New perspectives in space of COGARCH(1,1) models with fractional derivative

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Abstract: It is known to us that the fractional derivatives are the general case of differentiation, that is to say the normal derivative is a special case of fractional derivative. So, our contribution in this paper based on two paths, the first one aims is to study and estimate the parameters of nonlinear time series models sample with continuous case, known by COGARCH(1,1) with a fractional derivative. Where this study allows testing the generalization estimation algorithm on the fractional COGARCH, second path based to present an estimation comparison between classic COGARCH(1,1) and COGARCH(1, 1) with a fractional derivative. Then, we illustrate our study with simulation to answer some questions and extract some properties.

Keywords:
COGARCH(1,1), fractional derivatives, Caputo fractional derivatives.

1. Introduction

The continuous models have excited substantial interest to the most of the research communities, because its wide range of applications in real life are endless. On the other hand, in general the continuous case is always more generalized than the discrete case in mathematics. Especially in economics and finance, the most of the techniques and approaches to problem solving take a continuous form. There is a continuous case. model of which he wrote his fame in mathematical literature by gold, and known by COGARCH models, and among the researchers who have been specialized in the type of these models [kluppelberg, Lindner and Maller] see (Andersen, T., Bollerslev, T., & Hadi, A., 2014). This sample of models appeared as an urgent case in economic modeling due to the limitation of diffusion of GARCH models and their insufficiency to translate the data, and the stochastic expression of most phenomena is continuous.

Otherwise, all applications in time series spaces will be a special case of a space known by fractional analysis, the concept of fractional derivative appeared for the first time in a famous correspondence in 1733 see (Francq, C., & Zakoian, J. M. ,2019), our goal here to create a model with fractional derivative exactly Caputo form

The paper is organized as follows;

Second section: it is an extension of studying the COGARCH model of a classical form known, and some notions about fractional notions.

Third section: Estimation of the model coefficients from the moments approach, the study is enriched by clear fractional.

Fourth section: this section is based on the quote algorithm for estimating coefficients of fractional models as a generalized case of COGARCH classic.

Fifth section: Numerical illustration to prove the value of theoretical study.

2. Preliminaries

This section is devoted to recall the essentials of COGARCH (1,1) models. Firstly, the GARCH process governed by a discrete case, this property does not solve most of the major problems in economics, because the very large number of statistical data by very small variable jumps pushes statisticians to create a model with a generalized continuous case. So, the generalized model here with two Brownian motions with fractional derivative $w_1(t)$ and $w_2(t)$ defined by its expression

$$\begin{cases} dh(t) = \sigma(t)\,dw_1(t) \\ d\sigma^2(t) = \left(a - b\sigma^2(t)\right)dt + c\sigma^2(t)\,dw_2(t) \end{cases}$$

To enrich our study, we need to give a global overview on the Lévy processes. Let the GARCH(1,1) model defined by its expression

$$\begin{cases} h_t = z_t\sigma_t, \quad t \in \mathbb{N} - \{0\} \\ \sigma_t = \left(\gamma_0 + \gamma_1 h_{t-1}^2 + \gamma_2\sigma_{t-1}^2\right)^{\frac{1}{2}} \end{cases} \qquad (1)$$

such as the constants $(\gamma_i)_{i=0,\dots,2}$ must be positive, and white noise z_t follows a Gaussian random variable with 0 mean and a variance equal to 1, we can iterate the expression (1), we use recurrent mathematical relations in the original expression

$$\begin{aligned} \sigma_t^2 &= \gamma_0 + \gamma_1 h_{t-1}^2 + \gamma_2\sigma_{t-1}^2 \\ &= \gamma_0 + \gamma_1 z_{t-1}^2\sigma_{t-1}^2 + \gamma_2\sigma_{t-1}^2 \\ &= \gamma_0 + \left(\gamma_1 z_{t-1}^2 + \gamma_2\right)\sigma_{t-1}^2 \end{aligned}$$

