STRONG SPLIT DOMINATION POLYNOMIAL OF PATHS

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Abstract

Let $G = (V(G), E(G))$ be a simple graph. A dominating set $D \subseteq V(G)$ is a strong split dominating set if the induced subgraph $< V - D >$ is totally disconnected with at least two vertices. Let $\mathcal{D}_{ss}(G,i)$ be the family of strong split dominating sets of G of cardinality i and $|\mathcal{D}_{ss}(G,i)| = d_{ss}(G,i)$. We define the strong split domination polynomial of a graph G of order n as the polynomial $D_{ss}(G,x) = \sum_{i=\gamma_{ss}(G)}^{n-2} d_{ss}(G,i)x^i$. In this paper, we determine the strong split domination polynomial of paths and obtain some of its properties.[1]

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a γ-set. For a detailed treatment of this parameter the reader is referred to [2]. In [1], S Alikhani and Y H Peng has found the recursive relation for the domination polynomial of paths. Now in the same way we find the recursive relation for the strong split domination polynomial of paths.

A dominating set $D \subseteq V(G)$ is a strong split dominating set if the induced subgraph $< V - D >$ is totally disconnected with at least two vertices. The strong split domination number is the minimum size of a strong split dominating set of G and is denoted by $\gamma_{ss}(G)$. Strong split domination in graph was introduced by Kulli and Janakiraman in [3]. For more details on strong split domination we refer [4]. It is immediate that for any path P_n, $\gamma_{ss}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ [4].

Definition 1. [5] Let $\mathcal{D}_{ss}(G,i)$ be the collection of strong split dominating sets of G of cardinality i and $|\mathcal{D}_{ss}(G,i)| = d_{ss}(G,i)$. The strong split domination polynomial of G is defined as $D_{ss}(G,x) = \sum_{i=\gamma_{ss}(G)}^{n-2} d_{ss}(G,i)x^i$.

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2. Construction of Strong Split Dominating Sets of Paths

A Path is a graph whose vertices can be listed in the order $\{u_1, u_2, \dots, u_n\}$ such that the edges are $\{u_i u_{i+1}\}$ where $i = 1,2,3,\dots,n-1$. Let $\mathcal{D}_{ss}(P_n, i)$ be the collection of strong split dominating sets of P_n with cardinality i.

Observation 1. *For any path P_n,*

1. $\mathcal{D}_{ss}(P_n, i) = \emptyset$ *if and only if* $i > n-2$ *or* $i < \left\lfloor \frac{n}{2} \right\rfloor$
2. $\mathcal{D}_{ss}(P_n, i) \neq \emptyset$ *if and only if* $\left\lfloor \frac{n}{2} \right\rfloor < i < n-2$
3. *To find a strong split domination polynomial of P_n with cardinality i, it is enough to consider $\mathcal{D}_{ss}(P_{n-1}, i-1)$ and $\mathcal{D}_{ss}(P_{n-2}, i-1)$. Thus we have to consider four combinations of whether these two collections are empty or not.*

Lemma 1. If $\mathcal{D}_{ss}(P_{n-1}, i-1) = \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) = \emptyset$, then $\mathcal{D}_{ss}(P_n, i) = \emptyset$.

Proof. Suppose that $\mathcal{D}_{ss}(P_n, i) \neq \emptyset$. Let $\mathcal{A} \in \mathcal{D}_{ss}(P_n, i)$. Then at least one vertex n or $n-1$ is in $\mathcal{A}$. If $n \in \mathcal{A}$ then, at least one vertex $n-2$ or $n-3$ is in $\mathcal{A}$. If $n-2 \in \mathcal{A}$, then $\mathcal{A} - \{n\} \in \mathcal{D}_{ss}(P_{n-1}, i-1)$, which is a contradiction. If $n-3 \in \mathcal{A}$, then $\mathcal{A} - \{n\} \in \mathcal{D}_{ss}(P_{n-2}, i-1)$, which is a contradiction.

Now suppose that, $n-1 \in \mathcal{A}$. Then at least one vertex $n-3$ or $n-4$ is in $\mathcal{A}$. If $n-2 \in \mathcal{A}$, then $\mathcal{A} - \{n-1\} \in \mathcal{D}_{ss}(P_{n-1}, i-1)$, which is a contradiction. If $n-3 \in \mathcal{A}$, then $\mathcal{A} - \{n-1\} \in \mathcal{D}_{ss}(P_{n-2}, i-1)$, which is a contradiction.

Hence $\mathcal{D}_{ss}(P_n, i) = \emptyset$.

