DUPLEX EQUITABLE DOMINATION NUMBER OF A GRAPH

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Abstract: Let G = (V, E) be a graph. A subset D $\subseteq V$ is said to be a dominating set of G if every vertex in V− D is adjacent to some vertex in D. The cardinality of a minimum dominating set D is called the domination number of and is denoted by $\gamma(G)$. A subset D of V is called duplex equitable dominating set if for every vertex $v \in V - D$, there exists two vertices $u_1, u_2 \in D$ such that u_1 dominates v and u_2 equitable dominates v.The minimum cardinality of duplex equitable dominating set is called duplex equitable domination number and it is denoted by $\gamma_{de}(G)$. In this paper, we obtain some results on Duplex Equitable domination number of a graph. we found the best possible upper and lower bounds for γ_{de} , characterize the graphs satisfies these bounds, obtained γ_{de} number for some standard graphs and found relationship between other domination parameters like χ, δ and Δ.

Key words: equitable, domination number, duplex, duplex equitable.
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1.Introduction

By a graph G = (V, E) we mean a finite, connected, unordered with neither loops or multiple edges the order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Berge and Harary [1], [2]. Let $v \in$ V the neighborhood of v are defined by N (v)={$u \in$V, uv $\in$E }. A degree of a vertex v in a graph G is the number of edges incident with the vertex v and it is denoted by deg (v). If the degree of each vertex in the graph is two, then it is called a **Cycle**. A simple graph G = (V, E) with vertex partition V = {V_1, V_2} is called a **bipartite graph** if every edge of E joins a vertex in V_1 to a vertex in V_2. A **complete bipartite** graph of the form $K_{1, n-1}$ is a star graph with n-vertices. A **star** is a complete bipartite graph if a single vertex belongs to one set and all the remaining vertices belong to the other set. A **Wheel** is a graph formed by connecting a single universal vertex to all vertices of a cycle. The **Helm** is a graph obtained from a wheel by attaching a pendent edge at each vertex of the cycle. Any cycle with a pendent edge attached at each vertex called **Crown graph**. A **fan graph** is obtained from a path by adding a new vertex and joining it to all vertices of the path by an edge. A **corona graph** is obtained from two graphs, G of order p and H, taking one copy of G and p copies of H and joining by an edge the ith vertex of G to every vertex in the i th copy of H. The **ladder graph** is defined by the Cartesian product $P_n \times P_2$, where P_n is the path of n vertices and P_2 is the path of 2 vertices. A **gear graph** is obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph. The concept of domination was first studied by Ore and C. Berge [3]. Various range of applications has been attributed to the domination in graph theory for several fields includes modelling networks, coding theory, communication networks, land surveying, transportation systems etc. A subset $D \subseteq V$ is said to be a dominating set of G if every vertex in $V-D$ is adjacent to some vertex in D. A restrained dominating set is a set $S \subseteq V$ where every vertex in $V-S$ is adjacent to a vertex in S as well as another vertex in $V-S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. A set $S \subseteq V(G)$ in a graph G is said to be [1, 2] triple connected dominating set, if for every vertex $\upsilon \in V-S$, $1 \leq |N(\upsilon) \cap S| \leq 2$ and $< S >$ is triple connected. The minimum cardinality taken over all set is denoted by $\gamma_{[1,2]tc}(G)$. The subset D of V is called equitable dominating set if for every $v \in V-D$ there exist a vertex $u \in D$ such that $uv \in (G)$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called the equitable domination number of G. A vertex $u \in V$ is said to be degree equitable vertex if $|\deg(u) - \deg(v)| \leq 1$. A vertex $u \in V$ said to be equitable isolate if $|\deg(u) - \deg(v)| \geq 2$ for all $v \in V$.Equitable neighborhood of a vertex is denoted by $N^e(u)$ is defined as $N^e(u) = \{v \epsilon V / v \epsilon N(u), |deg(u) - deg(v)| \leq 1\}$. Let G= (V, E) be a graph. A subset D of V is called double dominating set if for every vertex $v \in V-D$, there exists two vertices $u_1, u_2 \in D$ such that $u_1 v$ and $u_2 v \in E(G)$. The minimum cardinality of double dominating set is called double domination number and it is denoted by $\gamma_d(G)$. This concept is presented by Harary and Haynes [5]. In this paper we introduce and determine the new domination parameter γ_{de} for some graphs and theoretical parameters are discussed and investigated.

