when the predator's assault frequency decreases as a result of more food being available to it. Some theoretical research [19] have reached the conclusion that the provision of more food could increase the number of target predators and reduce the attack rate of prey biomass. Toaha investigated the stability analysis of the model of the prey-predator population with harvesting on the predator population [20].. The assumption that the habitat is stable permits the use of the differential equations with partial derivatives, which are frequently employed in the creation of mathematical models of continuous ecosystems with linear features [21], [22]. In this paper, we present a continuous and deterministic Lotka-Volterra model of the prey-predator population. The prey-predator system's equilibrium solution, analytical solution, logistic growth rate, and harvesting effect were all determined. We also use graphical representations to assess the growth rates of the prey and predator populations and what would happen if one of the populations (the prey or predator population) disappeared.

METHOD AND MATERIALS:

Formulation of prey-predator model:

Let $X(t)$and $Y(t)$ is the population of the prey and predator species at any time t.

Now, the following assumption for prey and predator system [23] [24] [25]:

i. If there are no predators, the number of prey will increase at a rate determined by nature.

$$\frac{dX}{dt} = aX \qquad a > 0 \qquad (1)$$

ii. If there were no prey, the predator population might decline naturally.

$$\frac{dY}{dt} = -pY \qquad p > 0 \qquad (2)$$

iii. The expansion of predator species is promoted by the presence of both prey and predators, while the growth of prey species is prevented. For example, the predator species grows and declines at a rate equivalent to the amount of the two populations.

Based on the above considerations, a system of differential equations that simulates the interaction of competitors or prey-predator populations has been developed. For the purpose of analysis, we may choose the following prey-predator system as the form [26] [23], [27]–[30].

$$\frac{dX}{dt} = aX - bXY$$

$$\frac{dY}{dt} = -pY + qXY \qquad (3)$$

with the initial conditions $X = X_0$ and $Y = Y_0$ at time t=0

The variables X and Y represent, respectively, the size of the prey and the predators at time t. These two populations cooperate in interaction and competition. Positive constants a, b, p, and q are present. A represents the rate of growth in the prey population X, while p represents the rate of decline in the predator population Y. It seems reasonable to assume that the proportion of contacts between prey and predator is proportionate. The frequency of contacts between predators and prey is considered to be proportional to X Times Y and it is acceptable to suppose that a certain percentage p of these encounters will result in the deaths of prey population members. As a result, the term bXY measures the decline in the prey population, and its subtraction must be adjusted for uncontrolled growth in the population, which could occur if there is enough food. Similarly, the word qXY must be added to the equation to change the rate at which the predator population is declining because more predators survive when they come into contact with their prey.

The first equation in (3) is known as the prey equation, and the second equation in (3) is known as the predator equation. The Lotka-Volterra prey-predator model is the name of this well-known solution to equation (3).

Solution of the Model

$X(t) > 0$ and $Y(t) > 0$ with initial conditions of $X(0) = X_0$ and $Y(0) = Y_0$ to determine the solution of the system of equations (1) and (2).

We can write,

$$\frac{dY}{dX} = \frac{\frac{dY}{dt}}{\frac{dX}{dt}} = \frac{Y(qX - p)}{X(a - bY)}$$

$$\frac{(a - by)}{y} dy = \frac{(qx - p)}{x} dx$$

$$(\frac{a}{y} - b)dy = (q - \frac{p}{x})dx$$

Integrating both sides, we get

$$a \ln Y - bY = qX - p \ln X + \ln C$$

$$a \ln Y - bY - qX + p \ln X = \ln C$$

Where log c is constant.

$$a \ln Y - b \ln e^Y - q \ln e^X + p \ln X = \ln C \qquad \Rightarrow \ln C = \ln(\frac{Y^a . X^p}{e^{bY} . e^{qX}}) \qquad (4)$$

$$C = (\frac{Y}{e^{\frac{b}{a}} . Y})^a . (\frac{X}{e^{\frac{q}{p}} . X})^p$$

Let $\frac{a}{b} = Y^*, \frac{p}{q} = X^*$, then

$$C = (\frac{Y}{e^{\frac{Y}{Y*}}})^a . (\frac{X}{e^{\frac{X}{X*}}})^p$$

$$\Rightarrow C = (\frac{Yy*}{e^Y})^a . (\frac{Xx*}{e^X})^p \ [Let \frac{Y}{Y*} = y, \frac{X}{X*} = x]$$

$$\Rightarrow C = (\frac{y}{e^y})^a . (\frac{x}{e^x}) . (Y*)^a . (X*)^p$$

$$\Rightarrow (\frac{e^y}{y})^a . (\frac{e^x}{x})^p = \frac{1}{c} (Y*)^a . (X*)^p = K(say)$$

Thus the value of K is known as the final solution of the prey-predator system.

