# Fractional Order Differential Equation In Banach Algebra

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**Abstract-** In this paper, we investigate the fractional order nonlinear quadratic differential equation with initial value condition in Banach algebras under existence of solution. The primary conclusion is established using basic hybrid fixed point theory methods for three operators under certain monotonicity criteria.

**Keywords:** Fractional order quadratic differential equation, fixed point theorem, locally attractivitty, existence of solution, Banach algebra.

## 1.INTRODUCTION

Fractional differential equations appear in a variety of scientific and engineering disciplines because the mathematical analysis of systems and processes in the fields of physics, aerodynamics of complex media, and others requires fractional order derivatives of the order. Many authors have recently examined fractional order differential equations from two perspectives: the theoretical characteristics of solution existence and uniqueness, and t he analytic and numerical methods for solving them. The fractional differential equations are very useful for describing the hereditary features of different materials and processes. As a result, the subject of fractional differential equations is becoming increasingly popular and play an essential role. Dealing with nonlinear differential equations can be done in a variety of ways.

Fixed point theory is a crucial aspect of non linear analysis. For the existence of a solution to a fractional order nonlinear differential equation, we applied the fixed point approach. This approach has been demonstrated to handle a wide range of nonlinear problems successfully, quickly, and precisely. Several fixed point theorems are now used in nonlinear differential and integral equations applications. The fixed point theorem is chosen based on the data that is provided.

We consider the following fractional order nonlinear quadratic differential as

$$D^{\alpha}\left[\frac{u(t)}{p(t,u(t),u(\delta(t)))}\right]+k\left[\frac{u(t)}{p(t,u(t),u(\delta(t)))}\right]=q(t,u(t),u(\delta(t))+r(t,u(t),u(\delta(t)),t\in R \quad (1.1)$$

$$u(t_0)=u_0,u(\delta(t))=w_0\in R \quad (1.2)$$

$$p(t_0,u(t_0),u(\delta(t_0)))=p(t_0,u_0,w_0)\in R. \quad (1.3)$$

for $k>0\in R,\alpha\in(0,1)\ A:U\to U$ ,

Where, $p: R_+\times R\times R\to R-\{0\}$ and $q, r: R_+\times R\times R\to R$are continuous functions.

By a solution of fractional order nonlinear quadratic differential equation(1.1,)we mean a function $u\in\ell(R_+,R)$ such that:

(i) The function $t\to\left[\frac{u(t)}{p(t,u(t),u(\delta(t)))}\right]$ is continuous for each $u\in R.$

(ii) $u$ satisfies(1.1),(1.2)and(1.3).

## 2.Auxiliary Results

We need some definitions, notation and hypothesis, listed here.

Let $U=\ell(R_+,R)$ be the space of continuous real valued function on $R_+$ and $\Omega$ be a subsetof $U$ .Let a mapping $A:U\to U$ be an operator and consider the following operator equation in $U$ ,namely,

**Definition2.1.**[4].We say that solution of the equation (4) are locally attractive if there ~~exists~~ a closed ball $\overline{B_k(0)}$ in the space $A\ell(R_+,R)$ and for some real number k>0suchthat for arbitrary solution $u=u(t)\,and\,v=v(t)$ of equation(4) belonging to $\overline{B_k(0)}\cap\Omega$ we have that

$$\lim_{x\to\infty}(u(t)-v(t))=0\cdot$$

**Definition2.2.**[1].TheRiemann-Liouville fractional derivative of order α>0,n−1<α<n∈N with lower limit zero for a function $p$ is defined as

$$D^{\alpha}p(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{p(s)}{(t-s)^{\alpha}}ds \quad ,t>0$$

Suchthat $D^{-\alpha}p(t)=I^{\alpha}p(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{p(s)}{(t-s)^{1-\alpha}}ds$ respectively.

**Definition2.3.**[9]TheRiemann-Liouville fractional integral of order ξ>0,n−1<ξ<n,n∈ N with lower limit zero for a function *p* is defined by

$$I^{\xi} p(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{p(s)}{(t-s)^{1-\xi}} ds \quad t > 0$$
,

where Γ(ξ) denote the Euler gamma function. The Riemann-Liouville fractional derivative. operator of order ξ defined by $D^{\alpha} = \frac{d}{dt}^{\xi} = \frac{d}{dt} \circ I^{1-\xi}$.

