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Research Article

Common Fixed Point Theorems For Weakly Compatible Mappings In Symmetric Spaces

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Abstract
In this paper some common fixed point theorem has been established by using weakly compatible mappings and E.A property which improves and extends similar known results in the literature.
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1 INTRODUCTION:

The fixed point theory has become a part of non-linear functional analysis since 1960. It serves as an essential tool for various branches of mathematical analysis and its applications. Polish mathematician S. Banach[3] published his contraction Principle in 1922. In 1928, K. Menger[8] introduced semi-metric space as a generalization of metric space. In 1976, M. Cicchese [6] introduced the notion of a contractive mapping in semi-metric space and proved the first fixed point theorem for this class of spaces. Jungck [7] introduced the concept of compatible mappings in metric spaces. This concept has been frequently used to prove existence theorem in common fixed point theory. However, the study of common fixed point theorems for non-compatible mappings has also become interesting concept. Pant et al. [9] intially proved some common fixed point theorems for non-compatible mappings. Aamri et al. [1] gave a notion of E.A. property and established some common fixed point theorems for non-compatible mappings under contractive conditions. Cho and Kim [4] generalized the paper of M. Aamri and Moutawakil by replacing W4 property by C.C property with different contractive conditions. Cho et al. [5] proved some common fixed point theorems for weakly compatible mappings in symmetric spaces and gave some counter examples.

In this paper, some common fixed point theorems for six mappings have been established by using weakly compatible mappings. Section 2.1 consists of brief introduction to the contents. In Section 2, we have recorded necessary definitions with examples which will be useful in the sequel. In Section 3, some common fixed point theorem has been established by using weakly compatible mappings and E.A property.

2. BASIC DEFINITIONS:

Let X be a non-empty set. A symmetric (semi-metric) on a set X is a nonnegative real valued function d on $X \times X$ such that

(i) $d(x,y) = 0$ if and only if $x = y$ for $x, y \in X$.

(ii) $d(x,y) = d(y,x)$, for $x, y \in X$.

Example 2.1: Consider $X = R$ be the set of all real numbers. Let a function d be defined as follows:

$$d(x,y) = \begin{cases} |x-y|, & x \text{ and } y \text{ both are rational or irrational} \\ |x-y|^{-1} & \text{Otherwise} \end{cases}$$

Then, (X,d) is a symmetric (semi-metric) space but not a metric space because the property of triangle inequality is not satisfied by d.

Example 2.2: Consider $X = [0,1]$. Let a function d be defined by $d(x,y) = (x-y)^2$. Then, (X,d) is a symmetric (semi-metric) space but not a metric space because the property of triangle inequality is not satisfied by d.

Let d be a symmetric on a set X and for $r>0$ and any $x \in X$, let $B(x,r)=\{y \in X : d(x,y)<r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x,r) \subset U$ for some $r>0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r>0$, $B(x,r)$ is a neighbourhood of x in the topology $t(d)$. Note that $lim_{n\to\infty} d(x_n,x)=0$ if and only if $x_n \to x$ in the topology $t(d)$.

The difference between symmetric and metric space occurring due to triangle inequality. To obtain fixed point theorems in a symmetric (semi-metric) space, we require some additional axioms.

The following two axioms were given by Wilson [206]. Let (X,d) be a symmetric space.

(W3) Given a sequence $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y) = 0$ implies $x = y$.

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(W4) Given sequences $\{x_n\}, \{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ implies that $\lim_{n\to\infty} d(y_n, x) = 0$.
It is easy to see that for a semi-metric d, if $t(d)$ is a Hausdorff, then (W3) holds.
The following axiom was given by Aamri et al. [2].
Let (X,d) be a symmetric space.
(H.E) for given seqences $\{x_n\}$, $\{y_n\}$ and x in X, $lim_{n\to\infty}\, d(x_n, x)=0$ and $lim_{n\to\infty}\, d(y_n, x)=0$ implies $lim_{n\to\infty}\, d(x_n,y_n)=0$.

Preposition 3: For axioms in symmetric space (X, d),
(a) $(W4) \Rightarrow (W3)$
Proof: Let $\{x_n\}$ be a sequence in X and $x, y \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$
By substituting $y_n = y$ for each $n \in N$, we get $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n, y) = 0$. By using (W4) we get,

$$0 = \lim_{n\to\infty} d(y_n, x) = d(y, x).$$

In the sequel ϕ :$R_+ \to R_+$ be a function defined by $0<\phi(t)<t$ for all $t > 0$.