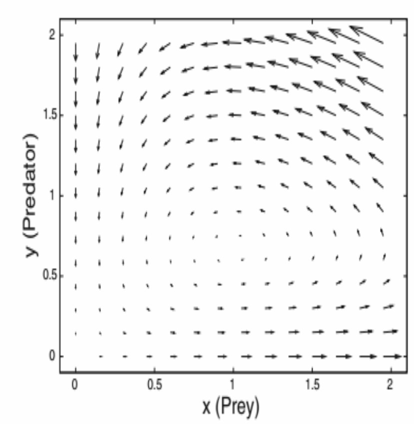


Figure-1 Lotka-Volterra prey-predator model vector field (3.3); values of the arbitrary parameter value a = 1, b = 1.5, p = 1, and q =1 [24]

Equilibrium Solution:

Special population values that result in zero change over time.

$$i.e. \frac{dX}{dt} = \frac{dY}{dt} = 0$$

system of equations:

$$0 = aX - bXY = X(a - byY) \qquad 0 = -pY + qXY = Y(qX - p)$$

One equilibrium solution: $(X,Y) = (0,0)$

If $X \neq 0$ then $Y = \frac{a}{b}$ and $X = \frac{p}{q}$

Second equilibrium solution:

$$(X,Y) = (\frac{p}{q}, \frac{a}{b})$$

We will linearize the functions

$$f_1 = X' = X(a - bY)$$

$$f_2 = Y' = Y(qX - p)$$

Let the stationary point be $(X^*, Y^*) = x^*$

$$X(t) = X^* + u(t)$$

$$Y(t) = Y^* + v(t)$$

Where $u(t)\, and\, v(t)$ are very small.

Now, the linear equation can be written as $\frac{dx}{dt} = Df(x^*).x$

Where $Df(X,Y) = \begin{pmatrix} a - bY & -bX \\ qY & qX - p \end{pmatrix}$

Case-1

For stationary point (0,0)

$$Df(0,0) = \begin{pmatrix} a & 0 \\ 0 & -p \end{pmatrix}$$

The eigenvalues are $\lambda_1 = a$ and $\lambda_2 = -q$

Now the linearize equation become

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{pmatrix} a & 0 \\ 0 & -q \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Therefore, $Y(t) = c_1 e^{at} \to \infty$ as $t \to \infty$

$$Y(t) = c_2 e^{-qt} \to 0 \text{ as } t \to \infty$$

Hence, an unstable saddle node is present at the stationary point (0,0).

Case-2

For the stationary point $(\frac{p}{q}, \frac{a}{b})$

$$Df\left(\frac{p}{q}, \frac{c}{d}\right) = \begin{pmatrix} a - b.\frac{a}{b} & -b.\frac{p}{q} \\ q.\frac{a}{b} & -p + q.\frac{p}{q} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{bp}{q} \\ \frac{aq}{b} & 0 \end{pmatrix}$$

Hence, the linear system's characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -\frac{bp}{q} \\ \frac{aq}{b} & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + ap = 0$$

$$\Rightarrow \lambda = \pm i\sqrt{ap}$$

As a result, the equilibrium point is stable. A center, definitely.

Harvesting of Prey Predator System

First we analyze harvesting. We harvest prey at a small but constant rate $\varepsilon > 0$ Thus the equations ultimately become

$$\frac{dX}{dt} = aX - bXY - \varepsilon = X(a - bY) - \varepsilon$$

$$\frac{dY}{dt} = -pY + qXY = Y(-p + qX)$$

For $\frac{dY}{dt} = 0,$ we still have or $X = \frac{p}{q}$. If $Y = 0,$ then $\frac{dY}{dt} = aX - \varepsilon = 0$ when $X = \frac{\varepsilon}{a}$. As a result, the equilibrium point at $(0,0)$ has moved to the right to $(\frac{\varepsilon}{a},0)$. This simple explanation: $\frac{\varepsilon}{a}$ is just a population of prey for whom the natural increase will just balance our harvesting rate ε in the absence of predators .

