Solving Singed Product Cordial Labeling of Corona Products of Paths and the Third Power of Lemniscate Graphs

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Abstract: A graph $G = (V, E)$ is called singed product cordial if it is possible to label the vertex by the function $f: V \rightarrow \{-1,1\}$ and label the edges by $f^*: E \rightarrow \{-1,1\}$, where $f^*(uv) = f(u)f(v)$, $u, v \in V$ so that $|v_{-1} - v_1| \leq 1$ and $|e_{-1} - e_1| \leq 1$. In our work we present necessary and sufficient conditions for which the singed product cordial labeling of corona product of paths and third power of lemniscate graph.
Keywords: Path, Lemniscate, Third power of graph, Corona Product, Singed product cordial labeling.

1. Introduction

Labeling graphs are used widely in different subjects including astronomy and communication networks. The concept of graph labeling was introduced during the sixties' of the last century by Rosa [1]. Many researches have been working with different types of labeling graphs [2] [3]. In 1954 Harray introduced S-cordiality [4]. An excellent reference for this purpose is the survey written by Gallian [5]. All graphs considered in this theme are finite, simple and undirected. The original concept of cordial graphs is due to Chait [3]. He showed that each tree is cordial; an Euerlian graph is not cordial if its size is congruent to $2(mod\ 4)$. Let $G = (V, E)$ be a graph and let $f: V \rightarrow \{-1,1\}$ be a labeling of its vertices, and let the induced edge labeling $f^*: E \rightarrow \{-1,1\}$ be given by $f^*(e) = f(u)f(v)$, where $e = uv$ and $u, v \in V$. Let v_{-1} and v_1 be the numbers of vertices that are labeled by (-1) and 1, respectively, and let e_{-1} and e_1 be the corresponding numbers of edges. Such a labeling is called *signed-cordial* if both $|v_{-1} - v_1| \leq 1$ and $|e_{-1} - e_1| \leq 1$ hold. A graph is called *signed-cordial* if it has a signed-cordial labeling. In [8] J.Devaraj and P.Delphy defined signed graphs, and started by labeling edges and then induced the labeling of vertices. In [9] Jayapal Baskar Babujee and Shobana Loganathan proved that path graph, cycle graphs, star-$K_{1,n}$, Bistar-$B_{n,n}$,P_n^+, $n \geq 3$ and C_n^+, $n \geq 3$ are signed product cordial.

Definition 1. A path with n vertices has $n - 1$ edge and cycle with n vertices has n edges.
Definition 2. The third power of cycle denoted by C_n^3 , is $C_n \cup J$, where J is the set of all edges of the form edges $v_i v_j$ such that $2 \leq d(v_i v_j) \leq 3$ and $i < j$ where $d(v_i v_j)$ is the shortest path from v_i to v_j .
Definition 3. The third power of lemniscate graph is the graph created by the union of two third power of cycles with a common vertex, and it denoted by $L_{n,m}^3 \equiv C_n^3 \sharp C_m^3$.
Definition 4. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona $G_1 \odot G_2$ of two graphs G_1 (with n_1 vertices , m_1 edges) and G_2 (with n_2 vertices , m_2 edges) is defined as the graph obtained by taking one copy of G_1 and copies of G_2 , and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2 . It is easy to see that the corona $G_1 \odot G_2$ that has $n_1 + n_1 n_2$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges.
In this paper we study the singed product cordial of the corona product $P_k \odot L_{n,m}^3$ of paths and the third power of lemniscate graphs, and show that this is singed product cordial for all positive integers k, n, m.

2. Terminologies and Notations

we can use these symbols of labeling as follows

L_{8r}	11(-1)(-1)(-1)(-1)11.....$(repeated\ r\ time)$
L'_{8r}	(-1)(-1)1111(-1)(-1).....$(repeated\ r\ time)$
S_{8r}	(-1)11(-1)1(-1)(-1)1.....$(repeated\ r\ time)$
S'_{8r}	1(-1)(-1)1(-1)11(-1).....$(repeated\ r\ time)$
M_{8r}	(-1)1(-1)11(-1)1(-1).....$(repeated\ r\ time)$
M'_{8r}	1(-1)1(-1)(-1)1(-1)1.....$(repeated\ r\ time)$
N_{4r}	11(-1)(-1).....$(repeated\ r\ time)$
N'_{8r}	(-1)(-1)11.....$(repeated\ r\ time)$
F_{4r}	(-1)1(-1)1.....$(repeated\ r\ time)$
F'_{4r}	1(-1)1(-1).....$(repeated\ r\ time)$
Q_{4r}	1(-1)(-1)1.....$(repeated\ r\ time)$
Q'_{4r}	(-1)11(-1).....$(repeated\ r\ time)$

Table 2.1. The labeling code.

Suppose that A_i, A'_i, A''_i andA'''_i is a collection of labeling of a path P_k where $k = i(mod4)$ and for the special P_k where $\forall k = 1,2,3$ we choose the labeling P_k, P'_k, P''_k andp'''_k.

Suppose that $j = 0,1,2,3$. let B^l_{oe} meaning the labeling of $L^3_{4t+l,4s+j}$ where t is odd and s is even, B^l_{oo} meaning the labeling of $L^3_{4t+l,4s+j}$ where t is odd and s is odd, B^l_{eo} meaning the labeling of $L^3_{4t+l,4s+j}$, where t is even and s is odd and B^l_{ee} finally the labeling of $L^3_{4t+l,4s+j}$ where t is even and s is even.

The labeling $B^j_e$and $B^{j'}_e$ for the particular cases $L^3_{3,4s+j}\forall j = 0,1,2,3$ where s is even and $B^j_o$and $B^{j'}_o$ for the odd s of $L^3_{3,4s+j}\forall j = 0,1,2,3$, let B_1 is the labeling of $L^3_{3,4}$. B^j_{1o} is the labeling of $L^3_{4,4s+j}, \forall j = 0,1,2,3$ where s is odd, B^j_{1e} meaning the labeling of $L^3_{4,4s+j}, \forall j = 0,1,2,3$ where s is even. Finally B_{11} is the labeling of $L^3_{4,4}$.

We use the notation $[A; (B; C)]$ where A is the labeling of P_k, B is the labeling of C^3_n, C is the labeling of C^3_m. if G and H are two graphs where G has m vertices, the labeling of corona $G \odot H$ denoted v_i and e_i ($i = 0,1$) to show the number of vertices and edges respectively labeled by i.

One can see that $v_{-1} - v_1 = (x_0 - x_1) + k.(Y_{-1} - Y_1)$ and $e_{-1} - e_1 = (a_{-1} - a_1) + k.(B_{-1} - B_1) - x_{-1}.(Y_{-1} - Y_1) + x_1.(Y_{-1} - Y_1)$, where k is the order of p. Consider that $Y_{-1} = (y_{-1} + y'_{-1})$ where y_{-1} and y'_{-1} are the number of zero vertices of C^3_nandC^3_m, $Y_1 = (y_1 + y'_1)$ where $y_1$andy'_1 are the number of ones vertices of $C^3_n$and C^3_m, $B_{-1} = (b_{-1} + b'_{-1})$ where b_{-1}andb'_0 are the number of negative ones edges of $C^3_n$and C^3_m and $B_1 = (b_1 + b'_1)$ where $y_o$andy'_o are the number of ones edges of $C^3_n$and C^3_m.