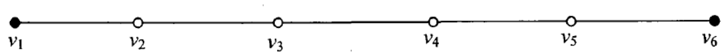
2. Duplex Equitable Domination Number

Definition 2.1

Let G= (V, E) be a graph. A subset D of V is called duplex equitable dominating set if for every vertex $v \in V-D$, there exists two vertices $u_1, u_2 \in D$ such that $u_1 v$ and $u_2 v \in E(G)$ and $|\deg(u_1) - \deg(v)| \leq 1$ or $|\deg(u_2) - \deg(v)| \leq 1$ The minimum cardinality of duplex equitable dominating set is called duplex equitable domination number and it is denoted by $\gamma_{de}(G)$.

Example:2.1.1

Consider P_6 :



Let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ then, $D = \{v_1, v_3, v_4, v_6\}$ and $V - D = \{v_2, v_5\}$

Let $v_2 \in V-D$, then $v_2 v_1, v_2 v_3 \in E(G)$ and $|\deg(v_1) - \deg(v_2)| \leq 1$

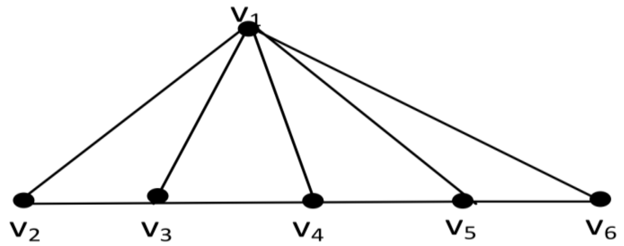
Let $v_5 \in V-D$, then $v_5 v_4, v_5 v_6 \in E(G)$ and $|\deg(v_5) - \deg(v_6)| \leq 1$

Hence $D = \{v_1, v_3, v_4, v_6\}$ is minimum duplex equitable dominating set.

Then duplex equitable domination number $\gamma_{de}(G) = 4$.

Example: 2.1.2

Consider the graph



Let V(G)=$\{v_1, v_2, v_3, v_4, v_5, v_6\}$ then, D=$\{v_1, v_3, v_6\}$ and $V - D = \{v_2, v_4, v_5\}$

Let $v_2 \in V - D$, then $v_2v_1, v_2v_3 \in E(G)$ and $|\deg(v_2) - \deg(v_3)| \leq 1$

Let $v_4 \in V - D$, then $v_4v_1, v_4v_3 \in E(G)$ and $|\deg(v_4) - \deg(v_3)| \leq 1$

Let $v_5 \in V - D$, then $v_5v_1, v_5v_6 \in E(G)$ and $|\deg(v_5) - \deg(v_6)| \leq 1$

Hence D=$\{v_1, v_3, v_6\}$ is minimum duplex equitable dominating set.

Then duplex equitable domination number $\gamma_{de}(G) = 3$.