Conversely, with $X = \frac{p}{q}, \frac{dX}{dt} = (a - bY)\frac{p}{q} - \varepsilon = 0$ when $Y = \frac{a}{b} - \frac{\varepsilon q}{bp}$. The second equilibrium point has moved to $(X^*, Y^*) = (\frac{p}{q}, \frac{a}{b} - \frac{\varepsilon q}{bp})$ as a result. Assumedly, ε is so little that $Y^* > 0$. Because we only added a constant to $f(X,Y)$ the Jacobian Matrix $J(X,Y)$, it is still identical, but the equilibrium points have changed. $J(\frac{\varepsilon}{a},0) = \begin{pmatrix} a & -\frac{b\varepsilon}{a} \\ 0 & -p + \frac{q\varepsilon}{a} \end{pmatrix}$ is the initial equilibrium point.

The positive eigenvalue is a, and the negative eigenvalue is $(-p+\frac{q\varepsilon}{a})$. (again, assuming is ε small). The equilibrium point is still a saddle as a result. The trajectory leaving this saddle point is still on the x-axis, and $\begin{pmatrix}1\\0\end{pmatrix}$ is still an eigenvector with the positive eigenvalue (it is still true that if $Y=0$ then $\frac{dY}{dt}=0$). When F is small, an eigenvector for the negative eigenvalue is $\begin{pmatrix}\frac{b\varepsilon}{a}\\ a+p-\frac{q\varepsilon}{a}\end{pmatrix}$, which (when ε is small) points up and slightly to the right.

Logistic Growth of Prey Predators system:

Suppose the system has a carrying for prey(Rabbits) **K**. For the population of unregulated prey (rabbits), logistic growth will be observed instead of exponential growth.

In the absence of predators(Foxes), $$\frac{dX}{dt}=aX(1-\frac{X}{K}) \qquad (5)$$

Couple back with the population,

$$\frac{dX}{dt}=\underbrace{aX(1-\frac{X}{K})}_{\text{logistic growth}}-\underbrace{bXY}_{\text{mass action}} \qquad (6)$$

$$\frac{dY}{dt}=\underbrace{-pY}_{\text{exponential decay}}+qXY \qquad (7)$$

Replacing 1/K with N, then

$$\frac{dX}{dt}=aX(1-NX)-bXY \qquad (8)$$

If $K>0$ exists, $N=\frac{1}{K}$ and if $N=0$ then it returns the original equation.

Compartmental model

- The ecosystem has a carrying capacity for prey (Rabbits) in the absence of predators (Foxes) $N=\frac{1}{K}$.
- Prey (Rabbits) decline at a pace proportional to how prey (Rabbits) and predators interact (Foxes).

- Predator (Foxes)population declines proportionate to their existing size when there are no prey (Rabbits).
- The interactions between prey (Rabbits) and predators (Foxes) determine how quickly predators (Foxes) grow (Foxes).

For equilibrium solutions,

$$0 = \frac{dX}{dt} = aX(1 - NX) - bXY \qquad (9)$$

$$0 = \frac{dY}{dt} = -pY + qXY \qquad (10)$$

$$\begin{cases} X(a - aNX - bY) = 0 \\ Y(-p + qX) = 0 \end{cases} \qquad (11)$$

We obtain from the second equation of (11)

$$Y = 0,\ X = \frac{p}{q}$$

Putting Y = 0 into the first equation of (11), we get

$$0 = a(1 - NX)$$

$$0 = \frac{p}{q}(a - aN\frac{p}{q} - bY)$$

$$\Rightarrow bY = (a - aN\frac{p}{q})$$

$$\Rightarrow Y = \frac{a}{b} - \frac{ap}{bq}N$$

Now, equilibrium solutions are: $(X, Y) = (0,0); (X, Y) = (K,0)$ (if K exists)

and $(X, Y) = \left(\frac{p}{q}, \frac{a}{b} - \frac{ap}{bq}N\right)$

Examples

In prey-predator system X represents Rabbits and Y represents Foxes in an area, where t(times) is measured in months. This relationship is defined by the differential equations:

$$\frac{dX}{dt} = 0.07X - 0.002XY$$

$$\frac{dY}{dt} = -0.03Y + 0.00004XY$$

Consider looking at the solutions

a) Find two equilibrium solutions

b) Using the differential equations explain what would happen to the Rabbits population if there were no Foxes.

c) Using the differential equations explain what would happen to the Rabbit population if there were no Rabbits.

d) If t = 0, there are initially 4000 Rabbits and 100 Foxes then what happens are the populations increasing or decreasing?