**Theorem2.1.**[2].(Arzela-Ascoli Theorem). If every uniformly bounded and equicontinuous squence{ $p_n$ }of functions in $\ell(R_+, R)$,then it has a convergent subsequence.

**Theorem2.2.**[2].A metric space $U$ is compact iff every sequence in $U$ has a convergent subsequence.

We apply new hybrid fixed pint theorem [2] which is the main tool in the existence theorem of solutions of fractional quadratic differential equation.

**Theorem2.3.**[2]:Let S be a non-empty,bounded and closed-convex subset of theBanac hspace

$U$ and let A:$U \to U$ and B:S→$U$ are two operators satisfying:

a) A is Lipschitz with a Lipschitz constant α.
b) B is completely continuous, and
c) $Au\, Bv$ ∈S for all $u$ ∈S, and

d) αM<1where M=‖B(S)‖:sup{‖B$u$‖: $u$ ∈S}.Then the operator equation A$u$ B$u$ = $u$ has a solution in S.

EXISTENCE THEORY

We seek the solution of 2.1) in the space $\ell(R_+, R, R)$ of continuous and real-valued function defined on $R_+$. Define a standard norm‖·‖ and a multiplication"·"in $\ell(R_+, R)$ by,

‖$u$‖=sup {|$u$(t)|:t∈$R_+$},($u$ $v$)(t)=$u$(t) $v$(t),t∈$R_+$

**Definition2.6.**[2]:A mapping $q$ :$R_+$×R×R→R is Caratheodory if:

i) $t \to q(t,u,v)$ is measurable for each $u, v$ ∈R and
ii) $u \to q(t,u,v)$ is continuous almost every where for t∈$R_+$.
iii) Furthermore a caratheodory function $q$ is $\ell'$-caratheodory if:

iv) For each real number $k$ >0there exists a function $h_k \in \ell'(R_+, R)$ such that | $q(t,u,v)| \leq h_k(t)$ a. e. t∈R+ for all $u, v$ ∈R with $|u|_k \leq k$ , $|v|_k \leq k$ .

Finally, a caratheodory function $q$ is $\ell'_X$ -caratheodory if:
X

v) There exists a function $h \in \ell'(R_+, R)$ such that $|q(t,u,v)| \leq h_k(t)$ a. e. t∈ R+ for all $u, v$ ∈R.

We need following lemma to prove our result.

**Lemma2.1.**Suppose that α∈(0,1)and the function $p, q$ satisfying fractional quadratic differential equation (1.1−1.3). Then $u$ is the solution of the fractional quadratic differential equation(1.1−1.3)if and only if it is the solution of integral equation

$$u(t)=p(t,u(t),u(\delta(t)))\left[\frac{u_0}{p(t_0,u_0,w_0)}-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u(s)}{p(s,u(s),u(\delta(s)))(t-s)^{1-\alpha}}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u(s),u(\delta(s)))}{(t-s)^{1-\alpha}}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u(s),u(\delta(s)))}{(t-s)^{1-\alpha}}ds\right] \tag{2.2}$$

for all t∈ R+

We need following hypothesis for existence the solution of fractional quadratic differential equation (1.1).

($A_1$) The function $p$ : R+ × R × R → R − {0} is continuous and bounded with bound P= $\sup_{p(t,u(t),u(\delta(t)))\in R_+\times R\times R} |p(t,u(t),u(\delta(t)))|$ There exist a bounded function $l : R_+ \to R$ with bound $\|l\|$ satisfying:

$$|p(t,u(t),u(\delta(t))) - p(t,v(t),v(\delta(t)))| \leq l(t)\max\{|u(t)-v(t)|,|u(\delta(t))-v((\delta(t))|\}$$

for all $u, v$ ∈R

($A_2$ ).The function $q(t,u,v): R_+ \times R \times R \to R$ is satisfying caratheodory condition with continuous function h(t):R+→R such that $q(t,u,v) \leq h(t)$ ∀t∈R+ and $u, v \in R$ .