2.2 Duplex Equitable Domination Number for General graphs

1. For a complete graph (K_n), $\gamma_{de}(K_n)$ =2
2. For a complete bipartite graph $(K_{p,q})$, $\gamma_{de}(K_{p,q}) = \begin{cases} 4 & if\ p - q \leq 1 \\ p + q & other\ wise \end{cases}$
3. For any star graph $(K_{1,n-1})$, $\gamma_{de}(K_{1,n-1}) = n$
4. For any path (P_n), $\gamma_{de}(P_n) = \begin{cases} \frac{n}{2} + 1 & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}$
5. For any cycle (C_n), $\gamma_{de}(C_n) = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}$
6. For any wheel $(W_{1,n})$, $\gamma_{de}(W_{1,n}) = \begin{cases} \frac{n+2}{2} & if\ n\ is\ even \\ \frac{n+3}{2} & if\ n\ is\ odd \end{cases}$
7. For any Helm graph (H_n), $\gamma_{de}(H_n) = \begin{cases} \frac{3n+2}{2} & if\ n\ is\ even \\ \frac{3n+3}{2} & if\ n\ is\ odd \end{cases}$
8. For any Fan graph $(F_{1,n})$, $\gamma_{de}(F_{1,n}) = \left\lceil \frac{n}{2} \right\rceil$
9. For any crown graph $(C_n{}^\circ P_2)$, $\gamma_{de}(C_n{}^\circ P_2) = \begin{cases} \frac{3n}{2} & if\ n\ is\ even \\ \frac{3n+1}{2} & if\ n\ is\ odd \end{cases}$

10. For a corona graph$(C_{n^\circ}P_3)$, $\gamma_{de}(C_{n^\circ}P_3) = 2n$

11.For any gear graph $G_n, \gamma_{de}(G_n) = n + 1$

3. Bounds of Duplex Equitable Domination Number

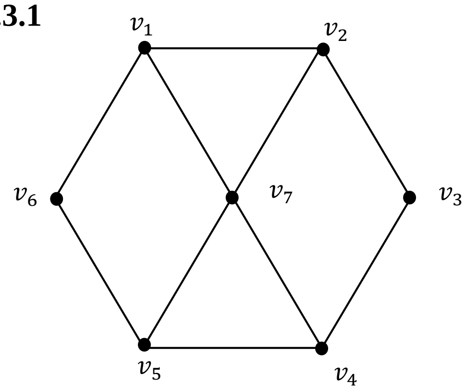
Theorem: 3.1
For any connected graph, $2 \leq \gamma_{de}(G) \leq n$.
Proof:
Since any duplex equitable dominating set has at least two vertices and at most n vertices.
Lower bound: For any complete graph $K_n, \gamma_{de}(K_n) = 2$

Upper Bound: For any star $K_{1,n-1}, \gamma_{de}(K_{1,n-1}) = n$

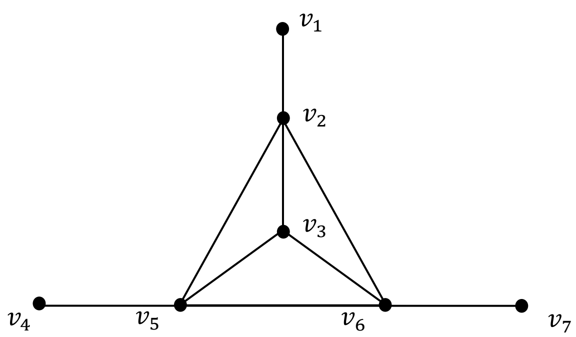
Theorem: 3.2
For any graph G, $\gamma(G) \leq \gamma_{de}(G)$
Proof:
For any graph, every duplex equitable dominating set is a dominating set but every dominating set need not be duplex equitable dominating set.
Theorem:3.3
For any graph G, $\gamma \leq \gamma_r \leq \gamma_c \leq \gamma_t \leq \gamma_e(G) \leq \gamma_{de}(G)) \leq \gamma_{[1,2]tc}(G)$

Proof:
For any graph, every duplex equitable dominating set is an equitable dominating set but every equitable dominating set need not be duplex equitable dominating set. For $\gamma_{de}(G)) \leq \gamma_{[1,2]tc}(G)$, sharpness satisfied on example 2.1.2 like graphs and other satisfied because lower bound for [1,2] triple connected domination number is 3.