Iterating equation (1), we get

$$\sigma_t^2 = \left\{\sigma^2(0) + \gamma_0\int_0^n \exp\left[-\underbrace{\sum_{i=0}^{[u]}\log(\gamma_2 + \gamma_1 z_i^2)}\right]du\right\} \times \exp\left[\underbrace{\sum_{i=0}^{n-1}\log(\gamma_2 + \gamma_1 z_i^2)}\right]$$

where $[u]$ integer part of u, we can write

$$\begin{aligned} c(n) &= \sum_{i=0}^{n-1}\log(\gamma_2 + \gamma_1 z_i^2) = \log\left\{\prod_{i=0}^{n-1}\gamma_2\left(1 + \frac{\gamma_1}{\gamma_2\%} z_i^2\right)\right\} \\ &= \log\prod_{i=0}^{n-1}\gamma_2 + \log\prod_{i=0}^{n-1}\left\{\left(1 + \frac{\gamma_1}{\gamma_2} z_i^2\right)\right\} = n\log(\gamma_2) + \sum_{i=0}^{n-1}\left(1 + \frac{\gamma_1}{\gamma_2} z_i^2\right) \end{aligned}$$

so, moving from the discrete state to the continuous state allows us to put

$\theta = \gamma_1/\gamma_2 - \log(\gamma_2) = \rho$ and $t = n$

$$c(t) = \rho t - \sum_{0<s\le t}\log\left\{1 + \theta(\Delta\ell(s))^2\right\},\ t > 0 \qquad (2)$$

the last equation represents the Lévy process auxiliary, and using Ito's formula it can be verified that the COGARCH (1,1) is a solution of the system

$$dh(t) = \sigma(t)\,d\ell(t),\ \varphi(0) = 0$$

And

$$\sigma^2(t) = \left(\sigma^2(0) + \gamma_0\int_0^t e^{X(t)}ds\right)\exp\left\{-X(\bar{t})\right\},\ t \ge 0$$

such as

$$X(\bar{t}) = \rho t - \sum_{0<s\le t} \log\left\{1+\theta(\Delta\ell(s))^2\right\}$$

Considering that the coefficients ρ and θ are positive, the COGARCH equation (2) is obtained by substituting the white noise by the jumps

$$\Delta\ell(s) = \ell(s) - \ell(\bar{s})$$

such as $\ell(\bar{t})$ is the left limit of the sample path of Lévy process ℓ at $t>0$, the COGARCH (1,1) model defined with two main recurring equations.

$$\begin{cases} dh(t) = \sigma(t)\,\mathrm{d}\ell(t), \quad h(0)=0 \\ d\sigma^2(t) = \left(\gamma_0 + \gamma_1\sigma^2(t)\right)dt + \gamma_2\sigma^2(t)\,d\,[\ell,\ell]^{(d)}(t) \end{cases} \qquad (3)$$

where the sequence of coefficients is always positive, and $[\ell,\ell]^{(d)}$ is the discrete part of quadratic variation of Lévy process which defined by its following expression

$$[\ell,\ell]^{(d)}(t) = \sum_{0<s\le t}\left[\Delta\ell(s)\right]^2$$

we can observe that the jumps of $\Delta h(t)$ is the same jumps of Lévy process $\Delta\ell(s)$ see (Behme, A., Klüppelberg, C., & Mayr, K. ,2014), where it is also worth that the stochastic process

$$[h,h]_t = \int_0^t \sigma^2(\bar{s})\,d\,[h,h]_s$$

Let us recall the fractional derivatives

$[a,\ b]$ $(-\infty < a < b < +\infty)$ be a finite interval of ℝ, the left and right Riemann -Liouville fractional integrals $I^r_{(a,t)}$ and $I^r_{(\mathrm{b},t)}$ of order *r* are defined as follows

$$I^r_{(a,t)}\ f(t) = \frac{1}{\Gamma(r)}\int_a^t (t-u)^{r-1} f(u)du,\ t>a,\ r>0$$

And

$$I^\beta_{(\mathrm{t,b})}\ f(t) = \frac{1}{\Gamma(\beta)}\int_t^b (u-t)^{\beta-1} f(u)du,\ t<b, \quad \beta>0$$

where the gamma function is defined

$$\Gamma(\beta) = \int_0^{+\infty} u^{\beta-1}\exp(-u)du,\ x>0$$

Definition 1 *The left and right Riemann- Liouville fractional derivatives* $d^\beta_{(a,t)}$ *and* $d^\beta_{(t,b)}$ *of order* $\beta>0$ *, are defined*

$$d^\beta_{(a,t)} = \frac{d^n}{dt^n}\left\{I^{n-\beta}_{(a,t)}\left(f(t)\right)\right\} = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_a^t (t-u)^{n-\beta-1} f(u)du,\ t>0$$

And

$$d^\beta_{(t,b)} = (-1)^n\frac{d^n}{dt^n}\left\{I^{n-\beta}_{(t,b)}\left(f(t)\right)\right\} = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_t^b (u-t)^{n-\beta-1} f(u)du,\ t>0$$

