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Research Article

Common Fixed Point Theorem For Rational Expressions In 2-Banach Spaces

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Abstract
The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non-linear analysis. Gahlar [6] introduced the concept of 2- Banach spaces. Badshah and Gupta [3] , Yadava, Rajput, Choudhary and Bhardwaj [26] proved some results on fixed point in 2-Banach spaces. Recently Yadava,Rajput, Bhardwaj [25] proved a result on fixed point in 2- Banach spaces for non- contraction mappings. In this paper we prove some common fixed point theorems for non-contraction mappings in 2-Banach spaces, which contains new rational expressions.
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1. Introduction:

Metric fixed point theory is a rich, interesting and exciting branch of mathematics. It is relatively young but fully developed area of research. Study of the existence of fixed points falls within several domains such as functional analysis, operator theory, general topology. Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as topology, mathematical economics, game theory, approximation theory and initial and boundary value problems in ordinary and partial differential equations. Moreover, recently, the usefulness of this concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major tool in the arsenal of mathematics.
In 1922, Polish mathematician Stephan Banach [2] established Banach's contraction principle (BCP) in his Ph.D. dissertation. It is also known to be Banach fixed point theorem or principle of contraction mapping. It has become milestone to all the students of mathematical analysis to establish new theorems by generalizing this theorem. The BCP has been considered to be very important as it is a source of existence and uniqueness theorem in different branches of sciences. This theorem provides an illustration of the unifying aspects in functional analysis. The important feature of the BCP is that it gives the existence, uniqueness and the sequence of the successive approximation converges to a solution of the problem. Browder [4] was the first mathematician to study non-expansive mappings. Meanwhile Brouwder [4] and Ghode [7] have independently proved a fixed point theorem for non-expansive mapping. Many other mathematicians viz; Datson [5] Goebel [8], Goebel and Zlotkienwicz [9], Goebel, Kirk and Simi [10], Iseki [11], Singh and Chatterjee [21], Sharma and Rajput [20], Rajput and Naroliya [18], Pathak and Maity [16], Qureshi and Singh [17], Sharma and Bhagwan [22], Ahmad and Shakil [1], Shahzad and Udomene [23] have done the generalization of non-expansive mappings as well as non-contraction mappings. Kirk [12, 13 and 14] gave the comprehensive survey concerning fixed point theorems for non- expansive mappings.

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non-linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally non-linear, therefore the fixed point methods specially Banach's contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods. Recently Verma [24] described about the application of Banach's contraction principle [2]. Gahlar [6] introduced the concept of 2-Banach spaces. Recently Badshah and Gupta [3], Yadava, Rajput, Choudhary and Bhardwaj [26], Saluja and Dhakde[19], Malcheski, Lukarevski and Anevska [15] also worked for 2-Banach spaces for non contraction mappings. Several researchers have proved fixed point theorems for contractive conditions satisfying rational inequalities. In this paper we prove some fixed point and common fixed point theorems for non-contraction mappings in 2-Banach spaces, which contains new rational expressions.
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Recently Yadava,Rajput, Bhardwaj [25] proved a results in 2- Banach spaces for non- contraction mappings as follows:

THEOREM 1: Let F be a mapping of Banach spaces X into itself. If F satisfies thefollowing conditions:
(i) $F^2 = I$

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(ii) $$\|F(x)-F(y),a\| \le \alpha \frac{\|x-F(x),a\|\|y-F(x),a\|+\|y-F(y),a\|\|x-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|} + \beta \frac{\|x-F(x),a\|\|y-F(y),a\|+\|y-F(x),a\|\|x-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|} + \gamma \frac{\|x-F(x),a\|\|y-F(y),a\|+\|y-F(x),a\|\|x-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|}$$

$\forall x, y\varepsilon X, x \neq y, a \succ 0, 0 \le \alpha, \beta, \gamma, \delta \prec 1$ and $\alpha + 7\beta + 8\gamma + 4\delta \prec 8$
Then F has fixed point, further if $\beta + 2\delta \prec 2$, then F has unique fixed point.
In this paper, we prove another common fixed point theorem in 2-Banach spaces in which multiplication of two mappings is an identity mapping.
That is *TG = I = GT*

2 PRELIMINARIES

2.1 DEFINITION: In a paper Gahler [6] defined a linear 2-normed space to be pair (L, ||., ||) where L is a Linear space and ||., . || is non negative, real valued functiondefined on L such that $a, b, c \in L$
(i) $\|a, b\| = 0$ if and only if *a* and *b* are linearly dependent
(ii) $\|a, b\| = \|b, a\|$
(iii) $\|a, \beta b\| = |\beta| \|a, \beta b\|$, β is real
(iv) $\|a, b + c\| \le \|a, b\| + \|a, c\|$
Hence ||.,. || is called a 2-norm..