Solution:

a) We have,

$$\frac{dX}{dt} = 0.07X - 0.002XY$$

$$\frac{dY}{dt} = -0.03Y + 0.00004XY$$

For equilibrium solutions,

$$\begin{cases} \frac{dX}{dt} = 0 \\ \frac{dY}{dt} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 0.007X - 0.002XY = 0 \\ -003Y + 0.00004XY = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X(0.007 - 0.002Y) = 0 \\ Y(-0.03 + 0.00004X) = 0 \end{cases}$$

Using the first equation shown before, we have X = 0 and Y = 35

Also, Using the second equation shown before, we have Y = 0 and X = 750

b) if there are no Rabbits, $\frac{dY}{dt} = -0.03Y$ → here, Foxes grow exponentially.

c) if there are no Foxes, $\frac{dY}{dt} = 0.007X$ → here, Rabbits grow exponential decay.

d) we have an initial population of Rabbits are 4000 and Foxes are 100, then

$$\frac{dX}{dt} = 0.07(4000) - 0.02(4000)(100)$$

= -7720 Rabbits/ month (decreasing)

$$\frac{dY}{dt} = -0.03(100) + 0.00004(100)(4000)$$

=16 Foxes/ month (increasing)

Result and Discussion

Result:

Numerical simulations with Graphical Representations

The way of analyzing numerical data is known as graphical representations. It illustrates in a diagram the relationship between facts, ideas, information, and concepts. It is easy to understand and also one of the most significant learning strategies.

By utilizing a representation in which state variables are continuously changing in connection to time, continuous simulation is the modeling of a realistic overview across time. To establish relationships for the rates of change of the state variables across time, differential equations are frequently used. As a result, biological differential equation models with various dependent variables are more likely to use a system of two or more connected differential equations.

In this study, we use **MATLAB** to illustrate the dynamical and complex characteristics of the system while representing certain numerical simulations to investigate the function of the prey-predator system. We fixed all parameters and the starting values for the populations of prey and predators at the beginning. The impact of parameters on the intricate behavior of a given system is explained via numerical simulations.

a) Think about the parameters listed below.

$a = 2, b = 0.001, p = 10 \, and \, q = 0.002$ and with an initial population, $x(0) = 4000, y(0) = 100$.

These values are used by the script to create the graph in **Figure 2**.

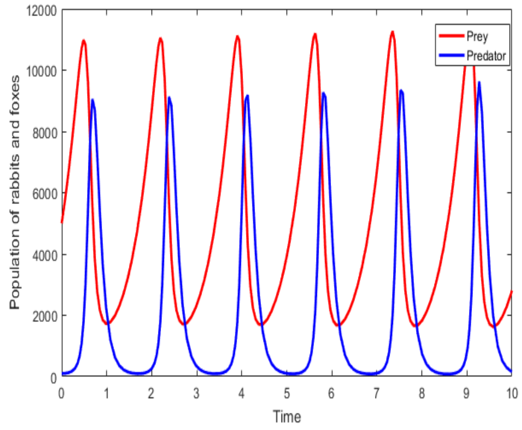
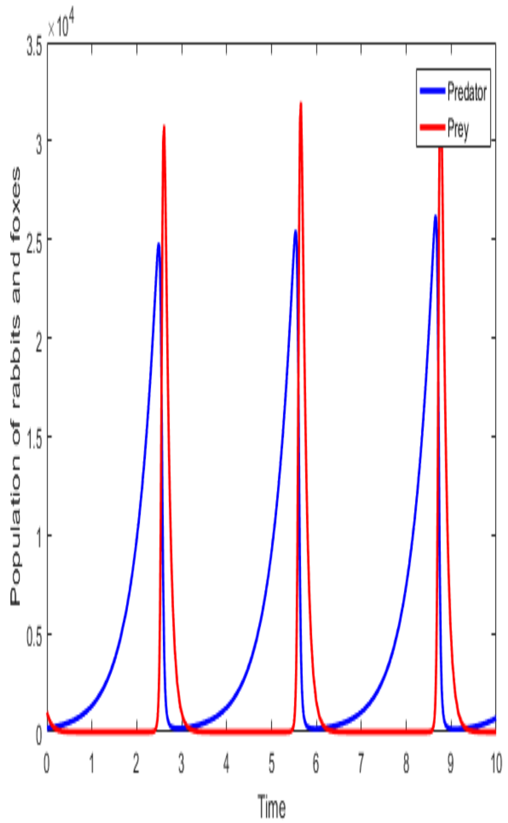