Lemma 2. *Suppose $\mathcal{D}_{ss}(P_n, i) \neq \emptyset$. Then*

1. $\mathcal{D}_{ss}(P_{n-1}, i-1) = \emptyset$ *and* $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$ *if and only if* $n = 2k+1$ *and* $i = k$ *for some* $k \in \mathbb{N}$.
2. $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ *and* $\mathcal{D}_{ss}(P_{n-2}, i-1) = \emptyset$ *if and only if* $i = n-2$.
3. $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ *and* $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$ *if and only if* $\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \leq i \leq n-2$.

Proof. 1. Assume that $\mathcal{D}_{ss}(P_{n-1}, i-1) = \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$. Since $\mathcal{D}_{ss}(P_{n-1}, i-1) = \emptyset$ by Observation 1, $i-1 > n-3$ or $i-1 < \left\lfloor \frac{n-1}{2} \right\rfloor$. If $i-1 > n-3$, then $i > n-2$ and by Observation 1, $\mathcal{D}_{ss}(P_n, i) = \emptyset$ a contradiction. So $i < \left\lfloor \frac{n-1}{2} \right\rfloor + 1$ and since $\mathcal{D}_{ss}(P_n, i) \neq \emptyset$ together with $\left\lfloor \frac{n}{2} \right\rfloor \leq i < \left\lfloor \frac{n-1}{2} \right\rfloor + 1$, we have $n = 2k+1$ and $i = k$ for some $k \in \mathbb{N}$.

Conversely, if $n = 2k+1$ and $i = k$ for some $k \in \mathbb{N}$, then by Observation 1, Assume that $\mathcal{D}_{ss}(P_{n-1}, i-1) = \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$.

2. Assume that $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) = \emptyset$. Since $\mathcal{D}_{ss}(P_{n-2}, i-1) = \emptyset$, by Observation 1, $i-1 > n-4$ or $i-1 < \left\lfloor\frac{n-2}{2}\right\rfloor$. If $i-1 < \left\lfloor\frac{n-2}{2}\right\rfloor$, then $i-1 < \left\lfloor\frac{n-1}{2}\right\rfloor$ and hence $\mathcal{D}_{ss}(P_{n-1}, i-1) = \emptyset$, a contradiction. So $i > n-3$ and also since $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$, $i-1 \leq n-3$. Therefore, $i = n-2$.
Conversely, if $i = n-2$, then by Observation 1, $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) = \emptyset$.

3. Let us assume that $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$. Then by Observation 1, $\left\lfloor\frac{n-1}{2}\right\rfloor \leq i-1 \leq n-3$ and $\left\lfloor\frac{n-2}{2}\right\rfloor \leq i-1 \leq n-4$. So, $\left\lfloor\frac{n-1}{2}\right\rfloor \leq i-1 \leq n-4$ and hence $\left\lfloor\frac{n-1}{2}\right\rfloor + 1 \leq i \leq n-2$.
Conversely, if $\left\lfloor\frac{n-1}{2}\right\rfloor + 1 \leq i \leq n-2$, then by Observation 1, $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$.

Theorem 1. *Let $n \geq 5$ and $i \geq \left\lfloor\frac{n}{2}\right\rfloor$*

1. *If $\mathcal{D}_{ss}(P_{n-1}, i-1) = \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$, then $\mathcal{D}_{ss}(P_n, i) = \{\{2,4,6,\dots,n-3,n-1\}\}$.*
2. *If $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) = \emptyset$, then $\mathcal{D}_{ss}(P_n, i) = \mathcal{S} \cup \{\{1,2,3,\dots,n-1\} - \{x\}/x \in \{1,2,3,\dots,n-2\}\}$ where $\mathcal{S} = \{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(P_{n-1}, n-3)\}$.*
3. *If $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$, then $\mathcal{D}_{ss}(P_n, i) = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 = \{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(P_{n-1}, i-1)\}$ and $\mathcal{S}_2 = \{X_2 \cup \{n-1\}/X_2 \in \mathcal{D}_{ss}(P_{n-2}, i-1)\}$.*

Proof. 1. Let $\mathcal{D}_{ss}(P_{n-1}, i-1) = \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$. By Lemma 2(1) $n = 2k+1$ and $i = k$ for some $k \in \mathbb{N}$. Then
$\mathcal{D}_{ss}(P_n, i) = \mathcal{D}_{ss}(P_{2k+1}, k) = \{\{2,4,6,\dots,n-3,n-1\}\}$.

2. Let $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) = \emptyset$. By Lemma 2(2), $i = n-2$. Therefore $\{\{1,2,3,\dots,n-1\} - \{x\}/x \in \{1,2,3,\dots,n-2\}\}$ be the collection of strong split dominating sets of P_n of cardinality $n-2$ with the vertex n not included and $\mathcal{S} = \{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(P_{n-1}, n-3)\}$.