($A_3$ ).The function $p$ :R+×R×R→R−{0}is satisfying caratheodory condition with continuous function i(t):R+→R such that $\frac{u(t)}{p(t,u(t),u(\delta(t)))} \leq i(t)$ $t \in R, and\ u \in R$ (

$A_4$ ).The function $u, v : R_+ \to R$ defined by the formulas $u(t) = \int_0^t \frac{P(s)}{(t-s)^{1-\alpha}}ds$ and $v(t) = \int_0^t \frac{h(s)}{(t-s)^{1-\alpha}}ds$ is bonded on $R_+$ and the functions $P(t), u(t) and\ v(t)$ vanish at infinity.

# 3.MAIN RESULT

**Theorem3.1.**Assume that conditions ($A_1$)−($A_4$) hold. Further if IK<k and $KK_1$<1,

Where K and $K_1$is defined as there exists a constant $K_1, K_2$>0 such that $K_1 = \sup\left\{\frac{ka(t)}{\Gamma(\alpha)} : t \in R_+\right\}$ and $K_2 = \sup\left\{\frac{b(t)}{\Gamma(\alpha)} : t \in R_+\right\}$, $I + K_1 + K_2 = K$ .Then functional quadratic differential equation (1.1) has a solution in the space $C(R_+,R)$, more over solution of (1.1) are locally attractive on $R_+$.

**Proof:** By a solution of fractional quadratic differential equation(1)we mean a continuous function $U:R_+ \to R$ that satisfies fractional quadratic differential equation(1) on $R_+$. Set $U=C(R_+,R)$ and define a subset S of U as $S=\{u \in U: \|u\| \leq k\}$. Where k at isfies the inequality, IK≤k

Let $U=C(R_+,R)$ be a Banach Algebra of all continuous real-valued function on $R_+$with The norm,

$$\|u\| = sup\,|\,u(t)\,|, t \in R_+ \tag{3.1}$$

We shall obtain the solution of fractional quadratic differential equation (1.1) under some suitable conditions involved in(1.1).Now the fractional quadratic differential equation (1.1)is equivalent to the fractional quadratic integral equation (2.2).

Let us define the two mappings A:U→UandB:S→Uby,

$$Au(t) = p(t, u(t), u(\delta(t))), t \in R_+ \tag{3.2}$$

$$Bu(t) = \frac{u(t_0)}{p(t_0, u(t_0), u(\delta(t_0)))} - \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{u(s)}{p(s, u(s), u(\delta(s)))} (t-s)^{\alpha-1} ds$$
$$+ \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{q(s, u(s), u(\delta(s)))}{(t-s)^{\alpha-1}} ds, + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{r(s, u(s), u(\delta(s)))}{(t-s)^{\alpha-1}} ds, \quad t \in R_+ \tag{3.3}$$

Thus from the fractional quadratic differential equation(1)we obtain the operator equation as follows:

$$u\,(t) = A\,u\,(t) B\,u\,(t)\ , t \in R_+. \tag{3.4}$$

If the operator A and B satisfy all the hypothesis of theorem (2.3), then the operator equation (3.4) has a solution on S.

**StepI:** Firstly, we show that A is Lipschitz on S.Let $u, v \in \overline{B_k(0)}$ ,then by($H_1$),

$$|\,Au(t) - Av(t)\big| \leq \big|\, p(t, u(t), u(\delta(t))) - p(t, v(t), v(\delta(t)))\,|$$
$$\leq \alpha(t)\,|\,u(t) - v(t)\,|$$

for all t∈R+, $u$ , $v$ ∈S

Taking suprimum over t we get, $\|Au-Av\| \le \|\alpha\|\|u-v\|$ forall $u,v\in S$. Thus, A is Lipchitz onS with Lipschitz constant $\alpha$ .

**Step II:** To show the operator B is completely continuous on U. Let $\{u_n\}$ be a sequence in S converging to a point $u$ .Then by lebesgue dominated convergence theorem for all t ∈ R+, we obtain

$$\lim_{n\to} Bu_n(t)$$

$$=\lim_{n\to\infty}\left\{\frac{u_n(t_0)}{p(t_0,u_n(t_0),u(\delta(t_0)))}-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u_n(s)}{p(s,u_n(s),u(\delta(s)))}(t-s)^{\alpha-1}ds+\frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u_n(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds+\frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u_n(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\right\}$$

$$=\frac{u(t_0)}{p(t_0,u_n(t_0),u(\delta(t_0)))}-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u_n(s)}{p(s,u_n(s),u(\delta(s)))}(t-s)^{\alpha-1}ds+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u_n(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u_n(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds$$

$$=Bu(t),\forall t\in R_+$$

This shows that B is continuous on S.

