EXISTENCE OF BEST PROXIMITY POINTS ON GEOMETRICAL PROPERTIES OF PROXIMAL SETS

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ABSTRACT: The notion of proximal intersection property and UC property is used to establish the existence of the best proximity point for mappings satisfying contractive conditions.

Keywords: Best Proximity point, Proximal sets, UC property, proximal intersection property.

1. Introduction and Preliminaries:
Let X be a nonempty set and T be a self map of X. An element x ∈ X is called a fixed point of T if Tx = x. Fixed point theorems deal with sufficient conditions on X and T ensures the existence of fixed points. Suppose the fixed point equation Tx = x does not possess a solution, then the natural interest to find an element x ∈ X, such that x is in proximity to Tx in some cases.

In other words we would like to get a desirable estimate for the quality d(x, Tx).

It is natural that some mapping, especially non-self mappings defined on a metric space (X, d), do not necessarily possess a fixed point that d(x, Tx) > 0 for all x ∈ X. In such situations, it is reasonable to search for the existence and uniqueness of the point x ∈ X such that d(x, Tx) = 0.

In other words, one intends to determine an approximate solution x ∈ X that is optimal in the sense that the distance between x and Tx is minimum. Here the point x is the proximity point. That is d(x, Tx) = d(A, B) where d(A, B) = inf{d(x, y): x ∈ A, y ∈ B}.

In Suzuki et al [1], UC property was introduced to prove some existence results on best proximity point. In Raj and Eldred [2], the author introduced p − property and proved strict convexity is equivalent to p − property.

We use proximal intersection property for a pair (A, B) where A and B are non empty closed subsets of a metric space. Then this property is used to prove the existence of the best proximity point for mapping satisfying some contractive conditions introduced by Wong [3].

In this section, we use some basic definitions and concepts that are related to the context of our main results.

Definition 1.1 [4] Let A and B be nonempty subsets of a metric space (X, d). Then, the pair (A, B) is said to satisfy the property UC if the following holds: If x_n and x'_n are sequence in A and y_n is a sequence in B such that lim d(x_n, y_n) = d(A, B) and lim d(x'_n, y_n) = d(A, B) then lim d(x'_n, x_n) = 0 holds.

Definition 1.2 Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy proximal intersection property if whenever A_n ⊂ A and B_n ⊂ B are a decreasing sequence of closed subsets such that d(A_n, B_n) → d(A, B), then ∩ A_n = {x}, ∩ B_n = {y} with d(x, y) = d(A, B).

Remark 1.1 d(A, B) → d(Ā, B̄) and δ(A, B) → d(Ā, B̄) where δ(A, B) = Sup{||x − y||: x ∈ A, y ∈ B}.

Definition 1.3 [2] Let X be a metric space and let T: X → X. Then d_T is the function on X × X defined by
d_T(x, y) = inf{d(T^n_x, T^n_y): n ≥ 1, x, y ∈ X}………………………………………………(1)

Definition 1.4 [3] Let A and B be nonempty subsets of a metric space X. We shall use X_d to denote the set {x': for some s > r', d(x, y) − d(A, B) ∈ [r', s] for some x ∈ A, y ∈ B}……………………………………..(2)

Remark 1.2 If r' ∈ X_d, then there exists x_n ∈ A, y_n ∈ B such that d(x_n, y_n) − d(A, B) → r'. Also if x ∈ A, y ∈ B, then d(x_n, y_n) − d(A, B) ∈ X_d if and only if x_n ∈ A, y_n ∈ B such that d(x_n, y_n) − d(A, B) → r', then r' ∈ X_d.

Lemma 1.1 [1] Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) has the property UC. Let {x_n} and {y_n} be sequence in A and B respectively such that either of the following holds:

lim_{n→∞} sup_n d(x_m, y_n) = d(A, B) or
lim_{n→∞} sup_n d(x_m, y_n) = d(A, B)

Then {x_n} is Cauchy.
2. Results:

**Theorem 2.1** Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ satisfying UC property. Let $A_n, B_n$ be decreasing sequence of nonempty closed subsets of $X$ such that $\delta(A_n, B_n) \to d(A, B)$ as $n \to \infty$ then $\bigcap A_n = \{x\}, \bigcap B_n = \{y\}$ with $d(A, B)$ that is $(A, B)$ satisfies proximal intersection property.

**Proof:** Construct a sequence $x_n, y_n$ in $X$ by selecting $x_n \in A_n, y_n \in B_n$ for each $n \in N$. Since $A_{n+1} \subseteq A_n, B_{n+1} \subseteq B_n$ for all $n$, we have $x_n \in A_n \subseteq A_m, y_n \in B_n \subseteq B_m$ for all $n > m$. We claim that $x_n$ is a Cauchy sequence. Let $\varepsilon > 0$ be given.

