Research Article

BEST PROXIMITY POINT THOREM FOR GENERALIZED (ψ,)-WEAK CONTRACTIONS IN BRANCIARI TYPE GENERALIZED METRIC SPACES

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ABSTRACT : In this paper, we establish a new convergence theorem for best proximity of weak contractions in Branciari type generalized metric spaces under weak conditions.

Keywords: Branciari type generalized metric spaces, Best Proximity point, p −property

1. Introduction and Preliminaries:

The concept of generalized metric spaces, which is a generalization of netric spaces was first defined by Branciari [1] in 2000. The generalization is via the fact that the triangle inequality is replaced by rectangular inequality $d(x,y) \le d(x,u) + d(u,v) + d(v,y)$ for all pairwise distinct points $x,y,u,v \in X$.
Afterwards many authors studied and extended the existence of fixed point results in such spaces [1-18].
In thus paper, we are to generalize well known best proximity point theorems.
For this we recall some basic definitions.

Definition:1.1 Let X be nonempty set and $d: X \times X \to [0,\infty)$ be a mapping such that for all $x,y \in X$ and for all distinct points $u,v \in X$ each of them different from x and y resoectively satisfying the following conditions:

(i) $d(x,y) = 0$ iff $x = y$
(ii) $d(x,y) = d(y,x)$
(iii) $d(x,y) \le d(x,u) + d(u,v) + d(v,y)$ the rectangular inequality.

Then (X,d) is called a Branciari type generalized metric space.
Remark:1.1 Every metric spaces is a Branciari type generalized metric space, but the converse is not true [2].

Definition:1.2 Let (X,d) be a Branciari type generalized metric space and $\{x_n\}$ be a sequence in X and $x \in X$. We call that

(i) $\{x_n\}$ is convergent iff $d(x_n,x) \to 0$ as $n \to \infty$ (denoted by $x_n \to x$)
(ii) $\{x_n\}$ is a Cauchy sequence iff for each $\varepsilon > 0$ there exists a natural number N such that $d(x_n,x_m) < \varepsilon$ for all $n,m > N$.
(iii) X is complete iff every Cauchy sequence is convergent in X.

In 2012, Lakzian and Samet [4] obtained a fixed point theorem of the generalized metric spaces.

Theorem:1.1 Let (X,d) be a Hausdorff and complete generalized metric space and Let $T: X \to X$ be a self mapping satisfying $\psi\big(d(Tx,Ty)\big) \le \psi\big(d(x,y)\big) - \phi(d(x,y))$ for all $x,y \in X$ where

(i) $\psi: [0,\infty) \to [0,\infty)$ is a continuous and monotone nondecreasing function with $\psi(t) = 0$ iff $t = 0$.
(ii) $\phi: [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$.

Then T has a unique fixed point.
In 2013, Liu and Chai [8] gave a generalization of the above theorem.

Theorem:1.2 [5] Let (X,d) be a Hausdorff and complete generalized metric space and Let $T: X \to X$ be a self mapping satisfying $\psi\big(d(Tx,Ty)\big) \le \psi\big(a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty)\big) - \phi(a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty))$ for all $x,y \in X$ where

(i) $\psi: [0,\infty) \to [0,\infty)$ is a continuous and monotone nondecreasing function with $\psi(t) = 0$ iff $t = 0$.
(ii) $\phi: [0,\infty) \to [0,\infty)$ satisfying $\lim_{t\to r} \phi(t) > 0$ for $r > 0$ and $\lim_{t\to r} \phi(t) = 0$ iff $r = 0$

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(iii) $a_i \geq 0 (i = 1,2,3)$ with $a_1 + a_2 + a_3 \leq 1$.
Then T has a unique fixed point.
Later in 2021, [22] Zhicum and Guiwen found a result for generalized $(\psi - \phi)$-weak contractions in Branciari type generalized metric spaces.