Example 2.4: Let $X = [0, \infty)$. Consider

$$d(x,y) = \begin{cases} |x-y|, & (x \neq 0, y \neq 0) \\ \frac{1}{x}, & (x \neq 0) \end{cases}$$

Thus, (X, d) is a symmetric space which satisfies (W4) but does not satisfy (H.E) for $x_n = n,\ y_n = n + 1$.

Example 2.5: Let $X = [0,1] \cup \{2\}$. Consider

$$d(x,y) = \begin{cases} |x|, & (0 < x \le 1, y = 2) \\ |x-y| & (0 \le x \le 1, 0 \le y \le 1) \end{cases}$$

and $d(0,2) = 1$.
Thus, (X, d) is a symmetric space which satisfies (H.E). Let $x_n = \frac{1}{n}$
Then, $\lim_{n\to\infty} d(x_n, 0) = \lim_{n\to\infty} d(x_n, 2) = 0$ but $d(0,2) \neq 0$. Therefore, symmetric space (X, d) does not satisfy (W3).

Example 2.6:
(i) Every metric space (X,d) satisfies property (H.E).
(ii) Let $X=[0,\infty)$ with the symmetric function d defined by $d(x,y) = e^{|y-x|} - 1$. Clearly, symmetric space (X,d) satisfies property (H.E).

Definition 2.7: Let X be a non empty set and $S, T: X \to X$ be an arbitrary mappings. A point $x \in X$ is called a coincidence point of S and T if and only if $Sx = Tx$.

Example 2.8: Let S and T be two self maps on $X = R$ defined by $S(x) = x^2 + 1$ and $T(x) = e^x$. Here $S(0) = T(0) = 1$, this imply, $S(0) = T(0)$. Hence $0 \in X$ is a coincidence point of S and T.

Definition 2.9: Let S and T be two self mappings of a symmetric space (X,d). S and T are said to be compatible if $lim_{n\to\infty}\, d(STx_n, TSx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty}\, d(Sx_n, t)= lim_{n\to\infty}\, d(Tx_n, t)=0$ for some $t\in X$.

Definition 2.10: Two self mappings S and T of a symmetric space (X, d) will be non compatible if there exist at least one sequence $\{x_n\}$ in X such that $lim_{n\to\infty}\, d(Sx_n, t)= lim_{n\to\infty}\, d(Tx_n, t)=0$ for some $t\in X$ but $lim_{n\to\infty}\, d(STx_n, TSx_n)$ is either non-zero or does not exists. Therefore, two non-compatible self mapping of a symmetric space (X, d) satisfy the property (E.A).

Definition 2.11: Two self mappings S and T of a symmetric space (X,d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.12: Let S and T be two self mappings of a symmetric space (X,d). We say that S and T satisfy the property (E.A) if there exist a sequence $\{x_n\}$ such that

$$lim_{n\to\infty}\, d(Sx_n, t)= lim_{n\to\infty}\, d(Tx_n, t)=0 \text{ for some } t\in X.$$

Example 2.13: Let $X=[0,\infty)$. Let d be a symmetric on X defined by $d(x,y)=e^{|y-x|}-1$ for all x, y in X. Define S, T:$X\to X$ as follows:

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$Sx = 2x + 1$ and $Tx = x + 2$, for all $x \in X$.
Here function d is not a metric. Consider the sequence $x_n = 1 + 1/n$, $n = 1, 2, \ldots$
Clearly $\lim_{n\to\infty} d(Sx_n, 3) = \lim_{n\to\infty} d(Tx_n, 3) = 0$.
Then S and T satisfy property (E.A).

Definition 2.14: A subset A of a symmetric space (X, d) is said to be d −closed if for a sequence $\{x_n\}$ in A and a point $x \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ implies $x \in A$. For a symmetric space (X, d), d −closedness implies $t(d)$ −closedness.

3 MAIN RESULTS

In this section we prove coincidence point theorem for six mappings satisfying the property (E.A) under some contractive conditions.

Theorem 3.1:
Let (X, d) be a symmetric (semi-metric) space that satisfies axioms (W3) and (H.E). Let A, B, S, T, I and J be self mappings of X such that

(i) $AB(X) \subset J(X)$ and $ST(X) \subset I(X)$ … (1)
(ii) the pair (ST,J) satisfies property (E.A) (resp. (AB,I) satisfies property (E.A)) … (2)
(iii) for any $x, y \in X$ $(x \neq y)$, $d(ABx, STy) < m(x, y)$, where
$m(x, y) = \max\{d(Ix, Jy), \min\{d(ABx, Ix), d(STy, Jy)\}, \min\{d(ABx, Jy), d(STy, Ix)\}\}$ … (3)
(iv) I(X) is d-closed ($\tau(d)$-closed) subset of X (resp., J(X) is a d-closed ($\tau(d)$-closed) subset of X. … (4)

Then, the pair (AB,I) as well as (ST,J) have a coincidence point.
Moreover, if the pairs (AB,I) and (ST,J) are weakly compatible then AB, ST, I and J have a unique common fixed point in X.
Further, if AB=BA, AI=IA, BI=IB, ST=TS, SJ=JS and TJ=JT then A, B,S, T, I and J have a unique common fixed point.

