Restrained Isolate Edge Geodetic Number of a Graph

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Abstract

Here we study the new concept of restrained isolate edge geodetic set for a graph $G = (V\,(G), E(G))$ of order $n \geq 4$. The edge geodetic set $S \subseteq V$ of a connected graph G is said to be a restrained isolate edge geodetic set if the induced sub graph $< S >$ has at least one isolate vertex and the induced sub graph $< V(G) - S >$ has no isolate vertex. The restrained isolate edge geodetic number denoted by $g_{1ri}(G)$ is the minimum cardinality of a restrained isolate edge geodetic set of G. Here we determine the restrained isolate edge geodetic number of some standard graphs and by adding leaf vertices to a graph C_n. In addition, we determine the restrained isolate edge geodetic number for some graphs using cartesian product and corona product.

Keywords: Isolate edge geodetic number, restrained edge geodetic set, extreme vertex, cartesian product, corona product.

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1. Introduction

We consider only finite, connected and simple graph $G = (V\,(G), E(G))$. In a connected graph G, the distance between any two vertices u, v is the length of the shortest $u - v$ path and the

$u-v$ path of length $d(u,v)$ is called an $u-v$ geodesic. This concept was introduced in [2]. The closed interval $I[u,v]$ consists of all vertices lying on some $u-v$ geodesic of G and for a non empty subset $S \subseteq V(G)$, $I[S] = \cup_{u,v \in S} I[u,v]$. The set S of vertices is a geodetic set in G if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the geodetic number $g(G)$. The edge geodetic set $S \subseteq V(G)$, in which all edges of G is included in a geodesic connecting some pair of vertices in S and $g_1(G)$ is the edge geodetic number that represents the cardinality of minimum edge geodetic set. The vertex v is termed as an isolated vertex if $deg(v) = 0$ and the vertex v is called as the pendant vertex if $deg(v) = 1$.

The geodetic set S is said to be an isolate geodetic set in G, if the induced sub graph $< S >$ has at least one isolate vertex. The cardinality of an isolate geodetic set which is minimum is the isolate geodetic number and is denoted as $g_o(G)$. In [5], isolate geodetic number of a graph was introduced.

A set of vertices S in a connected graph G is a restrained edge geodetic set if S is an edge geodetic set and if either $V = S$ or the induced sub graph $< V(G) - S >$ has no isolated vertex. The minimum cardinality of a restrained edge geodetic set of G is the restrained geodetic number $eg_r(G)$. In [6], the edge geodetic number of a graph and [1,7] restrained edge geodetic number of a graph was introduced. The vertex v in the graph G is said to be an extreme if a sub graph induced by its neighbours is a complete graph. The vertex v is called as the full vertex or dominating vertex of a graph G if $deg(v) = n-1$. The vertex adjacent to the pendant vertex is called a support vertex. Any undefined terms may be found in [3, 4].

Theorem 1.1.[6] If G has exactly one vertex v of degree $\mathrm{p}-1$, then $\mathrm{g}_1(\mathrm{G}) = \mathrm{p}-1$.

Theorem 1.2.[6]Any graph G with at least two vertices of degree $\mathrm{p}-1$, $\mathrm{g}_1(\mathrm{G}) = \mathrm{p}$.

2. Restrained Isolate Edge Geodetic Number of a Graph

Definition 2.1. The edge geodetic set $\mathrm{S} \subseteq \mathrm{V}(\mathrm{G})$ in G is said to be a restrained isolate edge geodetic set if the induced sub graph $< \mathrm{S} >$ has at least one isolate vertex and the induced sub graph $< V(G) - S >$ has no isolate vertex. The restrained isolate edge geodetic number $g_{1ri}(G)$ is the cardinality of a minimum restrained isolate edge geodetic set in G.