3. Main result

In this section we study the necessary and sufficient condition of the singed product cordial labeling of a corona product of paths and a third power of lemniscate graphs $P_k \odot L^3_{n,m}$ for all $k \geqslant 1$ and $n, m \geqslant 3$. we labeling each cycle separately, and we considered the common vertex as apart of the second cycle, and the labeling of the second cycle is starting from the common vertex. To achieved our target we study the following series of cases.

Table illustrate the vertex and edges of the path P_k where $k = 1,2,3$

$P_k, k = 1,2,3$	$x_{(-1)}$	x_1	a_1	$a_{(-1)}$
$P_1 = (-1)$ $P'_1 = 1$	1 0	0 1	0 0	0 0
$P_2 = (-1)1$ $P'_2 = (-1)_2$ $P''_2 = 1_2$	1 2 0	1 0 2	0 1 1	1 0 0

$P_3 = (-1)1(-1)$ $P_3' = (-1)_3$ $P_3'' = 1_3$ $P_3''' = (-1)_2 1$ $P_3'''' = 1_2(-1)$	2 3 0 2 1	1 0 3 1 2	0 2 2 1 1	2 0 0 1 1

Table (2) Vertex and edges of a path. P_k

The next table illustrate the vertex and edges of a path P_k where $k \equiv i(mod4)\ \forall i = 0,1,2,3$.

$P_k, k \equiv i(mod4) i = 0,1,2,3$	$x_{(-1)}$	x_1	a_1	$a_{(-1)}$
$A_0 = (-1)_{4r}$ $A_0' = F_{4r}$ $A_0'' = 1_{4r}$ $A_0''' = N_{4r}'$	$4r$ $4r-2$ 0 $4r-2$	0 $4r-2$ $4r$ $4r-2$	$4r-1$ 0 $4r-1$ $4r-2$	0 $4r-1$ 0 $4r-3$
$A_1 = (-1)_{4r}(-1)$ $A_1' = F_{4r}(-1)$ $A_1'' = 1_{4r}1$ $A_1''' = N_{4r}'(-1)$	$4r+1$ $4r-1$ 0 $4r-1$	0 $4r-2$ $4r+1$ $4r-2$	$4r$ 0 $4r$ $4r-2$	0 $4r$ 0 $4r-2$
$A_2 = (-1)_{4r}(-1)_2$ $A_2' = F_{4r}(-1)1$ $A_2'' = 1_{4r}1_2$ $A_2''' = N_{4r}'1(-1)$	$4r+2$ $4r-1$ 0 $4r-1$	0 $4r-1$ $4r+2$ $4r-1$	$4r+1$ 0 $4r+1$ $4r-1$	0 $4r+1$ 0 $4r-2$
$A_3 = (-1)_{4r}(-1)_3$ $A_3' = F_{4r}(-1)1(-1)$ $A_3'' = 1_{4r}1_3$ $A_3''' = N_{4r}'(-1)_2 1$	$4r+3$ $4r$ 0 $4r$	0 $4r-1$ $4r+3$ $4r-1$	$4r+2$ 0 $4r+2$ $4r-1$	0 $4r+2$ 0 $4r-1$

Table (3) Vertex and edge of a path. P_k where $k \equiv i(mod4)\ \forall i = 0,1,2,3$

<u>Case (1):</u> $n = 3$, then $P_k \odot L_{3,m}^3$ is singed product cordial .

j	labeling of lemniscate $L_{3,4s+j}^3$	$Y_{(-1)}$	Y_1	B_1	$B_{(-1)}$
0	$B_1 = N_4 1(-1)$ $B_e = 1_2(-1); (-1)L_{8s-8}'N_4'1(-1)$ $B_o = 1_2(-1); 0S_{8s}'1(-1)$	3 $4s+1$ $4s+3$	3 $4s+1$ $4s+3$	4 $12s-1$ $12s+5$	4 $12s-1$ $12s+5$
1	$B_o^1 = (-1)_3; (-1)1_2 L_{8s-8}'1$ $B_o^{1'} = 1_3; 1(-1)_2 L_{8s-8}(-1)$ $B_e^1 = 1_2(-1); (-1)L_{8s-8}'N_4'(-1)1_2$ $B_e^{1'} = (-1)_2 1; 1L_{8s-8}N_4 1(-1)_2$	$4s$ $4s-1$ $4s+1$ $4s+2$	$4s-1$ $4s$ $4s+2$ $4s+1$	$12s-5$ $12s-5$ $12s+1$ $12s+1$	$12s-6$ $12s-6$ $12s$ $12s$
2	$B_o^2 = 1_2(-1); (-1)L_{8s-8}'N_4'$ $B_e^2 = 1_2(-1); (-1)M_{8s}$	$4s$ $4s+2$	$4s$ $4s+2$	$12s-4$ $12s+2$	$12s-4$ $12s+2$
3	$B_o^3 = 1_2(-1); (-1)L_{8s-8}'N_4'1$ $B_o^{3'} = (-1)_2 1; 1L_{8s-8}N_4(-1)$ $B_e^3 = (-1)_3; 1_2 L_{8s-8}N_4 1(-1)_2$ $B_e^{3'} = 1_3; (-1)_2 L_{8s-8}'N_4'(-1)1_2$	$4s$ $4s+1$ $4s+2$ $4s+3$	$4s+1$ $4s$ $4s+3$ $4s+2$	$12s-2$ $12s-2$ $12s+4$ $12s+4$	$12s-3$ $12s-3$ $12s+3$ $12s+3$

Table 1.1. Vertex and edge of a lemniscate $L_{3,4s+j}^3$

By using table (2), we study the singed product cordial of $P_k \odot L_{3,4s+j}^3$ when $k = 1,2,3$.

j	P_k	$L_{3,4s+j}^3$ $j=0,1,2,3$	$v_{(-1)}-v_1$	$e_{(-1)}-e_1$
0	P_1	B_1 B_o B_e	1	1 0 0
1	p_1'	B_o	0	0
2	P_1	B_o B_e	1	0
3	P_1	B_o	0	0
0	p_2	B_o, B_o B_e, B_e	0	1
1	P_2	B_o^1, B_o^1	0	1
2	P_2	B_o^2, B_o^2 B_e^2, B_e^2	0	1
3	p_2	B_o^2, B_o^2	0	1
0	p_3'''	B_o^3, B_o^3, B_o^3 B_e^3, B_e^3, B_e^3	1	0
1	P_3''''	$B_o^{1\prime}, B_o^{1\prime}, B_o^1$ $B_e^{1\prime}, B_e^{1\prime}, B_e^1$	0	0
2	P_3'''	B_o^2, B_o^2, B_o^2 B_e^2, B_e^2, B_e^2	1	0
3	P_3'''	$B_o^3, B_o^3, B_o^{3\prime}$ $B_e^3, B_e^3, B_e^{3\prime}$	0	0

Table 1.2. Vertex and edge of $P_k \odot L_{3,4s+j}^3$

By using table (3), we study the singed product cordial of $P_k \odot L_{3,4s+j}^3$ when $k = i(mode)4\ \forall i = 0,1,2,3$.