Proof: In view of (2), the pair (ST,J) satisfy property (E.A) therefore, there exists a sequence $\{x_n\}$ in X and a point $u \in X$ such that
$$\lim_{n\to\infty} d(STx_n, u) = \lim_{n\to\infty} d(Jx_n, u) = 0$$
From (2.1), since $ST(X) \subset I(X)$ there exists a sequence $\{y_n\}$ in X such that
$$STx_n = Iy_n.$$
Hence, $\lim_{n\to\infty} d(Iy_n, u) = 0$
By using the property (H.E), we have
$$\lim_{n\to\infty} d(STx_n, Jx_n) = 0 \text{ and } \lim_{n\to\infty} d(Iy_n, Jx_n) = 0$$
From (2.4), since I(X) is a d-closed, hence
$$\lim_{n\to\infty} Iy_n = u \in I(X).$$
Therefore, there exists a point $v \in X$ such that $Iv = u$.
Now, we assert that $ABv = Iv$.
From condition (3), we have
$$d(ABv, STx_n) < max\{d(Iv, Jx_n), min\{d(ABv, Iv), d(STx_n, Jx_n)\}, min\{d(ABv, Jx_n), d(STx_n, Iv)\}\}$$
By taking $n \to \infty$, we have
$$\lim_{n\to\infty} d(ABv, STx_n) = 0$$
By using (W3), we have
$$\lim_{n\to\infty} d(STx_n, ABv) = 0 \text{ and } \lim_{n\to\infty} d(STx_n, Iv) = 0 \text{ implies } ABv = Iv$$
This shows that v is a coincidence point of pair (AB,I).
Again, $AB(X) \subset J(X)$, there exists a point $w \in X$ such that $ABv = Jw$.
We assert that $Jw = STw$.
If not, then from (3), we have,
$$\begin{aligned} d(ABv, STw) &< max\{d(Iv, Jw), min\{d(ABv, Iv), d(STw, Jw)\}, min\{d(ABv, Jw), d(STw, Iv)\}\} \\ &= max\{d(Jw, Jw), min\{d(ABv, ABv), d(STw, Jw)\}, min\{d(ABv, ABv), d(STw, Iv)\}\} \\ &= 0 \end{aligned}$$
which is a contradiction.
Hence, $ABv = STw$ and $ABv = Iv = STw = Jw$. This shows that w is a coincidence point of pair (ST, J).
Consequently, $ABv = Iv = STw = Jw$ which shows that the pairs (AB, I) and (ST, J) have a coincidence point v and w respectively.

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Now, we have $ABv = Iv = STw = Jw = z$ (say)
Since the pair (AB, I) is weakly compatible, hence
$ABz = ABIv = IABv = Iz.$
Suppose, $ABz \neq z$. From (3), we have

$$\begin{aligned} d(ABz, z) = d(ABz, STw) &< max\{d(Iz, Jw), min\{d(ABz, Iz), d(STw, Jw)\}, \\ &\quad min\{d(ABz, Jz), d(STw, Iz)\}\} \\ &< max\{d(ABz, STw), min\{d(ABz, ABz), d(STw, STw)\}, \\ &\quad min\{d(ABz, STw), d(STw, ABz)\}\} \\ &< \max\{d(ABz, STw), 0, d(ABz, STw)\} \\ &< d(ABz, STw), \text{ which is a contradiction.} \end{aligned}$$

Thus, $ABz = z$, which implies that $ABz = z = Iz$.
Again, since the pair (ST,J) is weakly compatible, we have
$STz = STJw = JSTw = Jz.$
Suppose, $STz \neq z$. From (3), we have

$$\begin{aligned} d(z, STz) = d(ABz, STz) &< max\{d(Iz, Jz), min\{d(ABz, Iz), d(STz, Jz)\}, \\ &\quad min\{d(ABz, Jz), d(STz, Iz)\}\} \\ &< max\{d(ABz, STz), min\{d(ABz, ABz), d(STz, STz)\}, \\ &\quad min\{d(ABz, STz), d(STz, ABz)\}\} \\ &< max\{d(ABz, STz), \min\{0,0\}, d(ABz, STz)\} \\ &< d(ABz, STz), \text{ which is a contradiction.} \end{aligned}$$

Thus, $STz = z = Jz$.
Consequently, $ABz = Iz = STz = Jz = z$ which follows that z is a common fixed point of AB, ST, I and J.