Definition 2 *The left and right Caputo fractional derivative* $d_{(a,t)}^{(C,r)}$ *and* $d_{(t,b)}^{(C,r)}$ *of order* $\beta > 0$ *are defined by*

$$d_{(a,t)}^{(C,\beta)} f(t) = d_{(a,t)}^{\beta}\left\{ f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(t-a)^j \right\}$$

$$= d_{(a,t)}^{\beta} f(t) - d_{(a,t)}^{\beta} \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(t-a)^j$$

$$= d_{(a,t)}^{\beta} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(j-\beta+1)}(t-a)^{j-\beta}$$

And the same for $d_{(t,b)}^{(C,\beta)} f(t)$:

$$d_{(t,b)}^{(C,\beta)} f(t) = d_{(t,b)}^{\beta} f(t) - \sum_{j=0}^{n-1} \frac{\%f^{(j)}(b)}{\Gamma(j-\beta+1)}(b-t)^{j-\beta}$$

where $n = [\beta] + 1,$ in the case where $0 < \beta < 1$ we find

$$d_{(a,t)}^{(C,\beta)} f(t) = d_{(a,t)}^{\beta} f(t) - \frac{f(a)}{\Gamma(1-\beta)}(t-a)^{-\beta}$$

and

$$d_{(t,b)}^{(C,\beta)} f(t) = d_{(t,b)}^{\beta} f(t) - \frac{f(b)}{\Gamma(1-\beta)}(b-t)^{-\beta}$$

Definition 3 *We define in the general case the fractional model COGARCH (1,1) with its expression*

$$\begin{cases} dh(t) = \sigma(t) d\ell(t) \\ d_{(t_0,t)}^{(C,\beta)} \sigma^2(t) = \left(\gamma_0 + \gamma_1 \sigma^2(t)\right) dt + \gamma_2 \sigma^2(t) d[\ell,\ell]^{(d)}(t), t > t_0 \\ \sigma^2(t_0) = \Phi \end{cases} \quad (4)$$

where $d_{(a,t)}^{(C,r)}$ is Caputo fractional derivative of order $\beta \in]0,1[,$ and the part $\ell(t)$ is a Lévy process, to determine the

solution of system (4), it is necessary to build some conditions. Let $C(I, \mathbb{R})$ be the space of continuous functions on I we define the norm

$$\|\Phi\| = \sup_{t \in I} |\Phi(t)|$$

In addition

$$g(\sigma^2(t), t) = (\gamma_0 + \gamma_1 \sigma^2(t)) dt + \gamma_2 \sigma^2(t) d[\ell,\ell]^{(d)}(t), \; t > t_0$$

Let

$$A = \{\sigma^2(t) \in C([t_0 - r, t_0 + r], \mathbb{R}), \sup_{t \in [t_0 - r, t_0 + r]} \left|\sigma^2(t) - \sigma^2(0)\right| \le R\} \quad (5)$$

where r and R are positive constants, we introduce the
following hypotheses

- (H-1) g is measurable with b respect to t on I ,

- (H-2) g is continuous with respect to Φ on $C([t_0-r,t_0+r]$.
- (H-3) there exist K constant where $|g|\leq K$.

Lemma 1 *If there exist r and R defined in (5), such that (H-1) and (H-3) are satisfied, then the problem (4) accepts a solution written in the form*

$$\sigma^2(t)=\Phi(0)+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}g(\sigma^2(s),s)ds,\ 0<\beta<1 \quad (6)$$

Proof

First it is easy to obtain that $g(\sigma^2(t),t)$ is Lebesgue measurable on $[t_0-r,t_0+r]$ according to the hypotheses (H-1) and (H-2), we can also deduce according to the hypothesis (H-3) that $(t-s)^{\beta-1}g(\sigma^2(s),s)$ is Lebesgue integrable where $t_0\leq s\leq t$, *and* $\sigma^2(t)\ \in A$, so

$$\left|\int_{t_0}^{t}(t-s)^{\beta-1}g(\sigma^2(s),s)ds\right|\leq\ \int_{t_0}^{t}\left|(t-s)^{\beta-1}g(\sigma^2(s),s)\right|ds\leq\ K\left|(t-s)^{\beta-1}\right|$$

to prove that (6) is a solution we replace it in the expression (4)