j	P_k	$L_{3,4s+j}^3$ $j=0,1,2,3$	$v_{(-1)}-v_1$	$e_{(-1)}-e_1$
0	A_0'''	$B_o, B_o, B_o, B_o\ldots$ $B_e, B_e, B_e, B_e\ldots$	0	-1
1	A_0'''	$B_o^{1\prime}, B_o^{1\prime}, B_o^1, B_o^1\ldots$ $B_e^{1\prime}, B_e^{1\prime}, B_e^1, B_e^1\ldots$	0	-1
2	A_0'''	$B_o^2, B_o^2, B_o^2, B_o^2\ldots$ $B_e^2, B_e^2, B_e^2, B_e^2\ldots$	0	-1
3	A_0'''	$B_o^3, B_o^3, B_o^{3\prime}, B_o^{3\prime}\ldots$ $B_e^3, B_e^3, B_e^{3\prime}, B_e^{3\prime}\ldots$	0	-1
0	A_1'''	$B_o, B_o, B_o, B_o\ldots, B_o$ $B_e, B_e, B_e, B_e\ldots, B_e$	-1	0
1	A_1'''	$B_o^{1\prime}, B_o^{1\prime}, B_o^1, B_o^1\ldots, B_o^{1\prime}$ $B_e^{1\prime}, B_e^{1\prime}, B_e^1, B_e^1\ldots, B_e^{1\prime}$	0	0
2	A_1'''	$B_o^2, B_o^2, B_o^2, B_o^2\ldots, B_o^2$ $B_e^2, B_e^2, B_e^2, B_e^2\ldots, B_e^2$	-1	0
3	A_1'''	$B_o^3, B_o^3, B_o^{3\prime}, B_o^{3\prime}\ldots, B_o^3$ $B_e^3, B_e^3, B_e^{3\prime}, B_e^{3\prime}\ldots, B_e^3$	0	0
0	A_2'''	$B_o, B_o, B_o, B_o\ldots, B_o, B_o$ $B_e, B_e, B_e, B_e\ldots, B_e, B_e$	0	1
1	A_2'''	$B_o^{1\prime}, B_o^{1\prime}, B_o^1, B_o^1\ldots, B_o^1, B_o^{1\prime}$ $B_e^{1\prime}, B_e^{1\prime}, B_e^1, B_e^1\ldots, B_e^1, B_e^{1\prime}$	0	1
2	A_2'''	$B_o^2, B_o^2, B_o^2, B_o^2\ldots, B_o^2, B_o^2$ $B_e^2, B_e^2, B_e^2, B_e^2\ldots, B_e^2, B_e^2$	0	1

3	A_2'''	$B_o^3, B_o^3, B_o^{3'}, B_o^{3'} \ldots, B_o^{3'}, B_o^3$ $B_e^3, B_e^3, B_e^{3'}, B_e^{3'} \ldots, B_e^{3'}, B_e^3$	0	1
0	A_3'''	$B_o, B_o, B_o, B_o \ldots, B_o, B_o, B_o$ $B_e, B_e, B_e, B_e \ldots, B_e, B_e, B_e$	-1	0
1	A_3'''	$B_o^{1'}, B_o^{1'}, B_o^1, B_o^1 \ldots, B_o^{1'}, B_o^{1'}, B_o^1$ $B_e^{1'}, B_e^{1'}, B_e^1, B_e^1 \ldots, B_e^{1'}, B_e^{1'}, B_e^1$	0	0
2	A_3'''	$B_o^2, B_o^2, B_o^2, B_o^2 \ldots, B_o^2, B_o^2, B_o^2$ $B_e^2, B_e^2, B_e^2, B_e^2 \ldots, B_e^2, B_e^2, B_e^2$	1	0
3	A_3'''	$B_o^3, B_o^3, B_o^{3'}, B_o^{3'} \ldots, B_o^3, B_o^3, B_{oo}^{3'}$ $B_e^3, B_e^3, B_e^{3'}, B_e^{3'} \ldots, B_e^3, B_e^3, B_e^{3'}$	0	0

Table 1.3. Vertex and edge of $P_k \odot L_{3,4s+j}^3$

It well known that $P_k \odot L_{3,4s+j}^3 \cong P_k \odot L_{4t+l,3}^3$.

<u>Case (2):</u> $n \equiv 0(mod4)$, then $p_k \odot L_{4t,m}^3$ is cordial.

consider $n = 4t$ and $m = 4s + j \forall j = 0,1,2,3$ is singed product cordial. the next table (1.1) illustrate the labeling of the lemniscate.

j	labeling of lemniscate $L_{4t,4s+j}^3$	$Y_{(-1)}$	Y_1	B_1	$B_{(-1)}$
0	$B_{oo} = L_{8t}'N_4'; 1L_{8s}(-1)1$ $B_{oe} = 1L_{8t}'1_2(-1); (-1)L_{8s-8}'N_4'1(-1)$ $B_{eo} = 1L_{8t-8}N_4(-1)_21; 1L_{8s}1(-1)$ $B_{ee} = L_{8t}'; L_{8s-8}'N_4'1_2(-1)$	$4t+4s+3$ $4t+4s+1$ $4t+4s+1$ $4t+4s-1$	$4t+4s+4$ $4t+4s+2$ $4t+4s+2$ $4t+4s$	$16t+12s+3$ $16t+8s+1$ $8t+16s+1$ $8t+8s+3$	$12t+16s+3$ $16t+8s+1$ $8t+16s+1$ $8t+8s+3$
1	$B_{oo} = L_{8t}N_4; (-1)_2L_{8s}'1_2$ $B_{oo}' = 1_2L_{8t}1_2; (-1)_2L_{8s-8}'N_4'1(-1)_31(-1)$ $B_{eo}' = (-1)L_{8t-8}'N_4'1_2(-1); (-1)_2L_{8s-8}'1_2$ $B_{oe} = L_{8t}N_4; (-1)L_{8s-8}'N_4'1_2(-1)$ $B_{ee} = L_{8t}'; (-1)L_{8s-8}'N_4'(-1)1_2$ $B_{eo} = L_{8t}'; (-1)_2L_{8s}'1_2$	$4t+4s+4$ $4t+4s+4$ $4t+4s-2$ $4t+4s+2$ $4t+4s$ $4t+4s+2$	$4t+4s+4$ $4t+4s+4$ $4t+4s-2$ $4t+4s+2$ $4t+4s$ $4t+4s+2$	$16t+16s$ $16t+16s+1$ $8t+4s+2$ $16t+12s-1$ $8t+12s+1$ $8t+16s+3$	$16t+16s+1$ $16t+16s$ $8t+4s+3$ $16t+12s-2$ $8t+12s$ $8t+16s+2$
2	$B_{oo} = 1L_{8t}1_2(-1); (-1)L_{8s-8}'N_4'$ $B_{oe} = 1_2L_{8t}1_2; (-1)_2L_{8s-8}'N_4'1(-1)_2$ $B_{eo} = 1L_{8t-8}N_4(-1)_21; L_{8s-8}N_41$ $B_{ee} = L_{8t}; L_{8s}1$	$4t+4s$ $4t+4s+2$ $4t+4s-2$ $4t+4s$	$4t+4s+1$ $4t+4s+3$ $4t+4s-1$ $4t+4s+1$	$16t+4s+2$ $16t+12s$ $8t+8s$ $8t+12s+2$	$16t+4s+2$ $16t+12s$ $8t+8s$ $8t+12s+2$
3	$B_{oo} = (-1)L_{8t}'(-1)_21; 1L_{8s-8}N_41$ $B_{oe} = (-1)_2L_{8t}'(-1)_2; 1_2L_{8s-8}N_41_3(-1)$ $B_{eo} = (-1)1L_{8t-8}'N_4'1_2; (-1)_2L_{8s-8}'N_4'1$ $B_{ee} = (-1)L_{8t-8}'N_4'1_2(-1); (-1)L_{8s}1$	$4t+4s+1$ $4t+4s+3$ $4t+4s-1$ $4t+4s+1$	$4t+4s+1$ $4t+4s+3$ $4t+4s-1$ $4t+4s+1$	$16t+8s$ $16t+12s+2$ $8t+8s+2$ $8t+12s+4$	$16t+8s-1$ $16t+12s+1$ $8t+8s+1$ $8t+12s+3$