Next,we will prove that the set B(S)is uniformly bounded in S, for an $u,v\in S$ ,we have,

$$|Bu(t)|=\left|\frac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))}-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|$$

$$\le\left|\frac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))}\right|+\left|-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds\right|$$

$$+\left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|+\left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|$$

$$\le P_0+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds$$

$$\le P_0+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{P(s)}{(t-s)^{\alpha-1}}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{\alpha-1}}ds$$

Taking supremum over t,we obtain

$$\|Bu\|\le P_O+\frac{ka(t)}{\Gamma(\alpha)}+\frac{b(t)}{\Gamma(\alpha)}\le\overline{P+K_1}+\overline{K_2=}K, \text{say.}$$

Therefore $\|Bu\|\le K,$ ,which shows that B is uniformly bounded on S. Now we will show that B(S) is equi-continuous set inu.Lett$_1$,t$_2$∈R+witht$_2$>t$_1$and $u\in S$ ,then we have

$$\left|Bu(t_2)-Bu(t_1)\right|=$$

$$\left|Bu(t_2)-Bu(t_1)\right|\left\|Bu(t_2)-Bu(t_1)\right\|\le K, u\in S$$

$$\left|\begin{array}{l}\left\{\begin{array}{l}\dfrac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))}-\dfrac{k}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds\\ \qquad +\dfrac{k}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds+\dfrac{k}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\end{array}\right\}\\ -\left\{\begin{array}{l}\dfrac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))}-\dfrac{k}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds\\ \qquad +\dfrac{k}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds+\dfrac{k}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\end{array}\right\}\end{array}\right|$$

$$\le\left|\frac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))}-\frac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))}\right|+\left|\begin{array}{l}\dfrac{k}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}\dfrac{+u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds\\ -\dfrac{\lambda}{\Gamma(\xi)}\displaystyle\int_{t_0}^{t}\dfrac{u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds\end{array}\right|$$

$$+\left|\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|$$
$$+\left|\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|$$

$$\leq \frac{k}{\Gamma(\alpha)}\|P\|\left|\begin{array}{l}\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}ds-\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\\ +\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}ds-\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\end{array}\right|+\frac{\|h\|}{\Gamma(\alpha)}\left|\begin{array}{l}\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}ds-\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\\ +\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}ds-\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\end{array}\right|$$

$$\leq \frac{k\|P\|}{\Gamma(\alpha)}\left(\begin{array}{r}\left|\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}ds-\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\right|\\ \left|+\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\right|\end{array}\right)+\frac{\|h\|}{\Gamma(\alpha)}\left(\begin{array}{r}\left|\int_{t_0}^{t_2}(t_2-s)^{\alpha-1}ds-\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\right|\\ \left|+\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}ds\right|\end{array}\right)$$

$$\leq \frac{k\|P\|}{\Gamma(\xi)}\left\{\begin{array}{r}\left|\left[\frac{(t_2-s)^{\alpha}}{-\alpha}\right]_{t_0}^{t_2}-\left[\frac{(t_1-s)^{\alpha}}{-\alpha}\right]_{t_0}^{t_2}\right|\\ +\left|\left[\frac{(t_1-s)^{\alpha}}{-\alpha}\right]_{t_0}^{t_2}\right|\end{array}\right\}+\frac{\|h\|}{\Gamma(\alpha)}\left\{\begin{array}{r}\left|\left[\frac{(t_2-s)^{\alpha}}{-\alpha}\right]_{t_0}^{t_2}-\left[\frac{(t_1-s)^{\alpha}}{-\alpha}\right]_{t_0}^{t_2}\right|\\ +\left|\left[\frac{(t_1-s)^{\alpha}}{-\alpha}\right]_{t_0}^{t_2}\right|\end{array}\right\}$$