2.2 DEFINITION: A sequence $\{x_n\}$ in a linear 2-Normed space L, is called aonvergent sequence if there is, $x \in L$, such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0 \text{ for all } y \in L.$$

2.3 DEFINITION: A sequence $\{x_n\}$ in a linear 2-Normed space L, is called aCauchy sequence if there exist $y, z \in L$, such that y and z are linearly independent and $\lim_{n\to\infty} \|x_m - x_n, y\| = 0$.

2.4 DEFINITION: A Linear 2-normed space in which every Cauchy sequence is convergent is called 2-Banach Spaces.

3 MAIN RESULT

3.1 THEOREM: Let T and G be two non-expansive mappings of a 2-Banach space X intoitself. T and G satisfy the following conditions:

$$TG = I = GT \text{ where } I \text{ is identity mapping.} \qquad \dots (1)$$

$$\|T(x) - G(y), a\| \le \alpha \frac{\|x-T(x),a\|\|y-G(y),a\|}{\|x-y\|} + \beta \frac{\|T-T(x),a\|\|x-G(y),a\|+\|y-T(x),a\|\|y-G(y),a\|+\|x-y,a\|^2}{\|x-T(x),a\|+\|y-G(y),a\|+\|x-G(y),a\|+\|y-T(y),a\|+\|x-y,a\|} + \gamma\|x-y,a\| + \delta[\|x-T(x),a\| + \|y-G(y),a\|] + \eta[\|x-G(y),a\| + \|y-T(x),a\|] \qquad \dots (2)$$

For all $x \neq y, \alpha, \beta, \gamma, \delta, \eta\varepsilon[0,1[$ with

$$[\|x-T(x),a\| + \|y-G(y),a\|] + \|x-G(y),a\| + \|y-T(x),a\| + \|x-y,a\| \neq 0$$

Then T and G have common fixed point.

PROOF:

Taking $y = \frac{1}{2}\|(T+I)(x)\|, z = G(y), u = 2y - z\|x-y,a\|$,
then $\|z-x,a\| = \|G(y) - TG(x)x,a\|$
So by using (1) and (2), we get,

$$\|z-x,a\| \le \alpha \frac{\|y-G(y),a\|\|T(x)-G(T(x)),a\|}{\|y-T(x)\|} + \beta \frac{\|y-G(y),a\|\|y-G(T(x)),a\|+\|T(x)-G(y),a\|\|T(x)-G(T(x)),a\|+\|y-T(x),,a\|^2}{\|y-G(y),a\|+\|T(x)-G(T(x)),a\|+\|y-G(T(x)),a\|+\|T(x)-G(y),a\|+\|y-T(x),a\|} + \gamma\|y-T(x),a\| + \delta\left[\|y-G(y),a\| + \left\|T(x) - G\big(T(x)\big),a\right\|\right] + \eta\left[\left\|y-G\big(T(x)\big),a\right\| + \|T(x)-G(y),a\|\right]$$

$$\le \alpha \frac{\|y-G(y),a\|\|T(x)-x,a\|}{\frac{1}{2}\|x-T(x),a\|} + \beta \frac{\|y-G(y),a\|\frac{1}{2}\|x-Tx,a\|+\|T(x)-y+y-G(y),a\|\|T(x)-x,a\|+\frac{1}{4}\|x-T(x),,a\|^2}{\|y-T(x),a\|+\|T(x)-x,a\|+\|y-x,a\|+\|y-T(x),a\|} + \gamma\|y-T(x),a\| + \delta[\|y-G(y),a\| + \|T(x)-x,a\|]$$

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$$+\eta[\|y-x\,,a\|+\|T(x)-y+y-G(y),a\|]$$
$$=2\alpha\|y-G(y),a\|+\tfrac{2}{5}\beta\left[\tfrac{3}{2}\|y-G(y),a\|+\tfrac{3}{4}\|T(x)-x,a\|\right]$$
$$+\tfrac{1}{2}\gamma\|x-T(x),a\|+\delta\|y-G(y),a\|+\|T(x)-x,a\|$$
$$+\eta[\|x-T(x)\,,a\|+\|y-G(y),a\|]$$
$$\|z-x,a\|\leq\|x-T(x),a\|\left[\frac{3\beta}{10}+\frac{\gamma}{2}+\delta+\eta\right]$$
$$+\|y-G(y),a\|\left[2\alpha+\tfrac{6\beta}{10}+\delta+\eta\right] \quad \ldots (3)$$

