Research Article

EXISTENCE OF BEST PROXIMITY POINTS ON GEOMETRICAL PROPERTIES OF PROXIMAL SETS

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ABSTRACT : The notion of proximal intersection property and UC property is used to establish the existence of the best proximity point for mappings satisfying contractive conditions.

Keywords: Best Proximity point, Proximal sets, UC property, proximal intersection property.

1. Introduction and Preliminaries:

Let X be a nonempty set and T be a self map of X. An element $x \in X$ is called a fixed point of T if $Tx = x$. Fixed point theorems deal with sufficient conditions on X and T ensures the existence of fixed points. Suppose the fixed point equation $Tx = x$ does not posses a solution, then the natural interest to find an element$x \in X$, such that x is in proximity to Tx in some cases.

In other words we would like to get a desirable estimate for the quality $d(x, Tx)$.

It is natural that some mapping, especially non-self mappings defined on a metric space (X, d), do not necessarily possess a fixed point that $d(x, Tx) > 0$ for all $x \in X$. In such situations, it is reasonable to search for the existence and uniqueness of the point $x \in X$ such that $d(x, Tx) = 0$.

In other words, one intends to determine an approximate solution $x \in X$ that is optimal in the sense that the distance between x and Tx is minimum. Here the point x is the proximity point. That is $d(x, Tx) = d(A, B)$ where $d(A, B) = \inf\{d(x, y): x \in A, y \in B\}$.

In Suzuki et al [1], UC property was introduced to prove some existence results on best proximity point. In Raj and Eldred [2], the author introduced p −property and proved strict convexity is equivalent to p −property.

We use proximal intersection property for a pair (A, B) where A and B are non empty closed subsets of a metric space. Then this property is used to prove the existence of the best proximity point for mapping satisfying some contractive conditions introduced by Wong [3].

In this section, we use some basic definitions and concepts that are related to the context of our main results.

Definition:1.1 [4] Let A and B be nonempty subsets of a metric space (X, d). Then, the pair (A, B) is said to satisfy the property UC if the following holds: If x_n and x_n' are sequence in A and y_n is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n\to\infty} d(x_n', y_n) = d(A, B)$ then $\lim_{n\to\infty} d(x_n, x_n') = 0$ holds.

Definition:1.2 Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy proximal intersection property if whenever $A_n \subset A$ and $B_n \subset B$ are a decreasing sequence of closed subsets such that $\delta(A, B) \to d(A, B)$, then $\bigcap A_n = \{x\}, \bigcap B_n = \{y\}$ with $d(x, y) = d(A, B)$.

Remark:1.1 $d(A, B) \to d(\bar{A}, \bar{B})$ and $\delta(A, B) \to d(\bar{A}, \bar{B})$ where $\delta(A, B) = Sup\{||x - y||: x \in A, y \in B\}$.

Definition:1.3 [2] Let X be a metric space and let $T: X \to X$. Then d_T is the function on $X \times X$ defined by

$d_T(x, y) = \inf\{d(T_x^n, T_y^n): n \geq 1, x, y \in X\}$…………………………………………….……….(1)

Definition:1.4 [3] Let A and B be nonempty subsets of a metric space X. We shall use X_d to denote the set

$\{r'$: for some $s > r', d(x, y) - d(A, B) \in [r', s]$ for some $x \in A, y \in B\}$……………...……(2)

Remark:1.2 If $r' \in X_d$, then there exists $x_n \in A, y_n \in B$ such that $d(x_n, y_n) - d(A, B) \to r'$. Also if $x \in A, y \in B$, then $d(x_n, y_n) - d(A, B) \in X_d$ and if $x_n \in A, y_n \in B$ such that $d(x_n, y_n) - d(A, B) \to r'$, then $r' \in X_d$.

Lemma:1.1 [1] Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) has the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequence in A and B respectively such that either of the following holds:

$\lim_{m\to\infty} Sup_{n\geq m} d(x_m, y_n) = d(A, B)$ or

$\lim_{n\to\infty} Sup_{m\geq n} d(x_m, y_n) = d(A, B)$

Then $\{x_n\}$ is Cauchy.

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2.Results:

Theorem:2.1 Let A and B be nonempty closed subsets of a complete metric space X satisfying UC property. Let A_n, B_n be decreasing sequence of nonempty closed subsets of X such that $\delta(A_n, B_n) \to d(A,B)$ as $n \to \infty$ then $\bigcap A_n = \{x\}, \bigcap B_n = \{y\}$ with $d(A,B)$ that is (A,B) satisfies proximal intersection property.