3. Let $\mathcal{D}_{ss}(P_{n-1}, i-1) \neq \emptyset$ and $\mathcal{D}_{ss}(P_{n-2}, i-1) \neq \emptyset$ ad assume that $X_1 \in \mathcal{D}_{ss}(P_{n-1}, i-1)$. Then $X_1 \cup \{n\} \in \mathcal{D}_{ss}(P_n, i-1)$. Take $\mathcal{S}_1 = \{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(P_{n-1}, i-1)\}$. Then $\mathcal{S}_1 \subseteq \mathcal{D}_{ss}(P_n, i)$.
Let us assume that $X_2 \in \mathcal{D}_{ss}(P_{n-2}, i-1)$. Then $X_2 \cup \{n-1\} \in \mathcal{D}_{ss}(P_n, i-1)$. Take $\mathcal{S}_2 = \{X_2 \cup \{n-1\}/X_2 \in \mathcal{D}_{ss}(P_{n-2}, i-1)\}$. Then $\mathcal{S}_2 \subseteq \mathcal{D}_{ss}(P_n, i)$. Thus $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{D}_{ss}(P_n, i)$.

Now suppose that $Y \in \mathcal{D}_{ss}(P_n, i)$. Then $n \in Y$ or $n \notin Y$. If $n \in Y$, then there exist $X_1 \in \mathcal{D}_{ss}(P_{n-1}, i-1)$ such that $Y = X_1 \cup \{n\}$. Hence $Y \in \mathcal{S}_1$. If $n \notin Y$, then $n-1 \in Y$

otherwise $Y \notin \mathcal{D}_{ss}(P_n, i)$. If $n-1 \in Y$, then there exist $X_2 \in \mathcal{D}_{ss}(P_{n-2}, i-1)$ such that $X_2 \cup \{n-1\} \in Y$. Thus $Y \in \mathcal{S}_2$. Therefore $\mathcal{D}_{ss}(P_n, i) \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$. Hence the Proof.

Example 1. Consider the Path P_7 on 7 vertices with $V(P_7) = \{1,2,3,4,5,6,7\}$. By using Theorem 1 we construct the strong split dominating sets of $\mathcal{D}_{ss}(P_7, i)$ for $i = 3,4,5$.

Since $\mathcal{D}_{ss}(P_6, 2) = \emptyset$ and $\mathcal{D}_{ss}(P_5, 2) \neq \emptyset$, by Theorem 1(1), $\mathcal{D}_{ss}(P_5, 2) = \{2,4\}$. Therefore $\mathcal{D}_{ss}(P_7, 3) = \{2,4,6\}$.

Since $\mathcal{D}_{ss}(P_6, 3) = \{\{1,3,5\},\{2,4,5\},\{2,4,6\},\{2,3,5\}\}$ and
$\mathcal{D}_{ss}(P_5, 3) = \{\{1,3,5\},\{2,3,4\},\{2,3,5\},\{1,2,4\},\{2,4,5\},\{1,3,4\}\}$, then by Theorem 1(3),

$$\mathcal{D}_{ss}(P_7, 4) = \left\{\begin{matrix} \{1,3,5,7\},\{2,4,5,7\},\{2,4,6,7\},\{2,3,5,7\}, \\ \{1,3,5,6\},\{2,3,4,6\},\{2,3,5,6\},\{1,2,4,6\},\{2,4,5,6\},\{1,3,4,6\} \end{matrix}\right\}.$$

Since $\mathcal{D}_{ss}(P_6, 4) = \left\{\begin{matrix} \{1,3,5,6\},\{2,3,4,6\},\{2,3,5,6\},\{1,2,4,6\},\{2,4,5,6\},\{1,3,4,6\}, \\ \{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}.\{2,3,4,5\} \end{matrix}\right\}$,

$\mathcal{D}_{ss}(P_5, 4) = \emptyset$, then by using Theorem 1(2),

$$\{X_1 \cup \{n\}/X_1 \in \mathcal{D}_{ss}(P_6, 4)\} = \left\{\begin{matrix} \{1,3,5,6,7\},\{2,3,4,6,7\},\{2,3,5,6,7\},\{1,2,4,6,7\}, \\ \{2,4,5,6,7\},\{1,3,4,6,7\},\{1,2,3,5,7\},\{1,2,4,5,7\}, \\ \{1,3,4,5,7\},\{2,3,4,5,7\} \end{matrix}\right\}$$

and $\{\{1,2,3,\dots,6\} - \{x\}/x \in \{1,2,3,4,5\}\} = \left\{\begin{matrix} \{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,4,5,6\}, \\ \{1,3,4,5,6\},\{2,3,4,5,6\} \end{matrix}\right\}$.

$$\text{Thus } \mathcal{D}_{ss}(P_7, 5) = \left\{\begin{matrix} \{1,3,5,6,7\},\{2,3,4,6,7\},\{2,3,5,6,7\},\{1,2,4,6,7\}, \\ \{2,4,5,6,7\},\{1,3,4,6,7\},\{1,2,3,5,7\},\{1,2,4,5,7\}, \\ \{1,3,4,5,7\},\{2,3,4,5,7\},\{1,2,3,4,6\},\{1,2,3,5,6\}, \\ \{1,2,4,5,6\},\{1,3,4,5,6\},\{2,3,4,5,6\} \end{matrix}\right\}.$$

3. Strong Split Domination Polynomial of Paths

In this section we determine the strong split domination Polynomial of Paths and some of its properties.