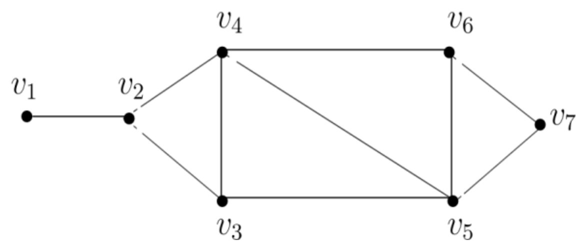

Figure 2.1

The connected graph G in Figure 2.1, $S = \{v_1, v_3, v_6, v_7\}$ is a minimum edge geodetic set with the induced sub graph $< S >$ has isolate vertices and $< V(G) - S >$ is connected has no isolate vertex. Thus the edge set S forms the restrained isolate edge geodetic set. Therefore $g_{1ri}(G) = 4$.

Observation 2.2. All graphs need not have a restrained isolate edge geodetic set. For example, complete graphs, complete bipartite graphs, wheel graphs have no restrained isolate edge geodetic set.

Theorem 2.3. There is no graph G of order n with restrained isolate edge geodetic number $g_{1ri}(G) = n - 1$.

Proof. Suppose that $g_{1ri}(G) = n - 1 = |S|$, the induced sub graph $< V(G) - S >$ contains an isolate vertex. Thus, there is no graph G of order n with $g_{1ri}(G) = n - 1$.

Theorem 2.4. If the graph G contains one or more full vertices, then there is no restrained isolate edge geodetic set in G.

Proof. Suppose the graph G contains one full vertex, then by Theorem 1.1, the edge geodetic set $g_1(G) = n - 1 = |S|$ and $< V(G) - S >$ contains an isolate vertex. If G has at least two full vertices, then by Theorem 1.2, the edge geodetic set $g_1(G) = n = |S|$. Clearly, the induced sub graph $< S >$ is connected has no isolate vertex and $< V(G) - S >$ is a order-zero graph K_0. Thus there is no restrained isolate edge geodetic set in G containing one or more full vertices.

Theorem 2.5. In a cycle $G = C_n$ with $n \geq 6$

$$g_{1ri}(C_n) = \begin{cases} 2, & \text{if n is even} \\ 3, & \text{if n is odd}. \end{cases}$$

Proof. Case 1: Suppose that n is even. The set $S = \{v_1, v_{r+1}\}$ of two antipodal vertices in a cycle $C_{2r}: v_1, v_2, \dots v_{2r}, v_1$ forms the edge geodetic set which is minimum with the induced sub graph $< S >$ has isolated vertices and $< V(C_{2r}) - S >$ has no isolate vertex. Thus set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(C_{2r}) = 2$.

Case 2: Suppose that n is odd. Let $S = \{v_1, v_{r+1}, v_{r+2}\}$ be a set of three vertices in a cycle $C_{2r+1}: v_1, v_2, v_{r+1}, v_{r+2}, \dots, v_{2r+1}, v_1$ forms an edge geodetic set with $< S >$ has isolate vertex and $< V(C_{2r+1}) - S >$ has no isolate vertex. Thus the set S forms a restrained isolate edge geodetic set. Therefore $g_{1ri}(C_{2r+1}) = 3$.

Theorem 2.6. The tree T containing n vertices with k pendant vertices also $\mathrm{n} - \mathrm{k} \geq 2$, the restrained isolate edge geodetic number $\mathrm{g}_{1\mathrm{ri}}(\mathrm{T}) = \mathrm{k}$.

Proof. The tree T of order n containing k pendant vertices with $n - k \geq 2$. The set S of k pendant vertices forms an edge geodetic set which is minimum with the induced sub graph $< S >$ has isolate vertices and $< V\ (T) - S >$ is connected has no isolate vertex. Thus the set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(T) = k$.

Theorem 2.7. In a graph G, $2 \leq g_{1ri}(G) \leq n - 2$.

Proof. Every restrained isolate edge geodetic set is an edge geodetic set and at least two vertices are required. So that $g_{1ri}(G) \geq 2$. Suppose that if $g_{1ri}(G) = n$, then the induced sub graph $< S >$ is connected has no isolate vertex and if $g_{1ri}(G) = n - 1$, then the induced sub graph $< V\ (G) - S >$ has isolate vertex. Further by Observation 2.2 and Theorem 2.3, there is no graph G of order n with restrained isolate edge geodetic number $g_{1ri}(G) = n$ and $g_{1ri}(G) = (n - 1)$ so that, $g_{1ri}(G) < (n - 1)$. Therefore $2 \leq g_{1ri}(G) \leq n - 2$.