$$\Theta=d_{(t_0,t)}^{(C,\beta)}\left\{\Phi(0)+\frac{1}{\Gamma(\beta)\%}\int_{t_0}^{t}(t-s)^{\beta-1}g(\sigma^2(s),s)ds\right\}$$

$$=d_{(t_0,t)}^{(C,\beta)}\left\{\frac{1}{\Gamma(\beta)}\%\int_{t_0}^{t}(t-s)^{\beta-1}g(\sigma^2(s),s)ds\right\},$$

we can notice according to the expression of Caputo with the case that $0<\beta<1$, we have

$$d_{(t,t_0)}^{(C,\beta)}f\ =d_{(t,t_0)}^{\beta}\{f\ (t)\}-\%\frac{f\ (t_0)}{\Gamma(1-\beta)}(t-t_0)^{-\beta}$$

and when

$$I_{(t_0,t)}^{\beta}=\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}g(\sigma^2(s),s)ds$$

then

$$\Theta=d_{(t_0,t)}^{\beta}\left\{I_{(t_0,t)}^{\beta}g(\sigma^2(t),t)\right\}-[I_{(t_0,t)}^{\beta}g(\sigma^2(t),t)]_{t=t_0}\ \frac{(t-t_0)^{-\beta}}{\Gamma(1-\beta)}$$

in this case we arrive that $d_{(t_0,t)}^{\beta}\left\{I_{(t_0,t)}^{\beta}g(\sigma^2(t),t)\right\}=g(\sigma^2(t),t),$ and we know that $[I_{(t_0,t)}^{\beta}g(\sigma^2(t),t)]_{t=t_0}=0,$ which means that

$$d_{(t_0,t)}^{(C,\beta)}\sigma^2(t)=g(\sigma^2(t),t)$$

and this completes the proof. □

After we made sure that represents a solution to the fractional equation. Now let’s analyze the solution itself, so

$$\sigma^2(t) = \Phi(0) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\%\beta-1} g(\sigma^2(s),s)ds,\ 0<\beta<1$$

$$= \Phi(0) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(\gamma_0+\gamma_1\sigma^2(s))ds$$

$$+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}\gamma_2\sigma^2(s)d\left[\ell,\ell\right]^{(d)}(s)$$

3 Estimation for fractional COGARCH (1,1)

In the stochastic literature there are several methods for estimating nonlinear models in the continuous case, but the most famous is the method of moments. For our sample COGARCH models were compiled by statisticians Haug, Kluperllberg, Linder and Zapp in 2005, see reference (Haug, S., Klüppelberg, C., Lindner, A., & Zapp, M. ,2007).

Definition 5 *The processes COGARCH(1,1) governed by the Poisson process corresponding to the Lévy process* (\`$(t))_{t\geq 0\geq}$ *obtained by the following expression*

The processes COGARCH (1,1) governed by the Poisson process corresponding to the Lévy process $(\ell(t))_{t\geq 0}$ obtained by the following expression

$$\ell(t) = \sum_{k=1}^{P(t)} N(t), t\geq 0$$

where $P(t)$ is a Poisson process of intensity $c>0$, $N(t)$ are independent random variables of the Poisson process. Let y be a random variable having the same distribution of $N(t)$, we denote by

by L_y the Lévy measure which verifies the relation $\nu(dx) = cL_y(dx)$

under this information we define the Laplace exponent according to the expression

under this information we define the Laplace exponent according to the
expression

$$\Psi(s) = -\gamma s - c\int_{\square} 1-(1+\theta y^2)^s L_y(dx)$$

We can deduce that

$$\begin{cases}\Psi(1) = -\gamma + c\theta E\left(y^2\right)\\ \Psi(2) = -2\gamma + 2c\theta E\left(y^2\right) + c\theta^2 E\left(y^4\right)\end{cases}$$

If we have $E(\ell(1))=0$ and $var(\ell(1))=1,$ the moment that $E(y^2)=1/c$ we can find a relation between $\Psi(1)$ and $\Psi(2)$

$$\begin{cases}\Psi(1) = \theta - \gamma\\ \Psi(2) = 2\Psi(1) + \theta^2\dfrac{E(y^4)}{E(y^2)}\end{cases}$$

We suppose that $\Psi(2)<0$ it pushes that

$$\theta^2 < \frac{-2(\theta-\gamma)}{cE\,(y^2)}$$

According to this extension we will cite a very important theorem which poses a fundamental approach for estimating the coefficients of a COGARCH model

Theorem 6 We consider that the Lévy process satisfies the following conditions

C_1: $E\,(\ell(1))=0$.