Definition 2. Let $\mathcal{D}_{ss}(P_n, i)$ be the collection of strong split dominating sets of P_n of cardinality i and $|\mathcal{D}_{ss}(P_n, i)| = d_{ss}(P_n, i)$. Then the strong split domination polynomial of path is defined as $D_{ss}(P_n, x) = \sum_{i=\left\lceil \frac{n}{2} \right\rceil}^{n-2} d_{ss}(P_n, i) x^i$.

Theorem 2. *If $\mathcal{D}_{ss}(P_n, i)$ is the collection of strong split dominating set of cardinality i of P_n, then $|\mathcal{D}_{ss}(P_n, i)| = |\mathcal{D}_{ss}(P_{n-1}, i)| + |\mathcal{D}_{ss}(P_{n-2}, i)| + |\mathcal{D}|$ where* $\mathcal{D} = \{\{1,2,3,\dots,n-1\} - \{x\}/x \in \{1,2,3,\dots,n-2\}\}$.

Proof. By using Theorem 1, the result follows.

Theorem 3. *For every path* $P_n (n \geq 5)$,
$D_{ss}(P_n, x) = x\big(D_{ss}(P_{n-1}, x) + D_{ss}(P_{n-2}, x)\big) + (n-2)x^{n-2}$ *with* $D_{ss}(P_3, x) = x$ *and* $D_{ss}(P_4, x) = 3x^2$.

Proof. By using Theorem 2 and the definition of strong split domination polynomial we get the result.

Theorem 4. *Let* $D_{ss}(P_n, x)$ *be the strong split domination polynomial of path* P_n. *Then the following properties hold.*

1. *For any positive integer* n, $d_{ss}(P_n, n-1) = 0$ *and* $d_{ss}(P_n, n) = 0$.
2. $d_{ss}(P_n, i) = d_{ss}(P_{n-1}, i-1) + d_{ss}(P_{n-2}, i-1)$, *for any positive integer* $\left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n-3$.
3. $d_{ss}(P_{2n+1}, n) = 1$, *for every positive integer* $n \geq 1$.
4. $d_{ss}(P_{2n}, n) = n+1$, *for every positive integer* $n \geq 2$.
5. $d_{ss}(P_n, n-2) = \frac{(n-1)(n-2)}{2}$, *for every positive integer* $n \geq 3$.

Proof. 1. The result follows from Definition 2.

2. It follows from Theorem 2.

3. By Theorem 2(1), $\big\{\{2,4,6, \dots, n-3, n-1\}\big\}$ is the only strong split dominating set of size n. Hence $d_{ss}(P_{2n+1}, n) = 1$.

4. For $= 2$, $\mathcal{D}_{ss}(P_4, 2) = \big\{\{1,3\}, \{2,4\}, \{2,3\}\big\}$. So, $d_{ss}(P_4, 2) = 3$ and $n+1 = 2+1 = 3$. Therefore the result is true for $n = 2$. Assume that the result is true for all positive integers less than n. By (2) and induction hypothesis

$$\begin{aligned} d_{ss}(P_{2n}, n) &= d_{ss}(P_{2n-1}, n-1) + d_{ss}(P_{2n-2}, n-1) \\ &= d_{ss}(P_{2n-1+1-1}, n-1) + d_{ss}\big(P_{2(n-1)}, n-1\big) \\ &= d_{ss}\big(P_{2(n-1)+1}, n-1\big) + d_{ss}\big(P_{2(n-1)}, n-1\big) \end{aligned}$$
$$d_{ss}(P_{2n}, n) = n+1.$$

5. There are $\binom{n}{n-2}$ sets of cardinality $n-2$. In any path P_n, exactly $(n-1)$ pair of vertices are adjacent. So, the number of strong split dominating sets of cardinality $n-2$ will be $d_{ss}(P_n, n-2) = \binom{n}{n-2} - (n-1) = \frac{(n-1)(n-2)}{2}$.

4. Conclusion

In this paper we have found the Strong Split Domination polynomial for Paths. In future we plan to investigate the polynomial for several graph products.

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