Figure 2: Variation of prey and predator population as time goes on ($x(0) = 4000, y(0) = 100$)

b) Think about the parameters listed below.

$a = 2, b = 0.001, p = 10$ *and* $q = 0.002$ and with an initial population, $x(0) = 2000, y(0) = 100$.

These values are used by the script to create the graph in **Figure 3.**



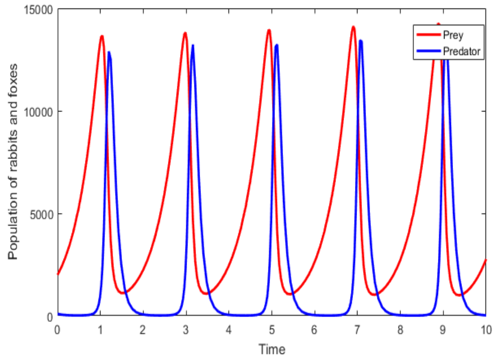


Figure 3: Variation of prey and predator population as time goes on ($x(0) = 2000, y(0) = 100$ **).**

c) Think about the parameters listed below.

$a = 2, b = 0.001, p = 10\ and\ q = 0.002$ and with an initial populatio $x(0) = 200, y(0) = 1000$.

These values are used by the script to create the graph in **Figure 4.**

Figure 4: **Variation of prey and predator population as time goes on (** $x(0) = 200, y(0) = 1000$ **).**

d) Think about the parameters listed below.

$a = 2, b = 0.001, p = 10\ and\ q = 0.002$ and with an initial population, $x(0) = 4000, y(0) = 0$.

These values are used by the script to create the graph in **Figure 5.**

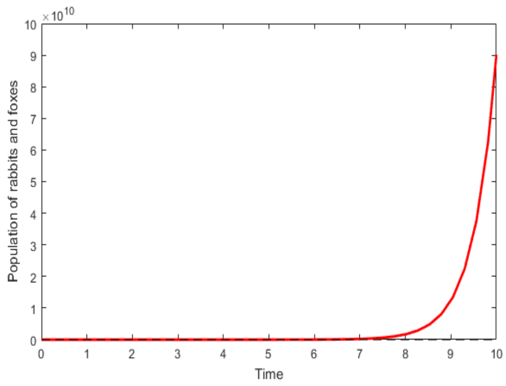


Figure 5: Represents the predator population absent as time

e) Think about the parameters listed below.

$a = 2, b = 0.001, p = 10\ and\ q = 0.002$ and with an initial population, $x(0) = 200, y(0) = 0$. These values are used by the script to create the graph in **Figure 6.**

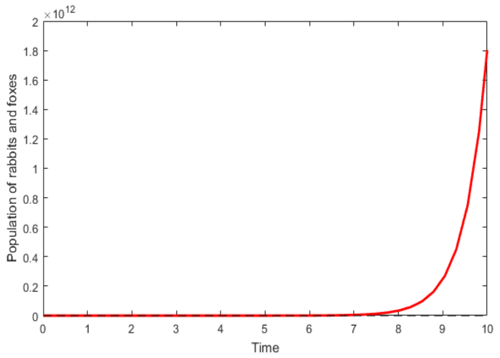


Figure 6: Represents the predator population absent as time goes on

f) Think about the parameters listed below.