C_2 : $\{Var(\ell(1))\}^r=1$ such as $r\in\{1,4\}$.

C_3: The Brownian motion variance $\tau_\ell^2 \in [0,1[$.

C_4 : $\int_{\mathbb{R}} x^3 \upsilon_\ell(dx)=0$.

C_5: $\Psi(2)<0$.

denoting by $\xi=\left\{h_i^{(1)}, i\in\mathbb{N}\right\}$ the sequence of stationary increments of processes COGARCH (1,1) with the parameters γ_0,γ_1 and γ_2, where

$$h_i^{(1)}=h_i-h_{i-1}$$

we assume the following estimation criteria

$$m=E\left\{(h_i^{(1)})^2\right\}$$

$$\omega(0)=var\left\{(h_i^{(1)})^4\right\}$$

$$p(r)=corr((h_{i+r}^{(1)})^2,(h_i^{(1)})^2)=ke^{-rd}, r\in\mathbb{N}$$

where $d=|\Psi(1)|$, we define the following two quantities expressed by the constants $d, m, \omega(0)$ and k

$$Q_1(k,d)=\omega(0)-2m^2-6\frac{1-d-e^{-d}}{(1-e^d)(1-e^{-d})}k\,\omega(0)$$

$$Q_2(k,d)=\frac{1}{Q_1}\frac{2kd\,\omega(0)}{(e^d-1)(1-e^{-d})}$$

It can be noted that $Q_1, Q_2>0,$ and the formulations of the parameters of Model which defined by its expression (4)

$$\begin{cases} \gamma_0=md \\ \gamma_2=d\,(\sqrt{1+Q_2}-1) \\ \gamma_1=d\,\tau_\ell^2+(1-\tau_\ell^2)d\,(1+Q_2)^{0.5} \end{cases} \qquad (7)$$

Proof see(Behme, A., Klüppelberg, C., & Mayr, K. ,2014).

4 Estimation Algorithm

Simulation can answer certain questions with theory, the tool we use in our digital illustration is Matlab, the main purpose of simulation is to make a comparison between the two models, the first is COGARCH(1 ,1) classical and the second is fractional COGARCH, first we set the generalized simulation steps with the same estimation method. Our estimation needed on the two benchmarks, sample size N and number of simulations NS , let the estimator

$$\hat{m}_N = \frac{1}{N}\sum_{k=1}^{N}(h_{i+k} - h_i)^2$$

We define the following function for any value $s \geq 2$ by its expression

$$\hat{f}_N(s) = \frac{1}{N}\sum_{k=1}^{N-s}\left\{(h_{i+s} - h_i)^2 - \hat{m}_N\right\}\left\{(h_{i+k} - h_i)^2 - \hat{m}_N\right\}$$

where

$$\hat{f}_N(0) = \frac{1}{N}\sum_{k=1}^{N}\left\{(h_{i+s} - h_i)^2 - \hat{m}_N\right\}^2$$

Let be the marginal autocorrelation defined by its vector

$$\hat{V}_{N,s} = \left(\frac{\hat{f}_N(1)}{\hat{f}_N(0)}, \frac{\hat{f}_N(2)}{\hat{f}_N(0)}, \cdots, \frac{\hat{f}_N(s)}{\hat{f}_N(0)}\right)^t$$

Where t is means transpose of a vector, to estimate the parameters γ_0, γ_1 and γ_2 of the model (3), we define the following vectors ×

$$A = (k, d),\ B = (\gamma_0, \gamma_1, \gamma_2)$$

Then, we calculate the estimator $\hat{\mathbf{H}}_N$ according to its generalized formula

$$\hat{\mathbf{H}}_N(\hat{V}_{N,s}, k, d) = \arg\min_{(k,d)\in\mathbb{R}^2} F(\hat{V}_{N,s}, k, d) \qquad (8)$$

where

$$F(\hat{V}_{N,s}, k, d) = \sum_{j=1}^{s}\left\{\log\left(\frac{\hat{f}_N(j)}{\hat{f}_N(0)}\right) - \log(k) + dj\right\}$$

We use the formula that specifies the minimum of $\hat{\mathbf{H}}_N(V_{N,s}, k, d)$, then we obtain

$$\hat{c}_N^* = \left\{\sum_{j=1}^{s}\left(\log\left(\frac{\hat{f}_N(j)}{\hat{f}_N(0)}\right) - \bar{p}_N\right)\left(j - \frac{s+1}{2}\right)\right\} \times \left\{\sum_{j=1}^{s}\left(j - \frac{s+1}{2}\right)^2\right\}^{-1}$$