Since $\delta(A_n, B_n) \to d(A, B)$, there exists a positive integer $N$ such that $\delta(A_n, B_n) < d(A, B) + \varepsilon$, for all $n \geq N$. Since $A_n, B_n$ are decreasing sequence, we have $A_n, A_N \subseteq A_N$ and $B_n, B_m \subseteq B_N$ for all $m, n \geq N$.

Therefore $x_n, x_m \in A_N$ and $y_n, y_m \in B_N$ for all $m, n \geq N$, and there we have

$$d(x_n, x_m) \leq \delta(A_n, B_n) < d(A, B) + \varepsilon,$$

for all $m, n \geq N$.

(3) since $A$ and $B$ satisfy UC property from lemma 1.1, $x_n$ is a cauchy sequence. There exists $x \in A$ such that $x_n \to x$.

Similarly there exists $y \in B$ such that $y_n \to y$

we claim that $x \in \bigcap A_n, y \in \bigcap B_n$.

since $A_n, B_n$ are closed for each $n$, $x \in A_n, y \in B_n$ for all $n \in N$

since $d(x_n, y_n) \to d(A, B)$ we have

$$d(x, y) = d(A, B)$$

finally to establish that $x$ is the only point in $\bigcap A_n$, if $x_1 \neq x_2 \in \bigcap A_n$, then $d(x, y) = d(A, B)$ UC property forces that $x_1 = x_2$, similarly $\bigcap B_n = \{y\}$.

**Lemma 2.1**

(i) Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $(A, B)$ satisfying UC property. Let $T: A \cup B \to A \cup B$ be continuous, suppose that $T(A) \subseteq B, T(B) \subseteq A$ be a continuous function such that $\inf d(x, Tx) : x \in A = d(A, B) = \inf d(x, Tx) : x \in A = d(A, B)$

(ii) There exists $\delta_n > 0$ such that $d(Tx, Ty) - d(A, B) < \frac{1}{n}$ whenever $\max(d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)) < \delta_n$ and $x \in A', y \in B'$ where $A'$ and $B'$ are any closed bounded sets of $A$ and $B$ respectively.

Then, there exists a best proximity point $x \in A$, such that $d(x, Tx) = d(A, B)$. Further, if $d(Tx, Ty) = d(x, y)$ for all $x \in A, y \in B$ then the best proximity point is unique.

**Proof:** Let $A_n = \{x \in A : d(x, Tx) - d(A, B) \leq \frac{1}{n}\}$. $B_n = \{y \in B : d(y, Ty) - d(A, B) \leq \frac{1}{n}\}$ since $T$ is continuous, $A_n, B_n$ are closed from (i) $A_n$ and $B_n$ are nonempty there exists $N$ for all $n \in N$

let $x \in A_n, y \in B_n$

then $d(x, Tx) - d(A, B) < \delta_n$ and $d(y, Ty) - d(A, B) < \delta_n$

from (ii) $d(Tx, Ty) - d(A, B) \leq \frac{1}{n}$ where $\delta_n \to 0$

for any $x \in A_n, y \in B_n$

$d(Tx, Ty) - d(A, B) \leq \frac{1}{n}$

which implies $\delta(T(A_n), T(B_n)) \to d(A, B)$

and hence $\delta(T(A_n), T(B_n)) \to d(A, B)$

By proximal intersection property,

we have $\bigcap_{n \geq 1} T(A_n) = y$ and $\bigcap_{n \geq 1} T(B_n) = x$ and $d(x, y) = d(A, B)$

Thus for each $n \geq 1$, there exists $x_n \in A_n$

such that $d(y, Tx_n) < \frac{1}{n}$

since $d(x_n, Tx_n) \to d(A, B)$ and $d(y_n, Ty_n) \to d(A, B)$

By UC property $x_n \to x$

Since $A_n$ is closed, $x \in A_n$ for each $n$

This implies that $d(x, Tx) \to d(A, B)$

Similarly $y_n \to y$ such that $d(y, Ty) \to d(A, B)$

To prove uniqueness,
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\[ d(x, Tx) = d(A, B) \]
Since \( T \) is non expansive \( d(T^2 x', Tx') = d(A, B) \) which implies that \( T^2 x' = x' \) as
\[ d(x, Tx) = d(Tx', T^2 x') = d(A, B) \]
from (ii) \( d(Tx, x') = d(Tx, T^2 x') = d(A, B) \)
which implies that \( x = x' \)