3. Adding a leaf vertex to a graph

Theorem 3.1. The graph G' is formed by adding the leaf vertex v to a cycle $G = C_n$ where $n \geq 4$. Then $g_{1ri}(G') = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd.} \end{cases}$

Proof. The graph G' formed by adding the leaf vertex $v \notin G$ to any vertex in a cycle $G = C_n$ where $n \geq 4$.

Case 1: Suppose that *n* is even.

In G', the set $S = \{v, u\}$ forms the edge geodetic set which is minimum where $d(v, u) = diam(G')$ with the induced sub graph $< S >$ has isolate vertices, $< V(G') - S >$ is connected has no isolate vertex. Thus the set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(G') = 2$.

Case 2: Suppose that n is odd.

The edge geodetic set $S = \{v, u, w\}$ where v is a leaf vertex and u, w are any two adjacent vertices in G' such that $d(v, u) = d(v, w)$ with the induced sub graph $< S >$ has isolate vertex, $< V(G') - S >$ is connected has no isolate vertex. Thus S in G' forms restrained isolate edge geodetic set. Hence $g_{1ri}(G') = 3$.

Theorem 3.2. If the graph G' is formed by joining k-leaf vertices $v_i \notin G$ where $1 \leq i \leq k$ to any vertex in $G = C_n$ where C_n is a cycle with $n \geq 4$, then

$$g_{1ri}(G') = \begin{cases} k+1, & \text{if n is even} \\ k+2, & \text{if n is odd.} \end{cases}$$

Proof. The graph G' formed by adding k-leaf vertices $X = \{v_i\}$ where $1 \leq i \leq k$ to any vertex in $G = C_n$.

Case 1: Suppose that n is even. The set $S = X \cup \{u\}$ forms an edge geodetic set in G' where $d(v_i, u) = diam(G'), 1 \leq i \leq k$. The induced sub graph $< S >$ has isolate vertices, $< V(G') - S >$ is connected has no isolate vertex. Thus the set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(G') = k + 1$.

Case 2: Suppose that n is odd. The set $S = X \cup \{u, w\}$ forms an edge geodetic set containing two adjacent vertices u, w with $d(v_i, u) = d(v_i, w)$, $1 \leq i \leq k$. The induced sub graph $< S >$ has isolated vertices and $< V(G') - S >$ is connected has no isolate vertex. Thus the set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(G') = k + 2$.

Theorem 3.3. The graph G' is formed by adding a leaf vertex to each vertex in a cycle $C_n = G$ where $n \geq 3$. Then $g_{1ri}(G') = n$.

Proof. The graph G' formed by joining a leaf vertex to each vertex in a cycle $C_n = G$ where $n \geq 3$. The set X of n end vertices in an extreme geodesic graph $G' = C_n \circ K_1$ forms minimum restrained isolate edge geodetic set in G'. Therefore $g_{1ri}(G') = n$.

Theorem 3.4. The graph G' is formed by joining k-leaf vertices $v_i \notin G$ where $1 \leq i \leq k$ to each vertex in $G = C_n$ where C_n is a cycle of order $n \geq 3$. Then $g_{1ri}(G') = nk$.

Proof. The graph G' is formed by joining k-leaf vertices $X = \{v_i\}$ where $1 \leq i \leq k$ to each vertex in $G = C_n, n \geq 3$. The set X of nk pendant vertices forms minimum restrained isolate edge geodetic set. Therefore $g_{1ri}(G') = nk$.

4. The restrained isolate edge geodetic number and corona product.

Definition 4.1. A graph formed by taking a single copy of G with $|V(G)| = n$ copies of H. By joining i^{th} vertex of G with each vertex in the i^{th} copy of H, then it is said to be a corona product of two graphs G and H and it is denoted as $G \circ H$.

Theorem 4.2. If $G = T$ is a connected non trivial tree of order n, then $g_{1ri}(T \circ K_1) = n$.

Proof. Let T be a connected tree of order n. By the definition of corona graph, $(T \circ K_1)$ is a tree with $2n$ vertices and n pendant vertices. By Theorem 2.6, the restrained isolate edge geodetic set $g_{1ri}(T \circ K_1) = n$.