Proof: Construct a sequence x_n, y_n in X by selecting $x_n \in A_n, y_n \in B_n$ for each $n \in N$.
Since $A_{n+1} \subseteq A_n, B_{n+1} \subseteq B_n$ for all n, we have $x_n \in A_n \subseteq A_m$, $y_n \in B_n \subseteq B_m$ for all $n > m$.
We claim that x_n is a Cauchy sequence.
Let $\varepsilon > 0$ be given.
Since $\delta(A_n, B_n) \to d(A,B)$, there exists a positive integer N such that $\delta(A_n, B_n) < d(A,B) + \varepsilon$, for all $n \geq N$.
Since A_n, B_n are decreasing sequence, we have $A_n, A_m \subseteq A_N$ and $B_n, B_m \subseteq B_N$ for all $m, n \geq N$.
therefore $x_n, x_m \in A_N$ and $y_n, y_m \subseteq B_N$ for all $m, n \geq N$,
and there we have
$d(x_n, x_m) \leq \delta(A_n, B_n) < d(A,B) + \varepsilon$, for all $m, n \geq N$……………………………………(3)
since A and B satisfy UC property from lemma 1.1, x_n is a cauchy sequence. There exists $x \in A$ such that $x_n \to x$.
similarly there exists $y \in B$ such that $y_n \to y$
we claim that $x \in \bigcap A_n, y \in \bigcap B_n$,
since A_n and B_n are closed for each n,
$x \in A_n, y \in B_n$ for all $n \in N$
since $d(x_n, y_n) \to d(A,B)$ we have
$d(x,y) = d(A,B)$
finally to establish that x is the only point in $\bigcap A_n$, if $x_1 \neq x_2 \in \bigcap A_n$, then $d(x,y) = d(A.B)$ UC property forces that $x_1 = x_2$, similarly $\bigcap B_n = \{y\}$.

Lemma:2.1

(i) Let A and B be nonempty closed subsets of a complete metric space X such that (A,B) satisfying UC property. Let $T: A \cup B \to A \cup B$ be continuous, suppose that $T(A) \subset B, T(B) \subset A$ be a continuous function such that $\inf\{d(x,Tx): x \in A\} = d(A,B) = \inf\{d(x,Tx): x \in A\} = d(A,B)$

(ii) There exists $\delta_n > 0$ such that $\mathrm{d(Tx,Ty)} - d(A,B) < \frac{1}{n}$ whenever $\max\{\mathrm{d(x,Tx)} - d(A,B), d(y,Ty) - d(A,B)\} < \delta_n$ and $x \in A', y \in B'$ where $A'$and B' are any closed bounded sets of A and B respectively.

Then, there exists a best proximity point $x \in A$, such that $d(x,Tx) = d(A,B)$, Further, if $d(Tx,Ty) = d(x,y)$ for all $x \in A, y \in B$ then the best proximity point is unique.

Proof: Let $A_n = \left\{x \in A: d(x,Tx) - d(A,B) \leq \frac{1}{n}\right\}$
$B_n = \left\{y \in B: d(y,Ty) - d(A,B) \leq \frac{1}{n}\right\}$ since T is continuous, A_n, B_n are closed
from (i) A_n and B_n are nonempty
there exists N for all $n \in N$
let $x \in A_n$, $y \in B_n$
then $d(x,Tx) - d(A,B) < \delta_n$ and
$d(y,Ty) - d(A,B) < \delta_n$
from (ii) $d(Tx,Ty) - d(A,B) \leq \frac{1}{n}$ where $\delta_n \to 0$
for any $x \in A_n$, $y \in B_n$
$d(Tx,Ty) - d(A,B) \leq \frac{1}{n}$
which implies $\delta(T(A_n), T(B_n)) \to d(A,B)$
and hence $\delta(\overline{T(A_n)}, \overline{T(B_n)}) \to d(A,B)$
By proximal intersection property,
we have $\bigcap_{n\geq 1} \overline{T(A_n)} = y$ and $\bigcap_{n\geq 1} \overline{T(B_n)} = x$ and
$d(x,y) = d(A,B)$
Thus for each $n \geq 1$, there exists $x_n \in A_n$
such that $d(y, Tx_n) < \frac{1}{n}$
since $d(x_n, Tx_n) \to d(A,B)$ and
$d(y_n, Ty_n) \to d(A,B)$
By UC property $x_n \to x$
Since A_n is closed, $x \in A_n$ for each n
This implies that $d(x,Tx) \to d(A,B)$
Similarly $y_n \to y$ such that $d(y,Ty) \to d(A,B)$
To prove uniqueness,

$d(x,Tx) = d(A,B)$
Since T is non expansive $d(T^2x',Tx') = d(A,B)$ which implies that $T^2x' = x'$ as
$d(x,Tx) = d(Tx',T^2x') = d(A,B)$
from (ii) $d(Tx,x') = d(Tx,T^2x') = d(A,B)$
which implies that $x = x'$