Corollary 7 When the conditions of the previous theorem hold, we can obtain when N

tends to ∞.

$$\hat{m}_N \to E(h^2(t)) \text{ and } \qquad \hat{f}_N \to f.$$

And finally, we use the estimators $k, d, \hat{m}_N$ and $\hat{f}_N$, and replace these parameters in the formula that presents the parameters γ_0, γ_1 and γ_2 (7)

$$\begin{cases} \hat{\gamma}_{0,N} = \hat{m}_N\ \hat{d}_N \\ \hat{\gamma}_{2,N} = \hat{d}_N\left(\sqrt{1 + Q_2(\hat{k}_N, \hat{d}_N)} - 1\right) \\ \hat{\gamma}_{1,N} = \hat{d}_N\ \tau_\ell^2 + (1 - \tau_\ell^2)\ \hat{d}_N\left(\sqrt{1 + Q_2(\hat{k}_N, \hat{d}_N)}\right) \end{cases}$$

5 Numerical illustration and simulations

This section is devoted to the simulation of the model

$$dh(t) = \sigma(t) d\ell(t)$$
$$d_{(t_0,t)}^{(C,\beta)} \sigma^2(t) = \left(\gamma_0 + \gamma_1 \sigma^2(t)\right)dt + \gamma_2 \sigma^2(t) d\,[\ell,\ell]^{(d)}(t),\ \beta \in \left]0, 1\right].$$

The case $\beta = 1$ represents the simulation of a classic COGARCH model, which shows that fractional derivation is a generalization of normal derivation. The study divided into two samples, the classic and fractional sample $\beta \in \left]0,1\right[$, we keep the same values for coefficients γ_0, γ_1 and γ_2 in order to make a

constructive comparison between the two situations. we use here some convergence criteria noted as follows, γ_i, $i = 0,...,2$ true values, but compared to the estimated values noted $\hat{\gamma}_{i,N}$, $i = 0,...,2$, where N sample size, NS (number of simulations) , $RMSE$ noted the root mean square error, Kurtosis (ku) , here we consider the true values to be $\gamma_0 = 0.005, \gamma_1 = 0.01, and\ \gamma_2 = 0.2$

Tableau 1: Estimation for COGARCH classic with true values $\gamma_0 = 0.0027, \gamma_1 = 0.013, and\ \gamma_2 = 0.3$

	N	NS	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
$\beta = 1$	300	250	0.0112	0.0145	0.3878
	1800	1200	0.0022	0.0276	0.2829
	3000	2400	0.0026	0.134	0.2764

Tableau 2: Estimation for fractional COGARCH with true values $\gamma_0 = 0.0027, \gamma_1 = 0.013, and\ \gamma_2 = 0.3$

	N	NS	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
$\beta = 0.7$	300	250	0.3225	0.1968	1.1245
	1800	1200	0.2658	0.9485	1.0569
	3000	2400	0.3654	0.6584	0.8974
$\beta = 1$	300	250	0.0065	0.1758	0.6521
	1800	1200	0.0054	0.1236	0.2589
	3000	2400	0.0030	0.0132	0.3211
$\beta = 1$	300	250	0.0106	0.0126	0.7856
	1800	1200	0.0033	0.0137	0.3952
	3000	2400	0.0029	0.0124	0.3087

Tableau 3: $RMSE$ for all situations of simulation

β	N	NS	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
0.7	1800	1200	0.2862	0.0919	0.0762
0.8	1800	1200	0.2761	0.0683	0.0623
0.9	3000	1200	0.0312	0.0413	0.0409

Tableau 4: ku for all situations of simulation

β	N	NS	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
0.7	**300**	**250**	**0.0364**	**1.0023**	**0.3658**
0.8	**1800**	**1200**	**0.1542**	**0.3654**	**0.1524**
0.9	**3000**	**2400**	**3.0468**	**3.3317**	**2.8741**

6 Comments and conclusion

Numerical illustration of our model simulation shows that the increase in the size of the samples N gives a better approximation of the estimators towards the true values, on the other hand also the increase in the number of simulations NS ensures a convergence between the estimated values and the real values, we observe when β tends to 1 then we will find the best approximation which shows that the COGARCH classic is the best approximation in our simulation.

We observe that the RMSE criterion tends to zero where β approaches 1 with some perturbations.

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