$a = 2, b = 0.001, p = 10$ *and* $q = 0.002$ and with an initial population, $x(0) = 0, y(0) = 100$. These values are used by the script to create the graph in **Figure 7.**

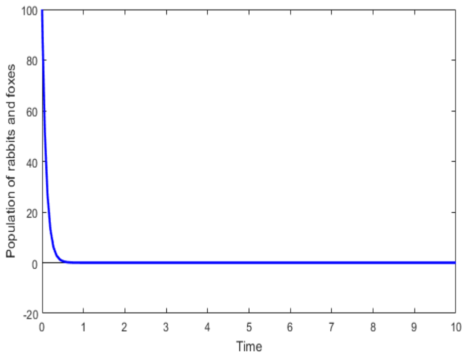


Figure 7: Represents the prey population absent as time goes on with constant prey.

g) Think about the parameters listed below.

$a = 2, b = 0.001, p = 10$ *and* $q = 0.002$ and with an initial population, $x(0) = 0, y(0) = 2000$. These values are used by the script to create the graph in **Figure 8.**

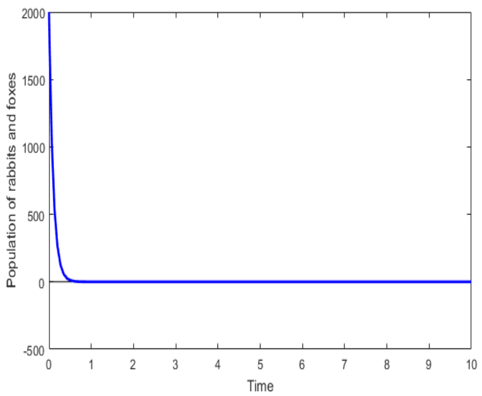


Figure 8: Represents the prey population absent as time goes on with increase predator.

Phase Plane Analysis

In the first quadrant, each trajectory is a closed curve. As a result, the numbers of prey and predator fluctuate in cycles. Having a small initial population (in the region

$0 < X < \frac{p}{q}, 0 < Y < \frac{a}{b}$) When trajectories don't intersect, the number of prey rises initially while the number of predators falls (but not to zero) until $X = \frac{p}{q}$. Predators increase while the prey declines until $X = \frac{p}{q}$. The following in the region $0 < X < \frac{p}{q}, Y > \frac{a}{b}$, the predators and prey both decrease until $Y = \frac{a}{b}$.

In case you're interested, below is the system's phase portrait for the first quadrant.

I. Taking the parameters $a = 1.667, b = 1.333, p = q = 1$ into consideration, we locate the following integral curve in the phase plane.

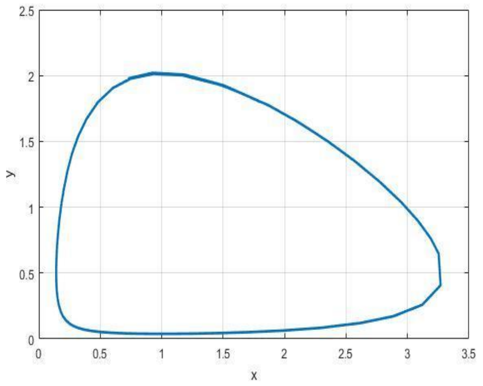


Figure 9: An integral curve in phase plane.

I. Taking the parameters $a = 0.667, b = 1.333, p = q = 1$ into consideration, we locate the following integral curve in the phase plane.

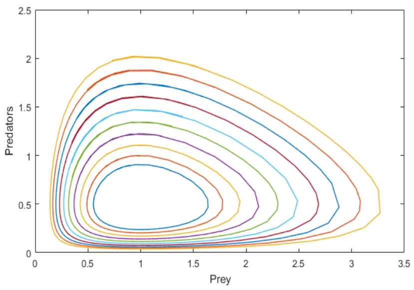


Figure 10: phase plot of Prey/Predator Populations

Discussion:

An important subject for study is the relationships between prey species and their predator species. Many researchers are still researching the multiple aspects of this relationship in the modern era. In order to achieve this, a mathematical model of the prey-predator system is created and examined in the present study, using numerical simulations involving both prey and predator species. By considering system parameters with an interval value rather than a single value, the model system is further improved.