Now we calculate $\|u-x,a\|$,

$$\|u-x,a\|=\|2y-z,a\|=\|T(x)-G(y),a\|$$
$$\leq\alpha\frac{\|x-T(x),a\|\|y-G(y),a\|}{\|x-y-a\|}$$
$$+\beta\frac{\|T-T(x),a\|\|x-G(y),a\|+\|y-T(x),a\|\|y-G(y),a\|+\|x-y,a\|^2}{\|x-T(x)\|+\|y-G(y)\|+\|x-G(y)\|+\|y-T(x),a\|+\|x-y\|}$$
$$+\gamma\|x-y,a\|+\delta[\|x-T(x)\,,a\|+\|y-G(y)\,,a\|]$$
$$+\eta[\|x-G(y)\,,a\|+\|y-T(x)\,,a\|]$$
$$\leq\alpha\frac{\|x-T(x),a\|\|y-G(y),a\|}{\frac{1}{2}\|x-Tx-a\|}$$
$$+\beta\frac{\|T-T(x),a\|[\|x-y,a\|+\|y-Gy,a\|]+\frac{1}{2}\|x-T(x),a\|+\|y-G(y),a\|+\frac{1}{4}\|x-T(x),a\|^2}{\frac{5}{2}\|x-T(x),a\|}$$
$$+\tfrac{1}{2}\gamma\|x-T(x),a\|+\delta[\|x-T(x)\,,a\|+\|y-G(y)\,,a\|]$$
$$+\eta\left[\tfrac{1}{2}\|x-T(x),a\|+\|G(y)-y,a\|+\tfrac{1}{2}\|x-T(x),a\|\right]$$
$$\|u-x,a\|\leq\|x-T(x),a\|\left[\frac{3\beta}{10}+\frac{\gamma}{2}+\delta+\eta\right]$$
$$+\|y-G(y),a\|\left[2\alpha+\tfrac{6\beta}{10}+\delta+\eta\right] \quad \ldots (4)$$

Now,

$$\|z-u,a\|\leq\|z-x,a\|+\|u-x,a\|$$
$$\|z-u,a\|\leq\|x-T(x),a\|\left[\tfrac{3\beta}{5}+\tfrac{\gamma}{1}+2\delta+2\eta\right]$$
$$+\|y-G(y),a\|\left[4\alpha+\tfrac{6\beta}{5}+2\delta+2\eta\right] \quad \ldots (5)$$

But $\|z-u,a\|=\|G(y)-2y+z\|=2\|G(y)-y\|$ … (6)

By (5) and (6)

$$2\|y-G(y)\,,a\|\leq\|x-T(x),a\|\left[\tfrac{3\beta}{5}+\gamma+2\delta+2\eta\right]$$
$$+\|y-G(y),a\|\left[4\alpha+\tfrac{6\beta}{5}+2\delta+2\eta\right]$$
$$\|y-G(y),a\|\leq S\|x-T(x),a\|,\text{ where } S=\frac{\left[\frac{3\beta}{5}+\frac{\gamma}{1}+2\delta+2\eta\right]}{2-\left[4\alpha+\frac{6\beta}{5}+2\delta+2\eta\right]}<1 \quad \ldots (7)$$

Because $20\alpha+9\beta+5\gamma+20\delta+20\eta<10$

Let $F=\frac{1}{2}[T+I]$, then for any $x\varepsilon X$

$$\|F^2(x)-F(x),a\|=\|F(Fx)-F(x),a\|=\|F(y)-y,a\|=\tfrac{1}{2}\|y-T(y),a\|$$
$$=\tfrac{1}{2}\|TG(y)-T(y),a\|\leq\tfrac{1}{2}\|G(y)-(y),a\|$$

Because T is non-expansive. So $\|F^2(x)-F(x),a\|\leq\frac{S}{2}\|x-T(x),a\|$

By the definition of S, we claim that $F_n(x)$ is a Cauchy sequence in X. Also by the completeness, $F_n(x)$ converges to some element (x_0) in X.

i.e $\lim_{n\to\infty}F^n(x)=x_0\Rightarrow F(x_0)=x_0$.

Hence $T(x_0)=x_0$. That is x_0 is fixed point of T.

Again

$$\|F^2(x)-F(x),a\|\leq\tfrac{S}{2}\|x-T(x),a\|$$
$$=\tfrac{S}{2}\|TG(x)-T(x),a\|$$
$$\leq\tfrac{S}{2}\|x-G(x),a\|$$

We can conclude that $G(x_0)=x_0$. That is x_0 is fixed point of G.

So $T(x_0)=G(x_0)=x_0$. So x_0 is common fixed point of T and G. The uniqueness part is trival.

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