Example:3.3.1



In the above example, $\gamma(G) = 3, \gamma_c(G) = 3, \gamma_t(G) = 3, \gamma_e(G) = 3$ and $\gamma_{de}(G) = 4$
Hence $\gamma \leq \gamma_c \leq \gamma_t \leq \gamma_e(G) \leq \gamma_{de}(G)$.
Note: 3.3.2
Let H be the Spanning subgraph of G, then $\gamma_{de}(H) \leq \gamma_{de}(G)$
Observation: 3.3.3
Support of a pendant vertex need not be in a duplex equitable dominating set.
Example : 3.3.4



Here D = $\{v_1, v_3, v_4, v_7\}$

Observation : 3.3.5
For any graph G, $\left\lceil \frac{n}{\Delta+1} \right\rceil \leq \gamma_{de}(G)$ and the bound is sharp.

We know that $\left\lceil\frac{n}{\Delta+1}\right\rceil \le \gamma(G)$ and by the theorem 2.2.2 it is clear that, $\left\lceil\frac{n}{\Delta+1}\right\rceil \le \gamma_{de}(G)$

Theorem: 3.4

Let G be a graph with even order n and without isolate vertices. Then $\gamma_{de}(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or corona graph $C_n{\circ}k_1$

Proof:

Suppose G≅ $C_n{\circ}k_1$, Let $\{v_1, v_2 \dots \dots v_n\}$ be the vertices of C_p. Corona graph formed by adding an k_1 to each p vertices of C_n. Let $\{v_{n1}, v_{n2,}v_{n3} \dots \dots v_{pm}\}$ be the vertices added to each vertices of cycle C_p. Now consider D= $\{v_{n1}, v_{n2}, v_{n3} \dots \dots v_{pm}\}$ is adjacent to remaining $\{v_1, v_2 \dots \dots v_n\}$ and each vertices of $v - D$ is adjacent to exactly two vertices in $v - D$. So clearly D is minimal dominating set and C_n is connected. Clearly D duplex equitable dominating set. Hence $\gamma_{de}(G) = \frac{n}{2}$ Conversely, $\gamma_{de}(G) = \frac{n}{2}$. Clearly G is the components are the cycle C_4 or corona graph $C_n{\circ}k_1$.

Observation: 3.4.1

Let G be a graph without isolates and let there exist a duplex equitable dominating set. Then $\gamma(G) + 1 \le \gamma_{de}(G)$ and the bound is sharp.

For Example, In $K_{4,}$ $\gamma(G) = 1$, $\gamma_{de}(G) = 2$ then $\gamma(G) + 1 = \gamma_{de}(G)$.

The sharpness is valid for all K_n.

Theorem: 3.5

For any integer n, there exists a graph G=$K_{1,n}$ such that $\gamma_{de}(G) - \left\lceil\frac{2n}{\Delta+2}\right\rceil = n$

Proof:

Let G =$K_{1,n}$, $n > 1$ then $\gamma_{de}(G) = n + 1$ and $\Delta(K_{1,n}) = n$

Now $\gamma_{de}(G) - \left\lceil\frac{2n}{\Delta+2}\right\rceil = (n+1) - \left\lceil\frac{2(n+1)}{n+2}\right\rceil$

$$= (n+1) - \left\lceil\frac{n+n+2}{n+2}\right\rceil$$

$$= (n+1) - \left\lceil\frac{n}{n+2} + 1\right\rceil$$

$$= n$$

Observation: 3.5.1

Every graph G without isolated vertices and δ is the minimum degree then

$\gamma_{de}(G) < n + \delta + 1$.