In our discussion, we studied continuous systems with a set of nonlinear differential equations determining their existence. Such a system has a number of parameters that need to be evaluated based on existing research. In this field, (for example, the predator-prey model's a, b, p, and q > 0). Point estimates are often computed and incorporated into the model. There is usually some ambiguity surrounding these estimates. Figure 1 represents the solutions, with prey illustrated in red (solid) and predators in blue (solid). The prey curve always leads the predator curve in this kind of scenario. The Predator-Prey model's periodic activity is seen in Figure 2, Figure 3 and Figure 4.

Figures 5 and 6 provide an expression for the prey's rate of change with respect to the predator and also illustrate the exponential growth of the prey population in the absence of predators. Similarly, the figure 7 and figure 8 which are represent the expression for the rate of change predator with respect to prey and also shows the exponential decay, when prey population are absent. The level curves in Figure 9 are closed, hence the solution is periodic. Although it is ideal, the numerical solution is not always periodic. Keep in mind that when there are no predators, the prey population increases, but there are reduced predators when there is no prey.

CONCLUSION

Consider a circumstance in which there are two populations present: prey and predators. The populations of predators and prey are both of importance to us. These populations do, however, communicate with one another. The Lotka-Volterra equations provide a description of the ecological predator-prey model (or parasite-host model). The dynamics of the nonlinear system are described by differential equations that are sensitive to changes in the Lotka—Volterra model.

Acknowledgements: The authors gratefully acknowledge the technical support provided by Department of mathematics in MawlanaBhashani Science and Technology University for carrying out this work.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest related to the publication of this article

Reference

[1] E. Almanza-Vasquez, R. D. Ortiz-Ortiz, and A. M. Marin-Ramirez, "Dynamic consequences of Lotka-Volterra predation model when the area inhabited by the prey and predator decreases," *Appl. Math. Sci.*, vol. 9, no. 37–40, pp. 1961–1970, 2015, doi: 10.12988/ams.2015.5172.

[2] P. Andayani and W. M. Kusumawinahyu, "Global stability analysis on a predator-prey model with omnivores," *Appl. Math. Sci.*, vol. 9, no. 33, pp. 1771–1782, 2015, doi:

10.12988/ams.2015.5120.

[3] E. P. Kolpak, S. Petersburg, and R. Federation, “A Predator-Prey Mathematical Model in a Limited Area,” *Glob. J. Pure Appl. Math.*, vol. 12, no. 5, pp. 4443–4453, 2016.

[4] J. Hofbauer, R. Kon, and Y. Saito, “Qualitative permanence of Lotka-Volterra equations,” *J. Math. Biol.*, vol. 57, no. 6, pp. 863–881, 2008, doi: 10.1007/s00285-008-0192-0.

[5] K. Pusawidjayanti, A. Suryanto, and R. B. E. Wibowo, “Dynamics of a predator-prey model incorporating prey refuge, predator infection, and harvesting,” *Appl. Math. Sci.*, vol. 9, no. 73–76, pp. 3751–3760, 2015, doi: 10.12988/ams.2015.54340.

[6] A. Edwin, “Modeling and analysis of a two prey-one predator system with harvesting, Holling Type II and ratio-dependent responses,” 2010.

[7] H. Almubayedh, R. Albannay, K. Alelq, R. Ahmad, N. Ahmad, and A. A. Naqvi, “Clinical uses and toxicity of Atropa belladonna; an evidence-based comprehensive retrospective review (2003-2017),” *Biosci. Biotechnol. Res. Commun.*, vol. 11, no. 1, pp. 41–48, 2018, doi: 10.21786/bbrc/11.1/6.

[8] A. J. Lotka, “Contribution to the theory of periodic reactions,” *J. Phys. Chem.*, vol. 14, no. 3, pp. 271–274, 1910, doi: 10.1021/j150111a004.

[9] N. S. Goel, S. C. Maitra, and E. W. Montroll, “On the Volterra and other nonlinear models of interacting populations,” *Rev. Mod. Phys.*, vol. 43, no. 2, pp. 231–276, 1971, doi: 10.1103/RevModPhys.43.231.

[10] L. N. Guin, M. Haque, and P. K. Mandal, “The spatial patterns through diffusion-driven instability in a predator-prey model,” *Appl. Math. Model.*, vol. 36, no. 5, pp. 1825–1841, 2012, doi: 10.1016/j.apm.2011.05.055.

[11] T. A. S. Obaid, “The Predator-Prey Model Simulation,” vol. 31, no. 2, pp. 103–109, 2013.