Table 2.1. Vertex and edge of a lemniscate $L_{4t,4s+j}^3$

By using table (2), we study the singed product cordial of $P_k \odot L_{4t,4s+j}^3$ when $k = 1,2,3$.

j	P_k	$L_{4t,4s+j}^3$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
0	P_1	B_{oo} B_{oe} B_{eo} B_{ee}	0	-1 1 -1 -1

1	P_1	B_{oo} B_{oe} B'_{eo} B_{ee}	1	1 -1 1 -1
2	P_1	B_{oo} B_{oe} B_{eo} B_{ee}	0	1
3	P_1	B_{oo} B_{oe} B_{eo} B_{ee}	1	-1
0	P'_2	B_{oo}, B_{oo} B_{oe}, B_{oe} B_{eo}, B_{eo} B_{ee}, B_{ee}	0	1
1	P_2	B'_{oo}, B'_{oo} B_{oe}, B_{oe} B_{eo}, B_{eo} B_{ee}, B_{ee}	0	-1
2	P'_2	B_{oo}, B_{oo} B_{oe}, B_{oe} B_{eo}, B_{eo} B_{ee}, B_{ee}	0	1
3	P_2	B_{oo}, B_{oo} B_{oe}, B_{oe} B_{eo}, B_{eo} B_{ee}, B_{ee}	1	-1
0	P'_3	B_{oo}, B_{oo}, B_{oo} B_{oe}, B_{oe}, B_{oe} B_{eo}, B_{eo}, B_{eo} B_{ee}, B_{ee}, B_{ee}	1	1
1	P_3	$B'_{oo}, B'_{oo}, B'_{oe}$ B_{oe}, B_{oe}, B_{oe} B_{eo}, B_{eo}, B_{eo} B_{ee}, B_{ee}, B_{ee}	1	-1
2	P'_3	B_{oo}, B_{oo}, B_{oo} B_{oe}, B_{oe}, B_{oe} B_{eo}, B_{eo}, B_{eo} B_{ee}, B_{ee}, B_{ee}	0	1
3	P_3	B_{oo}, B_{oo}, B_{oo} B_{oe}, B_{oe}, B_{oe} B_{eo}, B_{eo}, B_{eo} B_{ee}, B_{ee}, B_{ee}	1	-1

Table 2.2. Vertex and edge of $P_k \odot L^3_{4t,4s+j}$

By using table (3), we study the singed product cordial of $P_k \odot L^3_{4t,4s+j}$ when $k = i(mode)4\ \forall i = 0,1,2,3$.

j	P_k	$L^3_{4t,4s+j}$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
0	A_0	$B_{oo}, B_{oo}, B_{oo}, B_{oo}\dots$ $B_{oe}, B_{oe}, B_{oe}, B_{oe}\dots$ $B_{eo}, B_{eo}, B_{eo}, B_{eo}\dots$ $B_{ee}, B_{ee}, B_{ee}, B_{ee}\dots$	0	1

1	A'_0	$B'_{oo}, B'_{oo}, B'_{oo}, B'_{oo} \dots$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots$	0	0 -1 -1 -1
2	A_0	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots$	0	1
3	A'_0	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots$	0	-1
0	A_1	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}$	0	1
1	A'_1	$B'_{oo}, B'_{oo}, B'_{oo}, B'_{oo} \dots B'_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}$	1	-1
2	A_1	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}$	0	1
3	A'_1	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}$	1	-1
0	A_2	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots, B_{oo}, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}, B_{ee}$	0	1
1	A'_2	$B'_{oo}, B'_{oo}, B'_{oo}, B'_{oo} \dots B'_{oo}, B'_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}, B_{ee}$	0	0 -1 -1 -1
2	A_2	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots, B_{oo}, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}, B_{ee}$	0	1
3	A'_2	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots, B_{oo}, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}, B_{ee}$	0	-1
0	A_3	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots, B_{oo}, B_{oo}, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}, B_{oe}, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}, B_{eo}, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}, B_{ee}, B_{ee}$	0	1
1	A'_3	$B'_{oo}, B'_{oo}, B'_{oo}, B'_{oo} \dots B'_{oo}, B'_{oo}, B'_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}, B_{oe}, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}, B_{eo}, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}, B_{ee}, B_{ee}$	1	-1
2	A_3	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \dots, B_{oo}, B_{oo}, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \dots, B_{oe}, B_{oe}, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \dots, B_{eo}, B_{eo}, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \dots, B_{ee}, B_{ee}, B_{ee}$	0	1

3	A'_3	$B_{oo}, B_{oo}, B_{oo}, B_{oo} \ldots, B_{oo}, B_{oo}, B_{oo}$ $B_{oe}, B_{oe}, B_{oe}, B_{oe} \ldots, B_{oe}, B_{oe}, B_{oe}$ $B_{eo}, B_{eo}, B_{eo}, B_{eo} \ldots, B_{eo}, B_{eo}, B_{eo}$ $B_{ee}, B_{ee}, B_{ee}, B_{ee} \ldots, B_{ee}, B_{ee}, B_{ee}$	1	-1

Table 2.3. Vertex and edge of $P_k \odot L^3_{4t,4s+j}$

For a special case if $n = 4$, then $P_k \odot L^3_{4,m}$ is singed product cordial, except $P_k \odot L^3_{4,5}$.

consider $n = 4$ and $m = 4s + j \forall j = 0,1,2,3$ is singed product cordial. The next table (1.3) illustrate the labeling of the lemniscate

j	labeling of lemniscate $L^3_{4,4s+j}$	$Y_{(-1)}$	Y_1	B_1	$B_{(-1)}$
0	$B_{11} = 1N_4(-1)1(-1)$ $B'_{1e} = (-1)_3 1; 1L_{8s-8}N_4(-1)1$ $B_{1e} = 1_3(-1); (-1)L'_{8s-8}N'_4 1(-1)$ $B_{1o} = Q_4; 1L_{8s}(-1)1$ $B'_{1o} = Q'_4; (-1)L'_{8s}1(-1)$	3 $4s+2$ $4s+1$ $4s+3$ $4s+4$	4 $4s+1$ $4s+2$ $4s+4$ $4s+3$	6 $8s+5$ $8s+5$ $16s+2$ $16s+2$	6 $8s+4$ $8s+4$ $16s+2$ $16s+2$
1	$B_{1e} = (-1)_3 1; 1L_{8s-8}N_4(-1)1_2$ $B_{1o} = N_4; (-1)_2 L'_{8s} 1_2$	$4s+2$ $4s+4$	$4s+2$ $4s+4$	$12s+2$ $16s+4$	$12s+2$ $16s+4$
2	$B_{1o} = 1_3(-1); (-1)L'_{8s-8}N'_4$ $B'_{1o} = (-1)_3 1; 1L_{8s-8}N_4$ $B_{1e} = 1_4; (-1)_2 L'_{8s-8}N'_4 1(-1)_2$ $B'_{1e} = (-1)_4; 1_2 L_{8s-8}N_4(-1)1_2$	$4s$ $4s+1$ $4s+2$ $4s+3$	$4s+1$ $4s$ $4s+3$ $4s+2$	$8s+2$ $8s+2$ $20s+2$ $20s+2$	$8s+1$ $8s+1$ $20s+1$ $20s+1$
3	$B_{1o} = (-1)1(-1)_2; 1_2 L_{8s-8}N_4$ $B_{1e} = (-1)_4; 1_2 L_{8s-8}N_4 1_3(-1)$	$4s+1$ $4s+3$	$4s+1$ $4s+3$	$8s+3$ $16s+1$	$8s+3$ $16s+1$