**Theorem 2.2** Let \( A \) and \( B \) be nonempty closed subsets of a metric space \( X \) and let \( \{x_n\}_{n=1}^{\infty} \in A \cup B \) be continuous, such that \( T(A) \subset B, T(B) \subset A \). Suppose that there exists \( \phi: X_d \rightarrow [0, \infty) \) such that
\[ d(x, y) = d(A, B) \leq \phi(x, y) - d(A, B) \]
for all \( x \in A, y \in B \) and \( \sup_{x, y \in [r, s]} \phi(t - \phi(t)) > 0 \) for \( r \in X_d - \{0\} \). Then \( d(x, y) = d(A, B) \) for all \( x \in A, y \in B \) hence \( \phi(x, y) = d(A, B) \)

**Proof:** Suppose to the contrary that there exists \( x \in A, y \in B \) such that
\[ \phi(x, y) - d(A, B) > 0 \]
Then there exists \( s \in (r', \infty) \) such that \( u = \inf_{x, y \in [r', s]} (t - \phi(t)) > 0 \) where \( r' = r - d(A, B) \)
since there exists a sequence
\[ d(T^n x, T^n y) - d(A, B) \rightarrow r' \]
where \( r' \in X_d - \{0\} \)
Then from (2) we have
\[ d(T^n x, T^n y) - d(A, B) \rightarrow r' + t < s \text{ for some } n \geq 1. \]
Since \( d(T^n x, T^n y) - d(A, B) \in [r', s] \)
\[ u \leq d(T^n x, T^n y) - d(A, B) - \phi(d(T^n x, T^n y) - d(A, B)) \]
\[ \phi(d(T^n x, T^n y) - d(A, B)) \leq d(T^n x, T^n y) - d(A, B) - u \]
(5)

It follows that
\[ d_T(x, y) - d(A, B) \leq d_T(x, y) - d(A, B) \]
(6)
\[ \leq d(T^n x, T^n y) - d(A, B) \]
(7)
\[ \leq \phi(d(T^n x, T^n y) - d(A, B)) \]
(8)
\[ \leq d(T^n x, T^n y) - d(A, B) \]
from (5)
(9)
\[ \xi r' + t - u \]
(10)

Letting \( t=0 \), we have
\[ d_T(x, y) - d(A, B) \leq r' - u \]
(11)
\[ d_T(x, y) - d(A, B) \leq r' - d(A, B) - u \]
(12)
\[ d_T(x, y) \leq r - u \]
(13)

**Theorem 2.3** Let \( A \) and \( B \) be nonempty closed subsets of a metric space \( X \). Suppose \( (A, B) \) satisfies UC property. Let \( T \) be as in theorem 2.2 then \( T \) satisfies all the conditions of lemma 2.1 and therefore \( T \) has a unique best proximity point.

**Proof:** Clearly from theorem 2.2 and (i)2 of lemma are satisfied.

To prove (ii) of lemma 2.1 assume \( x_n \in A \) and \( y_n \in B \) bounded sequences, then \( d(x_n, Tx_n) \) and \( d(x_n, Ty_n) \)
\[ d(x_n, Tx_n) \rightarrow d(A, B) \]
where \( x_n \) and \( y_n \) are sequences in \( A \) and \( B \) respectively.

suppose \( d(x_n, Tx_n) \rightarrow d(A, B) \rightarrow 0 \)
with \( x_n, y_n \) bounded sequence, there exists subsequence \( n_k \) and \( r > 0 \) such that \( d(Tx_{n_k}, Ty_{n_k}) - d(A, B) \rightarrow r > 0 \)

clearly \( r \in X_d \)
let \( r_{n_k} = d(Tx_{n_k}, Ty_{n_k}) - d(A, B) \) and
\[ s_{n_k} = d(x_{n_k}, y_{n_k}) - d(A, B) \]
given \( r_{n_k} = s_{n_k} \rightarrow 0 \) as \( k \rightarrow \infty \)
\[ d(Tx_{n_k}, Ty_{n_k}) - d(A, B) \leq d(Tx_{n_k}, Ty_{n_k}) - d(A, B) \]
therefore \( r_{n_k} \leq \phi(s_{n_k}) \)
(13)
now from (13) we have
\[ 0 > \phi(s_{n_k}) - s_{n_k} \]
\[ = \phi(s_{n_k}) - r_{n_k} + r_{n_k} - s_{n_k} \]
\[ \geq r_{n_k} - s_{n_k} \]
since \( r_{n_k} - s_{n_k} \rightarrow 0 \) we have
\[ \liminf \phi(s_{n_k}) - s_{n_k} = 0 \]
contradicting \( inf_{(t, s, |r|)} (t - \phi(t)) > 0 \) where \( s_{n_k} \rightarrow r_0 \). This completes the proof.

**References:**