Theorem: 3.6

For any graph G, $\gamma_{de}(G) + \chi(G) \le 2n$ and the bounds is sharp if and only if G is isomorphic to K_2

Proof:

For any graph with n vertices. We know that χ (G) $\le n$. Also, we know that $\gamma_{de}(G) \le n$.
Hence $\gamma_{de}(G) + \chi(G) \le n + n$ and $\gamma_{de}(G) + \chi(G) \le 2n$. Suppose G is isomorphic to K_2.
Then γ_{de} (K_2) = 2 and χ (K_2) =2 then γ_{de} (K_2) + χ (K_2) = 2 + 2 = 4 = $2n$.Hence clearly
$\gamma_{de}(G) + \chi(G) = 2n$. Conversely, $\gamma_{de}(G) + \chi(G) = 2n$.To Prove $G \cong K_2$.This is possible only if
$\gamma_{de}(G)$ = 2 and $\chi(G)$ = n. But χ (G) = n and so $G \cong K_n$ for which $\gamma_{de}(G) = 2 = p - 2$
so that p = 4. Hence $G \cong K_2$

Observation:3.6.1

For any G with n vertices $\gamma_{de}(G)+\Delta(G)\leq 2n-1$ and the bound is sharp if and only if G is isomorphic to K_2

Note: 3.6.1

In a duplex equitable dominating set, every vertex in V – D contributes to degree sum of vertices in D.

Theorem: 3.7

For any graph G, $\left\lceil \frac{2n}{\Delta+2} \right\rceil \leq \gamma_{de}(G)$

Proof:

By definition of duplex equitable dominating set, every vertex in V-D contributes to degree sum of vertices in D. so $2|V-D| \leq \sum_{u\in D} \deg(u)$ where D is duplex dominating set.

⇨ $2|V-D| \leq \sum_{u\in D} \deg(u) \leq \gamma_{de}(G)\Delta$

⇨ $2|V-D| \leq \gamma_{de}(G)\Delta$

⇨ $2|V|-2|D| \leq \gamma_{de}(G)\Delta$

⇨ $2n-2\gamma_{de}(G) \leq \gamma_{de}(G)\Delta$

⇨ $\gamma_{de}(G)\ (2+\Delta) \geq 2n$

⇨ $\gamma_{de}(G) \geq \left\lceil \frac{2n}{\Delta+2} \right\rceil$

The above bound is sharp when $G = K_n$

We know that $\gamma_{de}(K_n) = 2$

$\left\lceil \frac{2n}{\Delta+2} \right\rceil = \left\lceil \frac{2n}{n-1+2} \right\rceil = \left\lceil \frac{2n}{n+1} \right\rceil = 2$

4. Characterization of Duplex equitable dominating set

Theorem: 4.1

Every duplex equitable dominating set contains all pendent vertices.

Proof:

Let D be duplex equitable dominating set. Let v be a pendent vertex with support vertex u.Suppose $v \notin D$, since D is duplex equitable dominating set then for every $v \in V-D$, there exists two vertices $u_1, u_2 \in D$ such that u_1v and $u_2v \in E(G)$ and $|\deg(u_1) - \deg(v)| \leq 1$. Hence u_1 and u_2 are adjacent to v. hence $\deg(v) \geq 2$ .which is a contradiction.Then v is a pendent vertex .Hence the theorem.

Theorem:4.2

Let G be a graph. Then $\gamma_{de}(G) \leq n - \Delta + k$,where k is the number of pendant vertices in G.

Proof:

Let u be a vertex with degree Δ. Let $v_1, v_2, v_3, \dots \dots$be the neighbors of u.
Let $v_1, v_2, v_3, \dots \dots \dots .\, v_k$ (k <Δ) be the vertices in N (u) which are pendant vertices.
Since every duplex equitable dominating set contains all pendent vertices,
$\gamma_{de}(G) < n - (\Delta - k) = n - \Delta + k$. Hence $\gamma_{de}(G) < n - \Delta + k$.
Also, Sharpness valid for a single pendent vertex.

Theorem :4.3
Let G be a graph. Then $\gamma_{de}(G) = n$ if and only if G is a star.