Table 2.4. Vertex and edge of a lemniscate $L^3_{4,4s+j}$

By using table (2), we study the singed product cordial of $P_k \odot L^3_{4,4s+j}$ where $k = 1,2,3$.

j	P_k	$L^3_{4,4s+j}$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
0	P_1	B_{11} B_{1e} B'_{1o}	0	1 0 0
1	P_1	B_{1e} B_{1o}	1	0
2	P'_1	B'_{1o} B'_{1e}	0	0
3	P_1	B_{1o} B_{1e}	0	0
0	P'_2	B_{11}, B_{11} B_{1e}, B_{1e} B_{1o}, B_{1o}	0	1 -1 -1
1	P_2	B_{1e}, B_{1e} B_{1o}, B_{1o}	0	-1
2	P_2	B_{1o}, B'_{1o} B_{1e}, B'_{1e}	0	1
3	P_2	B_{1o}, B_{1o} B_{1e}, B_{1e}	0	1
0	P'_3 P'''_3 P'''_3	B_{11}, B_{11} B'_{1e}, B'_{1e}, B_{1e} B'_{1e}, B'_{1e}, B_{1e}	0	1 0 0
1	P'''_3	B_{1o}, B_{1o}, B_{1o} B_{1e}, B_{1e}, B_{1e}	1	0

2	P_3'''	B_{1o}, B_{1o}, B_{1o}' B_{1e}, B_{1e}, B_{1e}'	0	0
3	P_3'''	B_{1o}, B_{1o}, B_{1o} B_{1e}, B_{1e}, B_{1e}	1	0

Table 2.5. Vertex and edge of $P_k \odot L^3_{4,4s+j}$ where $k = 1,2,3$.

By using table (3), we study the singed product cordial of $P_k \odot L^3_{4,4s+j}$.

j	P_k	$L^3_{4,4s+j}$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
0	A_0 A_0''' A_0'''	$B_{11}, B_{11}, B_{11}, B_{11}\dots$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}'\dots$ $B_{1o}, B_{1o}, B_{1o}', B_{1o}'\dots$	0	1 -1 -1
1	A_0'''	$B_{1e}, B_{1e}, B_{1e}, B_{1e}\dots$ $B_{1o}, B_{1o}, B_{1o}, B_{1o}\dots.$	0	-1
2	A_0'''	$B_{1o}, B_{1o}, B_{1o}', B_{1o}'\dots$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}'\dots$	0	-1
3	A_0'''	$B_{1o}, B_{1o}, B_{1o}, B_{1o}\dots$ $B_{1e}, B_{1e}, B_{1e}, B_{1e}\dots$	0	-1
0	A_1 A_1''' A_1'''	$B_{11}, B_{11}, B_{11}, B_{11}\dots, B_{11}$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}'\dots, B_{1e}$ $B_{1o}, B_{1o}, B_{1o}', B_{1o}'\dots, B_{1o}$	0	1 0 0
1	A_1'''	$B_{1e}, B_{1e}, B_{1e}, B_{1e}\dots, B_{1e}$ $B_{1o}, B_{1o}, B_{1o}, B_{1o}\dots, B_{1o}$	1	0
2	A_1'''	$B_{1o}, B_{1o}, B_{1o}', B_{1o}'\dots, B_{1o}$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}'\dots, B_{1e}$	0	0
3	A_1'''	$B_{1o}, B_{1o}, B_{1o}, B_{1o}\dots, B_{1o}$ $B_{1e}, B_{1e}, B_{1e}, B_{1e}\dots, B_{1e}$	1	0
0	A_2 A_2''' A_2'''	$B_{11}, B_{11}, B_{11}, B_{11}\dots, B_{11}, B_{11}$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}'\dots, B_{1e}', B_{1e}$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}^1\dots, B_{1e}', B_{1e}$	0	1 -1 -1
1	A_2'''	$B_{1o}, B_{1o}, B_{1o}, B_{1o}\dots, B_{1o}, B_{1o}$ $B_{1e}, B_{1e}, B_{1e}, B_{1e}\dots, B_{1e}, B_{1e}$	0	-1
2	A_2'''	$B_{1o}, B_{1o}, B_{1o}', B_{1o}'\dots, B_{1o}', B_{1o}$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}'\dots, B_{1e}', B_{1e}$	0	-1
3	A_2'''	$B_{1o}, B_{1o}, B_{1o}, B_{1o}\dots, B_{1o}, B_{1o}$ $B_{1e}, B_{1e}, B_{1e}, B_{1e}\dots, B_{1e}, B_{1e}$	0	-1
0	A_3 A_3''' A_3'''	$B_{11}, B_{11}, B_{11}, B_{11}\dots, B_{11}, B_{11}, B_{11}$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}'\dots, B_{1e}, B_{1e}, B_{1e}^1$ $B_{1o}, B_{1o}, B_{1o}', B_{o}'\dots, B_{1o}, B_{1o}, B_{1o}'$	0	1 0 0
1	A_3'''	$B_{1e}, B_{1e}, B_{1e}, B_{1e}\dots, B_{1e}, B_{1e}, B_{1e}$ $B_{1o}, B_{1o}, B_{1o}, B_{1o}\dots, B_{1o}, B_{1o}, B_{1o}$	1	0
2	A_3'''	$B_{1o}, B_{1o}, B_{1o}', B_{1o}'\dots, B_{1o}, B_{1o}, B_{1o}'$ $B_{1e}, B_{1e}, B_{1e}', B_{1e}'\dots, B_{1e}, B_{1e}, B_{1e}'$	0	0
3	A_3'''	$B_{1o}, B_{1o}, B_{1o}, B_{1o}\dots, B_{1o}, B_{1o}, B_{1o}$ $B_{1e}, B_{1e}, B_{1e}, B_{1e}\dots, B_{1e}, B_{1e}, B_{1e}$	1	0

Table 2.6. Vertex and edge of $P_k \odot L^3_{4,4s+j}$

Case (3): $n \equiv 1(mod4)$, then $P_k \odot L^3_{4t+1,m}$ is singed product cordial. except $P_k \odot L^3_{4t+1,4}$ and $P_k \odot L^3_{4t+1,5}$ consider $n = 4t + 1$ and $m = 4s + j, \forall j = 0,1,2,3$ is singed product cordial. The next table (2.1) illustrate the labeling of the lemniscate.