Theorem 4.3. If K_n is a complete graph of order n, then $g_{1ri}(K_n \circ K_1) = n$.

Proof. Let K_n be a complete graph of order n. By the definition of corona graph, $G = K_n \circ K_1$ is a connected graph of order $2n$ containing n pendant vertices. The set S of n pendant vertices forms an edge geodetic set which is minimum with the induced sub graph $< S >$ has isolated vertices, $< V(G) - S >$ is connected. Thus the set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(K_n \circ K_1) = n$.

Theorem 4.4. If the graph $K_{m,n}$ is a complete bipartite graph of order $(m + n)$, then $g_{1ri}(K_{m,n} \circ K_1) = m + n$.

Proof. Let $K_{m,n}$ be a complete bipartite graph of order $m + n$. By the definition of corona graph, $G = K_{m,n} \circ K_1$ is a connected graph of order $2(m+n)$. Let S be an edge geodetic set contains $m + n$ end vertices. It is clear that $< S >$ has isolated vertices and the induced sub graph $< V(G) - S >$ is connected complete bipartite graph has no isolate vertices. Thus the set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(K_{m,n} \circ K_1) = m + n$.

5. The Cartesian product and restrained isolate edge geodetic number

Definition 5.1. The Cartesian product of any two graphs G, H is a graph with vertex set $V(G) \times V(H) = (u_i, v_j)$where $u_i \in V(G)$, $v_j \in V(H), 1 \leq i \leq m$ and $1 \leq j \leq n$. Any two distinct vertices (u_i, v_j), (u_k, v_l) adjacent in $G \times H$ if and only if either $u_i = u_k$ and $v_j v_l \in E(H)$ or $v_j = v_l$ and $u_i u_k \in E(G)$.

Theorem 5.2. The restrained isolate edge geodetic number of the cartesian product of C_n, P_m is

$$g_{1ri}(C_n \times P_m) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd}. \end{cases}$$

Where C_n, $n > 3$ is a cycle and the path P_m, $m > 1$.

Proof. In C_n, $V = \{v_i\}, 1 \leq i \leq n$ is the vertex set and for P_m vertex set is $W = \{w_j\}, 1 \leq j \leq m$. By the definition of cartesian product, $G = (C_n \times P_m)$ formed from n-copies of P_m.

Case 1: Suppose that n is even.

Let $S = \{(v_i, w_m), (v_{i+\frac{n}{2}}, w_1)\}$ be the edge geodetic set, where $1 \leq i \leq \frac{n}{2}$ such that $diam(G) = d\{(v_i, w_m), (v_{i+\frac{n}{2}}, w_1)\}$ with the induced sub graph $< S >$ has isolate vertices and $< V(G) - S >$ is connected has no isolate vertex. Thus the set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(G) = 2$.

Case 2: Suppose that n is odd.

Let $S = \left\{(v_1, w_1), (v_{\lceil \frac{n}{2} \rceil}, w_m\right), \left(v_{\lceil \frac{n}{2} \rceil+1}, w_m\right), \left(v_{\lceil \frac{n}{2} \rceil+1}, w_1\right)\}$ be an edge geodetic set. The induced subgraph $< S >$ has isolated vertices and $< V(G) - S >$ is connected has no isolate vertex. Thus the set S forms restrained isolate edge geodetic set. Therefore $g_{1ri}(G) = 4$.

Theorem 5.3. If P_m and P_n are the paths with $m, n > 2$, then $g_{1ri}(P_m \times P_n) = 2$.

Proof. By the definition of graph product, $G = (P_m \times P_n)$ is obtained from m copies of P_n. The two vertex sets $V = \{v_1, v_2, \dots, v_m\}$ and $W = \{w_1, w_2, \dots, w_n\}$ of P_m and P_n respectively. The set containing any two antipodal vertices in a grid graph G forms restrained isolate edge geodetic set. Therefore $g_{1ri}(G) = 2$.

6.Conclusion

In this paper, we obtained results on the restrained isolate edge geodetic number of several special graphs also some general properties of cartesian product of two graphs.

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