Proof:
It is obviously true that if G is a star. Then $\gamma_{de}\,(G) = n$.
Conversely, Suppose $\gamma_{de}(G) = n$. To Prove G is a Star.

case *i:* Let V(G) = $\{v_1, v_2, v_3\}$ be the vertices of a graph G. Here $K_3, \overline{K}_3, P_2 \cup K_1$, $P_3, C_3, K_{1,2}$ be the possible three vertices graph. γ_{de} of C_n is $\left\lceil \frac{n}{2} \right\rceil$ and this is for C_3 is 2. γ_{de} of $K_3\, and\; P_2 \cup K_1$ are disconnected graph. γ_{de} of P_n is $\left\lceil \frac{n}{2} \right\rceil$ and this is for P_3 is 2. γ_{de} of $K_{1,2}$is 3.

G$\ncong$ $K_3, \overline{K}_3, P_2 \cup K_1, P_3$. Hence G$\cong$ $K_{1,2}$. $\gamma_{de}\,(K_{1,2})$=3. Hence G is a star.

case *ii:* Let V(G) = $\{v_1, v_2, v_3, v_4\}$ be the vertices of a graph G. Here $K_4, \overline{K}_4, P_2 \cup \overline{K}_2, P_3 \cup K_1, C_3 \cup K_1, P_2 \cup P_2, C_3(P_2), K_4 - e, P_4, C_4, K_{1,3}$ be the possible four vertices graph. γ_{de} of C_n is $\left\lceil \frac{n}{2} \right\rceil$ and this is for C_4 is 2. γ_{de} of $P_2 \cup \overline{K}_2$, $P_3 \cup K_1$, $C_3 \cup K_1, P_2 \cup P_2$, $\overline{K}_4$ are disconnected graph. γ_{de} of P_n is $\left\lceil \frac{n}{2} \right\rceil$ and this is for P_4 is 2. G$\ncong$ $K_4, \overline{K}_4, P_2 \cup \overline{K}_2, P_3 \cup K_1, C_3 \cup K_1, P_2 \cup P_2, C_3(P_2), K_4 - e, P_4$. Hence G$\cong$ $K_{1,3}$. $\gamma_{de}\,(K_{1,3})$=4. Hence G is a star.

case *iii:* Let V(G) = $\{v_1, v_2 \ldots\ldots\ldots. v_{n-3}\}$ be the vertices of a graph G. Here γ_{de} of P_{n-3} , $K_{n-3}, P_{n-5} \cup K_2, C_{n-5} \cup K_2, P_{n-5} \cup P_2$, $\overline{K}_{n-3}$ are disconnected graph.

Hence γ_{de} (G) = $\gamma(V(G_{n-3}) + V(G_3))$.

$For\ n = 3$, $\gamma_{de}\,(K_{1,2})$=3, $For\ n = 4$, $\gamma_{de}\,(K_{1,3})$=4 and so on gives $For\ n = n - 3$, γ_{de} $(K_{1,n-4})$=$n - 3$. Hence G$\cong$ $K_{1,n}$ implies that $\gamma_{de}\,(K_{1,n})$=n. Thus, G is a star.

Note: 4.3.1

Suppose every vertex of G is either pendant or equitable. Then G is a star, because if u is a equitable vertex then any neighbor of u is not equitable, in such a case the neighbor is pendant. Hence $\gamma_{de}(G)$= n.

Theorem :4.4
A graph G has a unique minimal duplex equitable dominating set if and only if every non-equitable vertex of G is either a pendant vertex of G or is adjacent to a equitable vertex of a graph G.