j	labeling of lemniscate $L^3_{4t+1,4s+j}$	$Y_{(-1)}$	Y_1	B_1	$B_{(-1)}$
1	$B^1_{oe}=(-1)_2L'_{8t-8}(-1)_21;1L_{8s-8}N_41_2(-1)$ $B^1_{oo}=1(-1)_2L'_{8t-8}(-1)_2;1_2L_{8s}(-1)1$ $B^1_{eo}=(-1)_2L'_{8t-8}N'_41_2(-1);(-1)L'_{8s}1_2(-1)$ $B^1_{ee}=1(-1)_2L'_{8t-8}N'_41_2;(-1)_2L'_{8s-8}N'_41(-1)$	$4t+4s-1$ $4t+4s+1$ $4t+4s+3$ $4t+4s+1$	$4t+4s-2$ $4t+4s$ $4t+4s+2$ $4t+4s$	$4t+8s+3$ $4t+16s+2$ $12t+16s$ $12t+12s-2$	$4t+8s+3$ $4t+16s+2$ $12t+16s$ $12t+12s-2$
2	$B^1_{oe}=L_{8t-8}N_4(-1);(-1)L'_{8s-8}N'_41_2(-1)1$ $B^1_{eo}=1_2L_{8t-8}N_4(-1)_21;1(-1)_2L'_{8s-8}(-1)1$ $B^{1\prime}_{oo}=(-1)1(-1)L_{8t-8}(-1)_21;12L_{8s-8}1(-1)1$ $B^1_{ee}=L_{8t}1;L_{8s-8}N_4(-1)F_4$	$4t+4s-1$ $4t+4s-1$ $4t+4s-3$ $4t+4s+1$	$4t+4s-1$ $4t+4s-1$ $4t+4s-3$ $4t+4s+1$	$4t+12s+2$ $4t+12s+2$ $4t+4s+3$ $12t+12s$	$4t+12s+1$ $4t+12s+1$ $4t+4s+4$ $12t+12s-1$
3	$B^1_{oo}=L_{8t-8}N_4(-1);(-1)L'_{8s-8}N'_41$ $B^1_{oe}=1(-1)_2L'_{8t-8}(-1)_2;1_2L_{8s}$ $B^1_{eo}=(-1)1(-1)L'_{8t-8}N'_41_2;(-1)_2L'_{8s-8}N'_41$ $B^1_{ee}=(-1)_21L'_{8t-8}N'_41_2;(-1)_2L'_{8s}$	$4t+4s-2$ $4t+4s$ $4t+4s$ $4t+4s+2$	$4t+4s-3$ $4t+4s-1$ $4t+4s-1$ $4t+4s+1$	$4t+8s+1$ $4t+12s+3$ $12t+8s-1$ $12t+12s+1$	$4t+8s+1$ $4t+12s+3$ $12t+8s-1$ $12t+12s+1$

Table 3.1. Vertex and edge of a lemniscate $L^3_{4t+1,4s+j}$

By using table (2), we study the singed product cordial of $P_k \odot L^3_{4t+1,4s+j}$ when $k=1,2,3$.

j	P_k	$L^3_{4t+1,4s+j}$ $j=0,1,2,3$	$v_{(-1)}-v_1$	$e_{(-1)}-e_1$
1	p'_1	B^1_{oo} B^1_{oe} B^1_{eo} B^1_{ee}	0	1
2	P_1	$B^{1\prime}_{oo}$ B^1_{oe} B^1_{eo} B^1_{ee}	1	1
3	P'_1	B^1_{oo} B^1_{oe} B^1_{eo} B^1_{ee}	0	-1
1	P''_2	B^1_{oe},B^1_{oe} B^1_{oo},B^1_{oo} B^1_{eo},B^1_{eo} B^1_{ee},B^1_{ee}	0	-1
2	P_2	B^1_{oo},B^1_{oo} B^1_{oe},B^1_{oe} B^1_{eo},B^1_{eo} B^1_{ee},B^1_{ee}	0	-1
3	p''_2	B^1_{oo},B^1_{oo} B^1_{oe},B^1_{oe} B^1_{eo},B^1_{eo} B^1_{ee},B^1_{ee}	0	1

1	P_3''	$B_{oo}^1, B_{oo}^1, B_{oo}^1$ $B_{oe}^1, B_{oe}^1, B_{oe}^1$ $B_{eo}^1, B_{eo}^1, B_{eo}^1$ $B_{ee}^1, B_{ee}^1, B_{ee}^1$	0	1
2	P_3	B_{oe}, B_{oe}, B_{oe} B_{eo}, B_{eo}, B_{eo} B_{ee}, B_{ee}, B_{ee}	1 -1 1	-1
3	P_3'''	$B_{oo}^1, B_{oo}^1, B_{oo}^1$ $B_{oe}^1, B_{oe}^1, B_{oe}^1$ $B_{eo}^1, B_{eo}^1, B_{eo}^1$ $B_{ee}^1, B_{ee}^1, B_{ee}^1$	0	1

Table 3.2. Vertex and edge of $P_k \odot L_{4t+1,4s+j}^3$

By using table (3), we study the singed product cordial of $P_k \odot L_{4t+1,m}^3$.

j	P_k	$L_{4t+1,4s+j}^3$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
1	A_0''	$B_{oo}^1, B_{oo}^1, B_{oo}^1, B_{oo}^1 \ldots$ $B_{oe}^1, B_{oe}^1, B_{oe}^1, B_{oe}^1 \ldots$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots$	0	1
2	A_0'	$B_{oo}^1, B_{oo}^1, B_{oo}^1, B_{oo}^1 \ldots$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots$	0	-1
3	A_0''	$B_{oo}^1, B_{oo}^1, B_{oo}^1, B_{oo}^1 \ldots$ $B_{oe}^1, B_{oe}^1, B_{oe}^1, B_{oe}^1 \ldots$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots$	0	1
1	A_1''	$B_{oo}^1, B_{oo}^1, B_{oo}^1, B_{oo}^1 \ldots, B_{oo}^1$ $B_{oe}^1, B_{oe}^1, B_{oe}^1, B_{oe}^1 \ldots, B_{oe}^1$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots, B_{eo}^1$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots, B_{ee}^1$	0	1
2	A_1'	$B_{oo}^1, B_{oo}^1, B_{oo}^1, B_{oo}^1 \ldots, B_{oo}^1$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots, B_{eo}^1$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots, B_{ee}^1$	1	-1
3	A_1''	$B_{oo}^1, B_{oo}^1, B_{oo}^1, B_{oo}^1 \ldots, B_{oo}^1$ $B_{oe}^1, B_{oe}^1, B_{oe}^1, B_{oe}^1 \ldots, B_{oe}^1$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots, B_{eo}^1$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots, B_{ee}^1$	0	1
1	A_2''	$B_{oo}^1, B_{oo}^1, B_{oo}^1, B_{oo}^1 \ldots, B_{oo}^1, B_{oo}^1$ $B_{oe}^1, B_{oe}^1, B_{oe}^1, B_{oe}^1 \ldots, B_{oe}^1, B_{oe}^1$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots, B_{eo}^1, B_{eo}^1$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots, B_{ee}^1, B_{ee}^1$	0	1
2	A_2'	$B_{oe}^1, B_{oe}^1, B_{oe}^1, B_{oe}^1 \ldots, B_{oe}^1, B_{oe}^1$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots, B_{eo}^1, B_{eo}^1$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots, B_{ee}^1, B_{ee}^1$	0	-1
3	A_2''	$B_{oo}^1, B_{oo}^1, B_{oo}^1, B_{oo}^1 \ldots, B_{oo}^1, B_{oo}^1$ $B_{oe}^1, B_{oe}^1, B_{oe}^1, B_{oe}^1 \ldots, B_{oe}^1, B_{oe}^1$ $B_{eo}^1, B_{eo}^1, B_{eo}^1, B_{eo}^1 \ldots, B_{eo}^1, B_{eo}^1$ $B_{ee}^1, B_{ee}^1, B_{ee}^1, B_{ee}^1 \ldots, B_{ee}^1, B_{ee}^1$	0	1