To prove that z is unique, let z' be another common fixed point of AB,ST, I and J. If, $z \neq z'$ then from (3), we have

$$\begin{aligned} d(z, z') = d(ABz, STz') &< max\{d(Iz, Jz'), min\{d(ABz, Iz), d(STz', Jz')\}, min\{d(ABz, Jz'), d(STz', Iz)\}\} \\ &< max\{d(z, z'), \min\{d(z, z), d(z', z')\}, min\{d(z, z'), d(z', z)\}\} \\ &< max\{d(z, z'), 0, d(z, z')\} \\ &< d(z, z'), \text{ which is a contradiction.} \end{aligned}$$

Thus, $z = z'$.

Now, we prove that z is a common fixed point of mappings A, B, S, T, I and J. Let both the pairs (AB, I) and (ST, J) have a unique common fixed point z. Then,

$$Az = A(ABz) = A(BAz) = AB(Az)$$
$$Az = A(Iz) = I(Az)$$
$$Bz = B(ABz) = B\big(A(Bz)\big) = BA(Bz) = AB(Bz)$$
$$Bz = B(Iz) = I(Bz)$$

which implies that (AB, I) has a common fixed points which are Az and Bz.
We get thereby $Az = z = Bz = Iz = ABz$, by virtue of uniqueness of common fixed point of pair (AB, I).
Similarly, on using $ST = TS, SJ = JS$ and $TJ = JT$, $Sz = z = Tz = Jz = STz$ can be shown.
Now, to show that $Az = Sz(Bz = Tz)$
We have,

$$\begin{aligned} d(Az, Sz) &= d(A(BAz), S(TSz)) \\ &= d(AB(Az), ST(Sz)) \\ &< max\{d(I(Az), J(Sz)), min\{d(AB(Az), I(Az)), d(ST(Sz), J(Sz))\}, \min\{d(AB(Az), J(Sz)), \\ &\quad d(ST(Sz), I(Az))\}\} \\ &< max\{d(Az, Sz), min\{d(Az, Az), d(Sz, Sz)\}, min\{d(Az, Sz), d(Sz, Az)\}\} \\ &< max\{d(Az, Sz), 0, d(Az, Sz)\} \\ &< \mathrm{d}(Az, Sz), \text{ which is a contradiction.} \end{aligned}$$

Therefore, $Az = Sz$. Similarly, $Bz = Tz$ can be shown.
Thus, z is a unique common fixed point of A, B, S, T, I and J.

Our first corollary is obtained by putting AB=A, ST=B, I=S and J=T in main theorem (3.1) which gives the result of Cho et al. [5].

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Corollary 3.2:
Let (X, d) be a symmetric (semi-metric) space that satisfies (W3) and (H.E). Let A, B, S and T be self mappings of X such that

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$

(ii) the pair (B,T) satisfies properties (E.A) (resp., (A,S) satisfies property (E.A))

(iii) for any $(x, y) \in X$ $(x \neq y)$, $d(Ax, By) < m(x, y)$, where
$m(x, y) = max\ \{d(Sx, Ty), min\{d(Ax, Sx), d(By, Ty)\}, min\{d(Ax, Ty), d(By, Sx)\}\}$

(iv) S(X) is a d-closed ($\tau(d)$ $-$ closed) subset of X (resp., T(X) is a d-closed ($\tau(d)$ $-$ closed) subset of X).

Then, the pairs (A,S) as well as (B,T) have a coincidence point.
Moreover, if the pairs (A,S) and (B,T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

Remark 3.3: By putting $AB = ST = g$ and $I = J = f$ in main therem (3.1), we can show that f and g have a coincidence point i.e. f and g have a unique common fixed point without making the assumption $g(X) \subset f(X)$.

Example 3.4: Let $X = R$, the set of all real numbers with d defined by
$d(x, y) = (x - y)^2$.
Let A, B, S, T, I and J be self mappings defined on X by

$$Ax = \frac{x+2}{3}, Bx = -3x + 4, Sx = \frac{x+1}{2},$$
$$Tx = -2x + 3, Ix = Jx = -x + 2$$

Then A, B, S, T, I and J satisfy all the conditions of the theorem (3.1) and have a unique common fixed point at $x = 1$.

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