Theorem: 1.3 [22] Let Let (X, d) be a Branciari type generalized metric space and Let $T: X \to X$ be a self mapping satisfying $\psi(d(Tx, Ty)) \leq \psi(a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)) - \phi(a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty))$ for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$ and $a_i \geq 0 (i = 1,2,3)$ with $a_1 + a_2 + a_3 \leq 1$.
Then T has a unique fixed point.

Definition:1.3 [21] $A_0 = \{x \in A: d(x, y) = d(A, B)$, for $y \in B$
$B_0 = \{y \in B: d(x, y) = d(A, B)$, for $x \in A$
where $d(A, B) = inf\{d(x, y): x \in A, y \in B\}$

Definition: 1.4[20] Let (A, B) be a pair of nonempty subsets of metric space (X, d) with $A_0 \neq 0$. Then the pair (A, B) is said to have p −property iff for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0, d(x_1, y_1) = d(A, B) = d(x_1, y_2)$
Remark: 1.2 [20] It is easy to that for any nonempty subsets A of X, the pair (A, A) has the p −property.

2 Main Results:

Theorem: 2.1 Let (A, B) be a pair of nonempty subsets of a metric space such that A_0 is nonempty. Let $T: A \to B$ be a mapping satisfying $T(A_0) \subset B_0$. Suppose $\psi(d(Tx, Ty)) \leq \psi((a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)) - d(A, B)) - \phi((a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)) - d(A, B))$ for all $x \in A, y \in B$ where $\psi \in \Psi$ and $\phi \in \Phi$ and $a_i \geq 0 (i = 1,2,3)$ with $a_1 + a_2 + a_3 \leq 1$.
Proof: Choose $x_0 \in A$.
Since $Tx_0 \in T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$.
Analogously, regarding the assumption, $Tx_1 \in T(A_0) \subseteq B_0$,
we determine $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$.
Recursively, we obtain a sequence (x_n) in A_0 satisfying $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in$………..(2)
Claim: $d(x_n, x_{n+1}) \to 0$
If $x_N = x_{N+1}$, then x_N is a best proximity point.
By the p-property, we have
$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$
Hence we assume that $x_n \neq x_{n+1}$ for all $n \in N$.
Since $d(x_{n+1}, Tx_n) = d(A, B)$, from (1), we have for all $n \in N$.
$\psi(d(x_{n+1}, x_{n+2})) = \psi(d(Tx_n, Tx_{n+1}))$
$\leq \psi((a_1 d(x_n, x_{n+1}) + a_2 d(x_n, Tx_n) + a_3 d(x_{n+1}, Tx_{n+1})) - d(A, B)) - \phi((a_1 d(x_n, x_{n+1}) + a_2 d(x_n, Tx_n) + a_3 d(x_{n+1}, Tx_{n+1})) - d(A, B))$…….....................…(3) $= \psi((a_1 + a_2 + a_3) d(x_n, Tx_n) - d(A, B)) - \phi((a_1 + a_2 + a_3) d(x_n, Tx_n) - d(A, B))$
$\leq \psi((d(x_n, Tx_n) - d(A, B)) - \phi((a_1 + a_2 + a_3) d(x_n, Tx_n) - d(A, B))$
ie $\phi((a_1 + a_2 + a_3) d(x_n, Tx_n)) = d(A, B)$ if $\sum_{i=1}^{3} a_i \neq 0$, we get $d(x_n, x_{n+1}) = 0$ a contradiction.
If $\sum_{i=1}^{3} a_i = 0$ we get from (3) that $\psi(d(x_n, x_{n+1})) = 0$
$d(x_n, x_{n+1}) = 0$, contradicting our assumption
Therefore $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for any $n \in N$ and hence $\{d(x_n, x_{n+1})\}$ is monotone decreasing sequence of nonnegative real numbers, hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.
In the view of the fact from (2), for any $n \in N$, we have
$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1}))$,
Taking the limit as $n \to \infty$ in the above inequality, and using the conditions of ψ and ϕ we have
$\psi(r) \leq \psi(r) - \phi(r)$ which implies $\phi(r) = 0$
Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$…………………………………………..………………...…(4)
Next we show that (x_n) is a Cauchy sequence.
If otherwise there exists $\varepsilon > 0$, for which we can find two sequences of positive integers (m_k) and (n_k) such that for all positive integers $m_k > n_k > k$, $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ and $d(x_{m_k}, x_{n_{k-1}}) < \varepsilon$.
Now $\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k})$,
that is $\varepsilon \leq d(x_{m_k}, x_{n_k}) < \varepsilon + d(x_{n_{k-1}}, x_{n_k})$
Taking the limit as $k \to \infty$ in the above inequality and using (4) we have
$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$……………………………………...………………………………(5)