1	A''_3	$B^1_{oo}, B^1_{oo}, B^1_{oo}, B^1_{oo}\dots, B^1_{oo}, B^1_{oo}, B^1_{oo}$ $B^1_{oe}, B^1_{oe}, B^1_{oe}, B^1_{oe}\dots, B^1_{oe}, B^1_{oe}, B^1_{oe}$ $B^1_{eo}, B^1_{eo}, B^1_{eo}, B^1_{eo}\dots, B^1_{eo}, B^1_{eo}, B^1_{eo}$ $B^1_{ee}, B^1_{ee}, B^1_{ee}, B^1_{ee}\dots, B^1_{ee}, B^1_{ee}, B^1_{ee}$	0	1
2	A'_3	$B^1_{oe}, B^1_{oe}, B^1_{oe}, B^1_{oe}\dots, B^1_{oe}, B^1_{oe}, B^1_{oe}$ $B^1_{eo}, B^1_{eo}, B^1_{eo}, B^1_{eo}\dots, B^1_{eo}, B^1_{eo}, B^1_{eo}$ $B^1_{ee}, B^1_{ee}, B^1_{ee}, B^1_{ee}\dots, B^1_{ee}, B^1_{ee}, B^1_{ee}$	1	-1
3	A''_3	$B^1_{oo}, B^1_{oo}, B^1_{oo}, B^1_{oo}\dots, B^1_{oo}, B^1_{oo}, B^1_{oo}$ $B^1_{oe}, B^1_{oe}, B^1_{oe}, B^1_{oe}\dots, B^1_{oe}, B^1_{oe}, B^1_{oe}$ $B^1_{eo}, B^1_{eo}, B^1_{eo}, B^1_{eo}\dots, B^1_{eo}, B^1_{eo}, B^1_{eo}$ $B^1_{ee}, B^1_{ee}, B^1_{ee}, B^1_{ee}\dots, B^1_{ee}, B^1_{ee}, B^1_{ee}$	0	1

Table 3.3. Vertex and edge of $P_k \odot L^3_{4t+1,4s+j}$

Case (4): $n \equiv 2(mod 4)$, then $P_k \odot L^3_{4t+2,m}$ is singed product cordial. except $p_k \odot L^3_{6,5}$ in all cases except when $k = 1$
consider $n = 4t + 2$ and $m = 4s + j,\ \forall j = 0,1,2,3$ is singed product cordial. The next table (4.1) illustrate the labeling of the lemniscate $L^3_{4t+2,4s+j}$.

j	labeling of lemniscate $L^3_{4t+2,4s+j}$	$Y_{(-1)}$	Y_1	B_1	$B_{(-1)}$
2	$B^2_{oo} = (-1)_2 L'_{8t-8} N'_4; L_{8s-8} N_4 1$ $B^2_{oe} = (-1)1(-1)L'_{8t-8}(-1)_2 1; 1L_{8s}$ $B^2_{eo} = (-1)1(-1)L'_{8t-8}N'_4 1_2(-1); (-1)L'_{8s-8}N'_4$ $B^2_{ee} = (-1)_2 L'_{8t}; L'_{8s} 1$	$4t + 4s - 2$ $4t + 4s$ $4t + 4s$ $4t + 4s + 2$	$4t + 4s - 3$ $4t + 4s - 1$ $4t + 4s - 1$ $4t + 4s + 1$	$8t + 8s - 3$ $8t + 12s - 1$ $12t + 8s - 1$ $12t + 12s + 1$	$8t + 8s - 3$ $8t + 12s - 1$ $12t + 8s - 1$ $12t + 12s + 1$
3	$B^2_{oo} = 1L_{8t-8} N_4 0; 0L'_{8s-8} N'_4 1$ $B^2_{oe} = 1L'_{8t-8} N'_4(-1); (-1)_2 L'_{8s-8} N'_4 1_2(-1)1$ $B^2_{eo} = L'_{8t} 1(-1); (-1)L'_{8s-8} N'_4 1$ $B^2_{ee} = L_{8t} 1_2; (-1)_2 L'_{8s}$	$4t + 4s - 2$ $4t + 4s$ $4t + 4s$ $4t + 4s + 2$	$4t + 4s - 2$ $4t + 4s$ $4t + 4s$ $4t + 4s + 2$	$4t + 8s + 3$ $8t + 12s + 1$ $12t + 8s + 1$ $12t + 12s + 3$	$8t + 8s - 1$ $8t + 12s$ $12t + 8s$ $12t + 12s + 2$

Table 4.1. Vertex and edge of a lemniscate $L^3_{4t+2,4s+j}$

By using table (2), we study the singed product cordial of $P_k \odot L^3_{4t+2,4s+j}$ when $k = 1,2,3$.

j	P_k	$L^3_{4t+2,4s+j}$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
2	P'_1	B^2_{oo} B^2_{oe} B^2_{eo} B^2_{ee}	0	1
3	P_1	B^2_{oo} B^2_{oe} B^2_{eo} B^2_{ee}	1	-1
2	P''_2	B^2_{oo}, B^2_{oo} B^2_{oe}, B^2_{oe} B^2_{eo}, B^2_{eo} B^2_{ee}, B^2_{ee}	0	1

3	p_2	B_{oo}^2, B_{oo}^2 B_{oe}^2, B_{oe}^2 B_{eo}^2, B_{eo}^2 B_{ee}^2, B_{ee}^2	0	-1
2	P_3'	$B_{oo}^2, B_{oo}^2, B_{oo}^2$ $B_{oe}^2, B_{oe}^2, B_{oe}^2$ $B_{eo}^2, B_{eo}^2, B_{eo}^2$ $B_{ee}^2, B_{ee}^2, B_{ee}^2$	-1	0
3	P_3	$B_{oo}^2, B_{oo}^2, B_{oo}^2$ $B_{oe}^2, B_{oe}^2, B_{oe}^2$ $B_{eo}^2, B_{eo}^2, B_{eo}^2$ $B_{ee}^2, B_{ee}^2, B_{ee}^2$	-1	-1

Table 4.2. Vertex and edge of $P_k \odot L_{4t+2,4s+j}^3$

By using table (3), we study the singed product cordial of $P_k \odot L_{4t+2,m}^3$. where $k = i(mod4)\ \forall i = 0,1,2,3$.