$$\leq \frac{k}{\Gamma(\alpha+1)}\|P\|\left\{\begin{array}{l}\left|-\left[(t_2-t_2)^{\alpha}-(t_2-t_0)^{\alpha}\right]\right|\\ \left|-\left[(t_1-t_2)^{\alpha}-(t_1-t_0)^{\alpha}\right]\right|\\ +\left|-\left[(t_1-t_2)^{\alpha}-(t_2-t_2)^{\xi}\right]\right|\end{array}\right\}+\frac{\|h\|}{\Gamma(\alpha+1)}\left\{\begin{array}{l}\left|-\left[(t_2-t_2)^{\alpha}-(t_2-t_0)^{\alpha}\right]\right|\\ \left|-\left[(t_1-t_2)^{\alpha}-(t_1-t_0)^{\alpha}\right]\right|\\ +\left|-\left[(t_1-t_2)^{\alpha}-(t_2-t_2)^{\alpha}\right]\right|\end{array}\right\}$$

$$\leq \left\{\frac{k}{\Gamma(\alpha+1)}\|P\|+\frac{\|h\|}{\Gamma(\alpha+1)}\right\}\left\{\left|(t_2-t_0)^{\alpha}-(t_1-t_0)^{\alpha}\right|\right\}\to 0$$

as $t_1 \to t_2, \forall n \in N$.Implies B is equi-continuous. Therefore by Arzela Ascoli theorem that B is

completely continuous operator on S.

**Step III:** To show $u = AuBv$ By$\Rightarrow u, v \in S$ .Let $u \in U$ and $v \in S$ such that $u = AuBv$ .By assumptions $(A_1, A_2, A_3)$

$$|u(t)| = |Au(t)Bu(t)|$$

$$\leq |Au(t)||Bu(t)|$$

$$\leq \left| p(f(t,u(t),u(\delta(t))) \right| \left| \begin{array}{l} \dfrac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))} - \dfrac{k}{\Gamma(\xi)} \displaystyle\int_{t_0}^{t} \dfrac{u(s)}{p(s,u(s),u(\delta(s)))} (t-s)^{\alpha-1} ds \\ \quad + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_{t_0}^{t} \dfrac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_{t_0}^{t} \dfrac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds \end{array} \right|$$

$$\leq \left\{ \begin{array}{l} P \left| \dfrac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))} \right| + \left| - \dfrac{k}{\Gamma(\xi)} \displaystyle\int_{t_0}^{t} \dfrac{u(s)}{p(s,u(s),u(\delta(s)))} (t-s)^{\alpha-1} ds \right| \\ \quad + \left| \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds \right| + \left| \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds \right| \end{array} \right\}$$

$$\leq P \left\{ P_0 + \frac{k}{\Gamma(\xi)} \int_{t_0}^{t} \frac{u(s)}{p(s,u(s),u(\delta(s)))} (t-s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{p(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds \right\}$$

Taking supremum over t,we obtain

$$\leq P \left\{ P_0 + \frac{k}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{P(s)}{(t-s)^{\alpha-1}} ds + \frac{1}{\Gamma(\xi)} \int_{t_0}^{t} \frac{h(s)}{(t-s)^{\alpha-1}} ds \right\}$$

$$\leq P \left\{ P_0 + \frac{ku(t)}{\Gamma(\alpha)} + \frac{v(t)}{\Gamma(\xi)} \right\} \leq P \left\{ P_0 + K_1 + K_2 \right\} = PK \leq k_1$$

Therefore $\|u\| \leq PK \leq k_1$.

That is we have, $\|u\| = \|AuBu\| \leq k_1,\ \forall\, u \in S$ .Hence assumption (c)of theorem (2.3) is proved.

**StepIV:** Also we have $M = \|B(s)\| = \sup\{\|Bu)\|\}$

$$= \sup \left\{ \sup_{t \in R_+} \left[ \left| \begin{array}{l} \dfrac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))} - \dfrac{k}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{u(s)}{p(s,u(s),u(\delta(s)))} (t-s)^{\alpha-1} ds \\ \quad + \dfrac{k}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds + \dfrac{k}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds \end{array} \right| \right] \right\}$$

$$\leq \sup \left\{ \sup_{t \in R_+} \left\{ \begin{array}{l} \left| \dfrac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))} \right| + \left| \dfrac{k}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{u(s)}{p(s,u(s),u(\delta(s)))} (t-s)^{\alpha-1} ds \right| \\ \quad \left| \dfrac{k}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds \right| + \left| \dfrac{k}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds \right| \end{array} \right\} \right\}$$