Again $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$.
Taking the limit as $k \to \infty$ in the above inequalities and using (4) and (5) we have
$\lim_{k\to\infty} d(x_{m_{k+1}}, x_{n_{k+!}}) = \varepsilon$……………………………..………………………………...(6)
Again $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$
$\leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})$
Letting $k \to \infty$ in the above inequalities and using (4) and (5) we have
$\lim_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) = \varepsilon$………………………...………………………………………(7)
$\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \varepsilon$ …………...…………………………………………………………(8)
For $x = x_{m_k}$, $y = y_{m_k}$ we have

$$d(x_{m_k}, Tx_{m_k}) - d(A,B) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, Tx_{m_k}) - d(A,B)$$
$$= d(x_{m_k}, x_{m_{k+1}})$$

Similarly $d(x_{n_k}, Tx_{n_k}) - d(A,B) = d(x_{n_k}, x_{n_{k+1}})$.
Also $d(x_{m_k}, Tx_{n_k}) - d(A,B) = d(x_{m_k}, x_{n_{k+1}})$ and
$d(x_{n_k}, Tx_{m_k}) - d(A,B) = d(x_{n_k}, x_{m_{k+1}})$.
From (1) we have $\psi(d(x_{m_{k+1}}, x_{n_{k+1}})) = \psi(d(Tx_{m_k}, Tx_{n_k}))$
$\leq \psi\Big(\big(a_1 d(x_{m_k}, x_{n_k}) + a_2 d(x_{m_k}, Tx_{m_k}) + a_3 d(x_{n_k}, Tx_{n_k})\big) - d(A,B)\Big) - \phi\Big(\big(a_1 d(x_{m_k}, x_{n_k}) +$
$a_2 d(x_{m_k}, Tx_{m_k}) + a_3 d(x_{n_k}, Tx_{n_k})\big) - d(A,B)\Big)$
$\leq \psi\Big(\big(a_1 d(x_{m_k}, x_{n_k}) + a_2 d(x_{m_k}, Tx_{m_k}) + a_3 d(x_{n_k}, Tx_{n_k})\big)\Big) - \phi\Big(\big(a_1 d(x_{m_k}, x_{n_k}) + a_2 d(x_{m_k}, Tx_{m_k}) +$
$a_3 d(x_{n_k}, Tx_{n_k})\big)\Big)$
It follows that

$$\psi(d(Tx_{m_k}, Tx_{n_k}))$$
$$\leq \psi\Big(\big(a_1 d(x_{m_k}, x_{n_k}) + a_2 d(x_{n_k}, Tx_{n_{k+1}}) + a_3 d(x_{m_k}, Tx_{m_{k+1}})\big)\Big)$$
$$- \phi\Big(\big(a_1 d(x_{m_k}, x_{n_k}) + a_2 d(x_{n_k}, Tx_{n_{k+1}}) + a_3 d(x_{m_k}, Tx_{m_{k+1}})\big)\Big)$$