Proof:
Suppose G has a unique minimal duplex equitable dominating set.
Suppose u is a non-equitable vertex which is neither a pendant nor adjacent to any equitable vertex of G. Then clearly deg (u) > 2.
Hence V− $\{u\}$ is a duplex equitable dominating set containing a minimal duplex equitable dominating set say D.
Since $u \in D$ and since D is a duplex equitable dominating set there exists
u_1, u_2 ∈D such that $u_1 v, u_2 v$ ∈ E (G) and $|\text{deg}\,(u_1) - \text{deg}\,(v)| \leq 1$
Since d (u_1) > 2, d (v) > 2.
By hypothesis u_1 is not a equitable vertex. Hence V− $\{u_1\}$ is a duplex equitable dominating set containing a minimal duplex equitable dominating set say D'. Since u_1 ∈ D and u_2 ∈ D',
D ≠ D'. Hence there exists two minimal duplex equitable dominating sets, a contradiction.
Conversely suppose every non-equitable vertex of G is either a pendant vertex of G or is adjacent to a equitable vertex of G.
Suppose D_1 and D_2 are two minimal duplex equitable dominating sets of G. Let $D_1 > D_2$.

Let $u \in D_1 - D_2$. Then u cannot be pendant and u cannot be equitable, since every duplex equitable dominating set contains all pendant and all equitable vertices.

Hence d (u) > 2 and u is adjacent to a equitable vertex v of G. Since D_1 is a minimal duplex equitable dominating set there exist v in V$-D_1$ such that N(v) ∩ D'= $\{u\}$ or there exist a v_2 in V$-D_1$ such that $|N^e(v_2) \cap D_1| = \{u\}$ where $N^e(v_2) = \{u/uv_2 \in E(G) and \mid \deg(v_2) - \deg(u) \mid \leq 1$.

Suppose there exist a v_1 in V$-D_1$ such that N $(v_1) \cap D_1$ = $\{u\}$. Hence v_1 is neither a equitable vertex nor a pendant vertex. Hence there exist we V(G) such that w is equitable and v_1 is adjacent to w. But $w \in D_1$. So, N $(v_1) \cap D_1 \supset \{u, w\}$, a contradiction.

Suppose there exist a $v_2 \in$ V$-D_1$ such that $|N^e(v_2) \cap D_1| = \{u\}$. Since $v_2 \in$ D_1, v_2 is neither pendant nor equitable. we get a contradiction. So $D_1 - D_2 = \emptyset$. Hence $D_1 = D_2$. So, G has a unique minimal duplex equitable dominating set.

Theorem:4.5

Let G= (V, E) be a simple graph. Let D⊂ V be duplex equitable dominating set of G , D is minimal duplex equitable dominating set if and only if for every $u \in D$ one of the following conditions hold

(i) u is a pendent vertex of G
(ii) u is isolated vertex or u is equitable isolated or N $(u) \cap D = \emptyset$
(iii) there exists a $v_1 \in V - D$ such that N $(v_1) \cap D = \{u\}$ or there exists a $v_2 \in V - D$ such that $N^e(v_2) \cap D = \{u\}$
(iv) there exists a $v \in V - D$ such that u∈ $N(v)$ and $|N(u) \cap D| = 2$

Proof:

Suppose D is minimal duplex equitable dominating set.

Let $u \in D$, To prove u satisfies all four conditions (i) to (iv).

suppose u not satisfies all the conditions (i) to (iv). Then u is not a pendent vertex that means

|N $(u)| \geq 2$,which is a contradiction to assumption. Hence u is a pendent vertex. Hence it satisfies (i). Suppose u is not an isolated vertex or not an equitable isolate. then u has some adjacent vertices N(u)∩ $D \neq \emptyset$, $|\deg(u) - \deg(v)| \geq 2$.which is a contradiction. Hence it satisfies condition (ii)

Since u does not satisfies condition (iii), then for every $v_1 \in V - D$ if $u \in$ N$(v_1) \cap D$ then there exists $w_1 \neq u$ such that $w_1 \in$ N$(v_1) \cap D$ also for every $v_2 \in V - D$ if $u \in N^e(v_2) \cap D$ then there exists $w_2 \neq u$ such that $w_2 \in$ N$(v_1) \cap D$. Consider $D - \{u\}$,let $x \in V - (D \cup \{u\})$.