$$\leq \sup_{t \in R_+} \left\{ P_0 + \frac{k}{\Gamma(\alpha)} \left| \int_{t_0}^{t} \frac{u(s)}{p(s,u(s),u(\delta(s)))} \right| (t-s)^{\alpha-1} ds + \frac{k}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds + \frac{k}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}} ds \right\}$$

$$\leq \sup_{t \in R_+} \left\{ P_0 + \frac{k}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{P(s)}{(t-s)^{\alpha-1}} ds + \frac{k}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{h(s)}{(t-s)^{\alpha-1}} ds \right\}$$

Taking supermum over t, we obtain

$$\leq\left\{P_0+\frac{ku(t)}{\Gamma(\alpha)}+\frac{v(t)}{\Gamma(\alpha)}\right\}$$
$$\leq\left\{P_0+K_1+K_2\right\}=K$$

And therefore MK=**K**K<1.

Thus the condition(d)of theorem(2.3)is satisfied.

Hence all the conditions of theorem(2.3)are satisfied and therefore the operator equation

$AuBu=u$ has a solution. As a result, the fractional quadratic differential equation(1)has a solution defined on $R_+$.

**Step V:** Finally, we have to show that the locally attractivity of the solution for fractional quadratic differential equation(1.1).Let $u$ and $v$ be two solutions of fractional quadratic differential equation (1.1) in S defined on $R_+$. Then, we have

$$|u(t)-v(t)|=\left|\begin{array}{l}\left\{\left[p(t,u(t),u(\delta(t)))\right]\left[\begin{array}{l}\frac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))}-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds\\+\frac{k}{\Gamma(\xi)}\int_{t_0}^{t}\frac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds+\frac{k}{\Gamma(\xi)}\int_{t_0}^{t}\frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\end{array}\right]\right\}\\-\left\{\left[p(t,v(t),v(\delta(t)))\right]\left[\begin{array}{l}\frac{v(t_0)}{p(t_0,v(t_0),v(\delta(t_0)))}-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{v(s)}{p(s,v(s),v(\delta(s)))}(t-s)^{\alpha-1}ds\\+\frac{k}{\Gamma(\xi)}\int_{t_0}^{t}\frac{q(s,v(s),v(\delta(s)))}{(t-s)^{\alpha-1}}ds+\frac{k}{\Gamma(\xi)}\int_{t_0}^{t}\frac{r(s,v(s),v(\delta(s)))}{(t-s)^{\alpha-1}}ds\end{array}\right]\right\}\end{array}\right|$$

$$\leq\left\{\left|p(t,u(t),u(\delta(t)))\right|\left[\begin{array}{l}\left|\frac{u(t_0)}{p(t_0,u(t_0),u(\delta(t_0)))}\right|+\left|-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u(s)}{p(s,u(s),u(\delta(s)))}(t-s)^{\alpha-1}ds\right|\\+\left|\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{q(s,u(s),(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|+\left|\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{r(s,u(s),(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|\end{array}\right]\right\}$$

$$+\left\{\left|p(t,v(t),v(\delta(t_0)))\right|\left[\begin{array}{l}\left|\frac{v(t_0)}{p(t_0,v(t_0),v(\delta(t_0)))}\right|+\left|-\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{v(s)}{p(s,v(s),v(\delta(s)))}(t-s)^{\alpha-1}ds\right|\\+\left|\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{q(s,v(s),v(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|+\left|\frac{\lambda}{\Gamma(\xi)}\int_{t_0}^{t}\frac{r(s,v(s),v(\delta(s)))}{(t-s)^{\alpha-1}}ds\right|\end{array}\right]\right\}$$

$$\leq P\left\{P_0+\frac{k}{\Gamma(\xi)}\int_{t_0}^{t}\left|\frac{u(s)}{p(s,u(s),u(\delta(s)))}\right|(t-s)^{\alpha-1}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,u(s),u(\delta(s)))}{(t-s)^{\alpha-1}}ds\right\}$$