Theorem:2.2 Let A and B be nonempty closed subsets of a metric space X and let Let $T: A \cup B \to A \cup B$ be continuous, such that $T(A) \subset B, T(B) \subset A$. Suppose that there exists $\phi: X_d \to [0,\infty)$ such that $d(x,y) - d(A,B) \leq \phi((x,y) - d(A,B))$ for all $x \in A, y \in B$ and $sup_{\delta > r} inf_{t\in[r,s]}\big(t - \phi(t)\big) > 0$ for $r \in X_d - \{0\}$. Then $d_T(x,y) = d(A,B)$ for all $x \in A$, $y \in B$ hence $inf\{d(x,Tx): x \in A\} = d(A,B)$

Proof: Suppose to the contrary that there exists $x \in A, y \in B$ such that
$Inf\{d(T^nx,T^ny): n \geq 1\} > d(A,B)$……………………………………………………….……(4)
by hypothesis there exists $s \in (r',\infty)$ such that $u = inf_{t\in[r',s]}\big(t - \phi(t)\big) > 0$ where $r' = r - d(A,B)$
since there exists a sequence
$d(T^nx,T^ny) - d(A,B) \to r'$ where $r' \in X_d - \{0\}$
Then from (2) we have
$d(T^nx,T^ny) - d(A,B) \to r' + t < s$ for some $n \geq 1$.
Since $d(T^nx,T^ny) - d(A,B) \in [r',s]$
$u \leq d(T^nx,T^ny) - d(A,B) - \phi\big(d(T^nx,T^ny) - d(A,B)\big)$
$\phi(d(T^nx,T^ny) - d(A,B)) \leq d(T^nx,T^ny) - d(A,B) - u$ ………………………………(5)
If $T^nx \in A, T^ny \in B$ and vice versa
It follows that
$d_T(x,y) - d(A,B) \leq d_T(T^nx,T^ny) - d(A,B)$…………………………………...……..(6)
$\leq d(T^nx,T^ny) - d(A,B)$……………………………………………………...………(7)
$\leq \phi(d(T^nx,T^ny) - d(A,B))$…………………………………………………….…(8)
$\leq d(T^nx,T^ny) - d(A,B)$from (5)…………………………………………….……(9)
$< r' + t - u$……………………………………………………………......……….(10)
Letting $t \to 0$, we have
$d_T(x,y) - d(A,B) \leq r' - u$ …………………………………………………………....(11)
$d_T(x,y) - d(A,B) \leq r' - d(A,B) - u$ ………………………………………………..(12)
$d_T(x,y) \leq r - u$ a contradiction.

Theorem:2.3 Let A and B be nonempty closed subsets of a metric space X. Suppose (A,B) satisfies UC property. Let T be as in theorem 2.2 then T satisfies all the conditions of lemma 2.1 and therefore T has a unique best proximity point.

Proof: Clearly from theorem 2.2 and (i)2.1 of lemma are satisfied.
To prove (ii) of lemma 2.1 assume $x_n \in A$, and $y_n \in B$ are bounded sequences, then $d(x_n,Tx_n)$ and $d(y_n,Ty_n) \to d(A,B)$ where x_n and y_n are sequences in A and B resoectively.
suppose $d(x_n,Tx_n) - d(A,B) \to 0$
since x_n, y_n are bounded sequence, there exists subsequence n_k and $r > 0$ such that $d(Tx_{n_k},Ty_{n_k}) - d(A,B) \to r > 0$
clearly $r \in X_d$
let $r_{n_k} = d(Tx_{n_k},Ty_{n_k}) - d(A,B)$ and
$s_{n_k} = d(x_{n_k},y_{n_k}) - d(A,B)$
given $r_{n_k} - s_{n_k} \to 0$ as $k \to \infty$
$d(Tx_{n_k},Ty_{n_k}) - d(A,B) \leq d(Tx_{n_k},Ty_{n_k}) - d(A,B)$
therefore$r_{n_k} \leq \phi(s_{n_k})$……………………………………………………………...(13)
now from (13) we have
$0 > \phi(s_{n_k}) - s_{n_k}$
$= \phi(s_{n_k}) - r_{n_k} + r_{n_k} - s_{n_k}$
$\geq r_{n_k} - s_{n_k}$
since $r_{n_k} - s_{n_k} \to 0$ we have
$\lim inff(\phi(s_{n_k}) - s_{n_k}) = 0$
contradicting $inf_{t\in[r_0,s]}\big(t - \phi(t)\big) > 0$ where $s_{n_k} \to r_0$.
This completes the proof.

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