From (4), (5), (6) and (7) and letting $k \to \infty$ in the above inequalities and using the conditions of ψ and ϕ, we have $\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$ which is contradiction by virtue of property ϕ.
Hence $\{x_n\}$ is a Cauchy sequence.
Since $\{x_n\} \subset A$ and A is a closed subset of the complete metric space (X,d), there exists x^* in A such that $x_n \to x^*$.
Putting $x = x_n$ and $y = x^*$ and since
$d(x_n, Tx^*) \leq d(x_n, x^*) + d(x^*, Tx_n)$ and
$d(x^*, Tx_n) \leq d(x^*, Tx^*)$+ $d(Tx^*, Tx_n)$
We have $\psi(d(x_{n+1}, Tx^*) - d(A,B) \leq \psi d(Tx_n, Tx^*))$
$\leq \psi\Big(\big(a_1 d(x_n, x^*) + a_2 d(x_n, Tx_n) + a_3 d(x^*, Tx^*)\big) - d(A,B)\Big) - \phi\Big(\big(a_1 d(x_n, x^*) + a_2 d(x_n, Tx_n) +$
$a_3 d(x^*, Tx^*)\big) - d(A,B)\Big)$
Taking the limit as $n \to \infty$ in the above inequalities and using the conditions of ψ and ϕ,we have

$$\psi(d(x^*, Tx^*) - d(A,B)) \leq \psi(d(x^*, Tx^*) - d(A,B)) - \phi(d(x^*, Tx^*) - d(A,B))$$

which implies that $d(x^*, Tx^*) = d(A,B)$
Hence x^* is a best proximity point of T.
For the uniqueness
Let p and q be two best proximity points and suppose that $p \neq q$,
Then putting $x = p$ and $y = q$ in (1) we obtain
$\psi\big(d(T_p, T_q)\big) \leq \psi\Big(\big(a_1 d(p,q) + a_2 d(p, T_p) + a_3 d(q, T_q)\big) - d(A,B)\Big) - \phi\Big(\big(a_1 d(p,q) + a_2 d(p, T_p) +$
$a_3 d(q, T_q)\big) - d(A,B)\Big)$
that is $\psi(d(p,q)) \leq \psi(d(p,q)) - \phi(d(p,q))$
which is contradiction by virtue of a property ϕ.
There $p = q$
This completes the proof.

Example:2.1 Let $X = A \cup B$,where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ and $B = \{1,2\}$
Define the generalized metric space on X as follows:

$$d(x,y) = \begin{cases} d(x,y), x \in A, y \in B \\ 0, \quad x \in A, y \in B \text{ with } x = y \\ 0.3, x = \frac{1}{2}, y = \frac{1}{3} \text{ or } x = \frac{1}{4}, y = \frac{1}{5} \\ 0.2, x = \frac{1}{2}, y = \frac{1}{5} \text{ or } x = \frac{1}{3}, y = \frac{1}{4} \\ 0.6, x = \frac{1}{2}, y = \frac{1}{4} \text{ or } x = \frac{1}{5}, y = \frac{1}{3} \\ |x-y|, x \in A, y \in B \end{cases}$$

Then (X, d) is a Branciari type generalized metric space, but it is not metric space.
In fact $0.6 = d\left(\frac{1}{2},\frac{1}{4}\right) > d\left(\frac{1}{2},\frac{1}{3}\right) + d\left(\frac{1}{3},\frac{1}{4}\right) = 0.5$
Let $T: A \to B$ is defined by

$$Tx = \begin{cases} \frac{1}{5}, \ x \in [1,2] \\ \frac{1}{4}, \ x \in [\frac{1}{2},\frac{1}{3},\frac{1}{4}] \\ \frac{1}{3}, \qquad x = \frac{1}{5} \end{cases}$$

Define $\psi(t) = t, \phi(t) = \frac{t}{5}, t \in [0, \infty)$
then T satisfies
$\psi\big(d(Tx,Ty)\big) \le \psi\big(a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty)\big) - \phi\big(a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty)\big)$ for all $x \in A, y \in B$ where $a_1 = 0.4, a_2 = 0.4, a_3 = 0.2$.
$\psi\big(d(Tx,Ty)\big) \le \psi(\big(a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty)\big) - d(A,B)) - \phi(\big(a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty)\big) - d(A,B))$
Thus all the hypothesis of theorem are satisfied and T has a best proximity point.

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