$+$

$$P\left\{P_0+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\left|\frac{v(s)}{p(s,v(s),v(\delta(s)))}\right|(t-s)^{\alpha-1}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{q(s,v(s),v(\delta(s)))}{(t-s)^{\alpha-1}}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{r(s,v(s),v(\delta(s)))}{(t-s)^{\alpha-1}}ds\right\}$$

$$\leq P\left\{P_0+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{P(s)}{(t-s)^{\alpha-1}}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{\alpha-1}}ds\right\}$$

$$\leq 2P\left\{P_0+\frac{k}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{P(s)}{(t-s)^{\alpha-1}}ds+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{\alpha-1}}ds\right\}$$

Taking supremum over t,we obtain

$$\leq P\left\{P_0+\frac{ka(t)}{\Gamma(\alpha)}+\frac{b(t)}{\Gamma(\alpha)}\right\}$$

Since $\lim_{n\to\infty}a(t)=0, \lim_{n\to\infty}b(t)=0 \; and \lim_{n\to\infty}P(t)=0$ for ε>0,there exist a real number $T'>0, T''>0, T'''>0$ such that if

WechooseT*=max $\{T', T'', T''\}$,forall $t \geq T^*$

$$P_0 \leq \frac{\varepsilon}{6P}, a(t) \leq \frac{\Gamma(\alpha)\varepsilon}{6kP} \; and \; b(t) \leq \frac{\Gamma(\xi)\varepsilon}{6kP}$$

Then from above inequality it follows that $|u(t)-v(t)| \leq \varepsilon$ for all t≥T*. Hence for all $t \geq T^*$ (1)has a locally attractive solution on $R_+$.

## 4.EXAMPLE

Consider the following fractional quadratic differential equation of type (1.1)

$$D^{\frac{1}{2}}\left[\frac{u(t)}{p(t,u(t),u(\delta(t)))}\right]+2\left[\frac{u(t)}{p(t,u(t),u(\delta(t)))}\right]=q(t,u(t),u(\delta(t)))+r(t,u(t),u(\delta(t))), t\in R_+$$

$u(0)=0$, $p(0,0,0)$

Where the functions $p(t,u(t),u(\delta(t)))=\sin t\left[\frac{u(t)}{1-u(t)}+e^{-t}\right]$,

$$q(t,u(t),u(\delta(t)))=\frac{1}{t^2(1+u(2t))}, r(t,u(t),u(\delta(t)))=\frac{1}{(1+ut))}, h(t)=\frac{1}{t^2} \; and \; P(t)=\frac{1}{\cos t} \; and \; \alpha=\frac{1}{2}, k=2$$

$(A_1)$. $Now\,|p(t,u(t),u(\delta(t)))-p(t,v(t),v(\delta(t)))|$ s

$$=\left|\left\{\sin t\left[\frac{u(t)}{1-u(t)}+e^{-t}\right]\right\}-\left\{\sin t\left[\frac{v(t)}{1-v(t)}+e^{-t}\right]\right\}\right|$$

$$=\left|\sin t\left[\frac{u(t)}{1-u(t)}-\frac{v(t)}{1-v(t)}\right]\right|.$$

$$\leq|\sin t|\left|\frac{u(t)v(t)+u(t)-v(t)-u(t)v(t)}{u(t)v(t)-u(t)-v(t)+1}\right|$$

$$\leq|\sin t||u(t)-v(t)|$$

$$\leq K(t)|u(t)-v(t)|$$

$$\leq\|K\||u(t)-v(t)|$$

Since $K(t)=\sin t$ say which is continuous and bounded on $R_+$ has bound ‖κ‖.

($A_2$).Take, $h(t)=\frac{1}{t^2}$, it is continuous on $R_+$.

Implies that $q(t,u(t),u(\delta(t)))+r(t,u(t),u(\delta(t)))\leq h(t)$ that is

Implies , caratheodory satisfy above condition.

($A_3$).Thefunction $\frac{u(t)}{p(t,u(t),u(\delta(t))}$ is again caratheodory function with continuous function P:$R_+\rightarrow$R such that $P(t)=\frac{1}{\cos t}$ and satisfying

$$\frac{u(t)}{p(t,u(t),u(\delta(t)))}\leq P(t).$$

It follows that all the conditions ($A_1$)−($A_3$) satisfied. Thus by theorem (2.3) above problem has a solution on $R_+$.

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