Case (i) Suppose $x = u$ since N(u)$\cap D = \emptyset$, $N^e\,(v_2) \cap D = \emptyset$ and $|N(u)\ ) \cap D| \geq 2$. Hence $D - \{u\}$ is duplex equitable dominating set.

Case (ii) Suppose $x \neq u$ then $x \in V - D$,there exists $y_1, y_2 \in D$ and $y_1 \neq y_2$ such that $y_1 \in$ N $(x) \cap D$ and $y_2 \in N^e\ (x) \cap D$. If $u \neq y_1$ and $u \neq y_2$ then $D - \{u\}$ is duplex equitable dominating set. which is a contradiction. Hence this satisfies condition (iii).

To prove (iv). Suppose u does not satisfies condition (iv) we get that if $u \in N(v)$ for any $v \in V - D$ then $|N(v) \cap D| \neq 1$. since D is duplex equitable dominating set for every $v \in V - D$ there exist $u_1, u_2 \in D$ such that $u_1v, u_2v \in E(G)$ and $|\deg(u_2) - \deg\,(v)| \leq 1$ Hence$|N(u) \cap D| \neq 2$ since D duplex equitable dominates v then $|N(u)\ ) \cap D| \neq 1$

Hence $|N(u)\ ) \cap D| \geq 3$.Suppose $y_1 \neq u$. y_2= u.Then $u \in N\ (x)$.

Hence $|N\ (x)\ \cap\ D|\ \geq\ 3$. So there exist $w\ \in\ N\ (x)\ \cap\ D, w\ \neq\ u$ and $w \neq\ y_2$. Hence $D - \{u\}$ is duplex equitable dominating set.

Suppose $y_2 = u$. Then $u \in N^e(x) \cap D \subseteq N(u) \cap D$. Hence $|N(x) \cap D| \geq 3$. So there exists $w \in N(x) \cap D$, $w \neq u$ and $w \neq y_1$. Hence $D - \{u\}$ is duplex equitable dominating set. By all above conditions $D - \{u\}$ is not an duplex equitable dominating set it is contradiction for the assumption is D is minimal duplex equitable dominating set.

Hence $u \in D$ one of the following conditions hold

(i) u is a pendent vertex of G

(ii) $u$is isolated vertex or u is equitable isolated or N(u)∩ $D = \emptyset$

(iii) there exists a $v_1 \in V - D$ such that N $(v_1) \cap D = \{u\}$ or there exists a $v_2 \in V - D$ such that $N^e(v_2) \cap D = \{u\}$

(iv) there exists a $v \in V - D$ such that $u \in N(v)$ and $|N(u) \cap D| = 2$

Conversly, Suppose D is a duplex equitable dominating set and $u \in D$ satisfies one of the conditions. Consider $D - \{u\}$. If u is pendent vertex then D− $\{u\}$ not a duplex equitable dominating set. If u is isolated vertex or u is equitable isolated or N (u)∩ $D = \emptyset$ then D− $\{u\}$ not a duplex equitable dominating set. If every $u \in D$ there exists a $v_1 \in V - D$ such that N $(v_1) \cap D = \{u\}$ or there exists a $v_2 \in V - D$ such that $N^e(v_2) \cap D = \{u\}$ then D−$\{u\}$is not a duplex equitable dominating set. Similarly, if every $u \in D$ there exists a $v \in V - D$ such that u ∈ $N(v)$ and $|N(u) \cap D| = 2$ then D−$\{u\}$ not a duplex equitable dominating set. Hence D is minimal duplex equitable dominating set.

CONCLUSION

In this paper, the new domination parameter duplex equitable domination number introduced, determined duplex domination number for some general graphs, some upper and lower bounds were discussed. Further we would like to extend our research work relate with other domination parameters which will help to find the real-life applications of the duplex equitable domination number.

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