[12] J. Ghosh, B. Sahoo, and S. Poria, “Prey-predator dynamics with prey refuge providing additional food to predator,” *Chaos, Solitons and Fractals*, vol. 96, pp. 110–119, 2017, doi: 10.1016/j.chaos.2017.01.010.

[13] H. I. Freedman and P. Waltman, “Persistence in models of three interacting predator-prey populations,” *Math. Biosci.*, vol. 68, no. 2, pp. 213–231, 1984, doi: 10.1016/0025-5564(84)90032-4.

[14] B. Mukhopadhyay and R. Bhattacharyya, “Effects of harvesting and predator interference in a model of two predators competing for a single prey,” *Appl. Math. Model.*, vol. 40, no. 4, pp. 3264–3274, 2016, doi: 10.1016/j.apm.2015.10.018.

[15] C. Science, “in a Predator-Prey Population Growth Model,” vol. 1221, no. 02, 2002.

[16] S. Mondal and G. P. Samanta, “Dynamics of an additional food provided predator–prey system with prey refuge dependent on both species and constant harvest in predator,” *Phys. A Stat. Mech. its Appl.*, vol. 534, p. 122301, 2019, doi: 10.1016/j.physa.2019.122301.

[17] M. Andersson, “on rodents of predation Influence populations,” vol. 29, no. 1977, pp. 591–

597, 2015.

[18] D. Holt, “and the Structure of Prey Communities,” vol. 229, pp. 197–229, 1977.

[19] M. Van Baalen, V. Křivan, P. C. J. Van Rijn, and M. W. Sabelis, “Alternative food, switching predators, and the persistence of predator-prey systems,” *Am. Nat.*, vol. 157, no. 5, pp. 512–524, 2001, doi: 10.1086/319933.

[20] S. Toaha, “Stability Analysis of Prey-Predator Population Model with Harvesting on The Predator Population,” vol. 12, no. 1, pp. 48–59, 2015.

[21] B. Dubey, B. Das, and J. Hussain, “A predator-prey interaction model with self and cross-diffusion,” *Ecol. Modell.*, vol. 141, no. 1–3, pp. 67–76, 2001, doi: 10.1016/S0304-3800(01)00255-1.

[22] A. Chakraborty, M. Singh, D. Lucy, and P. Ridland, “Predator-prey model with prey-taxis and diffusion,” *Math. Comput. Model.*, vol. 46, no. 3–4, pp. 482–498, 2007, doi: 10.1016/j.mcm.2006.10.010.

[23] G. F. Estabrook, *Mathematical Biology: I: An Introduction. Third Edition. Interdisciplinary Applied Mathematics, Volume 17. By J D Murray. New York: Springer. $59.95. xxiii + 551 p; ill.; index. ISBN: 0–387–95223–3. 2002. Mathematical Biology: II: Spatial Models an*, vol. 79, no. 1. 2004. doi: 10.1086/421587.

[24] J. Müller and C. Kuttler, *Methods and Models in Mathematical Biology Deterministic and Stochastic Approaches*. 2014.

[25] M. Danca, S. Codreanu, and B. Bakó, “Detailed Analysis of a Nonlinear Prey-predator Model,” *J. Biol. Phys.*, vol. 23, no. 1, pp. 11–20, 1997, doi: 10.1023/A:1004918920121.

[26] E. Society, “The Stability and the Intrinsic Growth Rates of Prey and Predator Populations Author (s): James T. Tanner Published by: Ecological Society of America Stable URL : http://www.jstor.org/stable/1936296. THE STABILITY AND THE INTRINSIC GROWTH RATES OF PR,” vol. 56, no. 4, pp. 855–867, 2013.

[27] J. D. Murray, *Mathematical Biology biomedical applications*. doi: 10.1007/b98869.

[28] *No Tit.צכןle*. [Online]. Available: https://www.ptonline.com/articles/how-to-get-better-mfi-results

[29] M. F. Elettreby, “Two-prey one-predator model,” *Chaos, Solitons and Fractals*, vol. 39, no. 5, pp. 2018–2027, 2009, doi: 10.1016/j.chaos.2007.06.058.

[30] H. A. Adamu, “Mathematical analysis of the predator-prey model with two prey and one predator,” no. March 2020.