j	P_k	$L_{4t+2,4s+j}^3$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
2	A_0''	$B_{oo}^2, B_{oo}^2, B_{oo}^2, B_{oo}^2 \ldots$ $B_{oe}^2, B_{oe}^2, B_{oe}^2, B_{oe}^2 \ldots$ $B_{eo}^2, B_{eo}^2, B_{eo}^2, B_{eo}^2 \ldots$ $B_{ee}^2, B_{ee}^2, B_{ee}^2, B_{ee}^2 \ldots$	0	1
3	A_0'	$B_{oo}^2, B_{oo}^2, B_{oo}^2, B_{oo}^2 \ldots$ $B_{oe}^2, B_{oe}^2, B_{oe}^2, B_{oe}^2 \ldots$ $B_{eo}^2, B_{eo}^2, B_{eo}^2, B_{eo}^2 \ldots$ $B_{ee}^2, B_{ee}^2, B_{ee}^2, B_{ee}^2 \ldots$	0	-1
2	A_1''	$B_{oo}^2, B_{oo}^2, B_{oo}^2, B_{oo}^2 \ldots, B_{oo}^2$ $B_{oe}^2, B_{oe}^2, B_{oe}^2, B_{oe}^2 \ldots, B_{oe}^2$ $B_{eo}^2, B_{eo}^2, B_{eo}^2, B_{eo}^2 \ldots, B_{eo}^2$ $B_{ee}^2, B_{ee}^2, B_{ee}^2, B_{ee}^2 \ldots, B_{ee}^2$	0	1
3	A_1'	$B_{oo}^2, B_{oo}^2, B_{oo}^2, B_{oo}^2 \ldots, B_{oo}^2$ $B_{oe}^2, B_{oe}^2, B_{oe}^2, B_{oe}^2 \ldots, B_{oe}^2$ $B_{eo}^2, B_{eo}^2, B_{eo}^2, B_{eo}^2 \ldots, B_{eo}^2$ $B_{ee}^2, B_{ee}^2, B_{ee}^2, B_{ee}^2 \ldots, B_{ee}^2$	1	-1
2	A_2''	$B_{oo}^2, B_{oo}^2, B_{oo}^2, B_{oo}^2 \ldots, B_{oo}^2, B_{oo}^2$ $B_{oe}^2, B_{oe}^2, B_{oe}^2, B_{oe}^2 \ldots, B_{oe}^2, B_{oe}^2$ $B_{eo}^2, B_{eo}^2, B_{eo}^2, B_{eo}^2 \ldots, B_{eo}^2, B_{eo}^2$ $B_{ee}^2, B_{ee}^2, B_{ee}^2, B_{ee}^2 \ldots, B_{ee}^2, B_{ee}^2$	0	1
3	A_2'	$B_{oo}^2, B_{oo}^2, B_{oo}^2, B_{oo}^2 \ldots, B_{oo}^2, B_{oo}^2$ $B_{oe}^2, B_{oe}^2, B_{oe}^2, B_{oe}^2 \ldots, B_{oe}^2, B_{oe}^2$ $B_{eo}^2, B_{eo}^2, B_{eo}^2, B_{eo}^2 \ldots, B_{eo}^2, B_{eo}^2$ $B_{ee}^2, B_{ee}^2, B_{ee}^2, B_{ee}^2 \ldots, B_{ee}^2, B_{ee}^2$	0	-1
2	A_3''	$B_{oo}^2, B_{oo}^2, B_{oo}^2, B_{oo}^2 \ldots, B_{oo}^2, B_{oo}^2, B_{oo}^2$ $B_{oe}^2, B_{oe}^2, B_{oe}^2, B_{oe}^2 \ldots, B_{oe}^2, B_{oe}^2, B_{oe}^2$ $B_{eo}^2, B_{eo}^2, B_{eo}^2, B_{eo}^2 \ldots, B_{eo}^2, B_{eo}^2, B_{eo}^2$ $B_{ee}^2, B_{ee}^2, B_{ee}^2, B_{ee}^2 \ldots, B_{ee}^2, B_{ee}^2, B_{ee}^2$	0	1
3	A_3'	$B_{oo}^2, B_{oo}^2, B_{oo}^2, B_{oo}^2 \ldots, B_{oo}^2, B_{oo}^2, B_{oo}^2$ $B_{oe}^2, B_{oe}^2, B_{oe}^2, B_{oe}^2 \ldots, B_{oe}^2, B_{oe}^2, B_{oe}^2$ $B_{eo}^2, B_{eo}^2, B_{eo}^2, B_{eo}^2 \ldots, B_{eo}^2, B_{eo}^2, B_{eo}^2$ $B_{ee}^2, B_{ee}^2, B_{ee}^2, B_{ee}^2 \ldots, B_{ee}^2, B_{ee}^2, B_{ee}^2$	1	-1

Table 4.3. Vertex and edge of $P_k \odot L_{4t+2,4s+j}^3$

For a special case suppose that $m = 4$. one can label the vertices of $L^3_{4t+2,4}$ by

labeling of lemniscate $L^3_{4t+2,4}$	$Y_{(-1)}$	Y_1	B_1	$B_{(-1)}$
$B^{2\prime}_{o1} = (-1)_2 L'_{8t-8} N'_4; 1_2(-1)$	$4t$	$4t+1$	$8t+2$	$8t+2$
$B^2_{o1} = 1_2 L_{8t-8} N_4; (-1)_2(-1)$	$4t+1$	$4t$	$8t+1$	$8t+1$
$B^{2\prime}_{e1} = (-1)_2 L'_{8t}; 1_2(-1)$	$4t+3$	$4t+2$	$16t$	$16t$
$B^2_{e1} = 1_2 L_{8t}; (-1)_2 1$	$4t+2$	$4t+3$	$16t-1$	$16t-1$

Table 4.4. Vertex and edge of $L^3_{4t+2,4}$

By using table (2), we study the singed product cordial of $P_k \odot L^3_{4t+2,4}$, where $k = 1,2,3$.

P_k	$L^3_{4t+2,4}$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
P'_1	$B^{2\prime}_{o1}$ $B^{2\prime}_{e1}$	0	0
P''_2	$B^{2\prime}_{o1}, B^{2\prime}_{o1}$ $B^{2\prime}_{e1}, B^{2\prime}_{e1}$	0	-1
P''_3	$B^{2\prime}_{o1}, B^{2\prime}_{o1}, B^{2\prime}_{o1}$ $B^{2\prime}_{e1}, B^2_{e1}, B^{2\prime}_{e1}$	0	1

Table 4.5. Vertex and edge of $P_k \odot L^3_{4t+2,4}$

By using table (3), we study the singed product cordial of $P_k \odot L^3_{4t+2,4}$, where $k = i(mod4)\ \forall i = 0,1,2,3$.

P_k	$L^3_{4t+2,4}$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
A'''_0	$B^2_{o1}, B^2_{o1}, B^{2\prime}_{o1}, B^{2\prime}_{o1} \dots$ $B^2_{e1}, B^2_{e1}, B^{2\prime}_{e1}, B^{2\prime}_{oe} \dots$	0	-1
A'''_1	$B^2_{o1}, B^2_{o1}, B^{2\prime}_{o1}, B^{2\prime}_{o1} \dots, B^2_{o1}$ $B^2_{e1}, B^2_{e1}, B^{2\prime}_{e1}, B^{2\prime}_{e1} \dots, B^2_{e1}$	0	0
A'''_2	$B^2_{o1}, B^2_{o1}, B^{2\prime}_{o1}, B^{2\prime}_{o1} \dots, B^{2\prime}_{o1}, B^2_{o1}$ $B^2_{e1}, B^2_{e1}, B^{2\prime}_{e1}, B^2_{e1} \dots, B^{2\prime}_{e1}, B^2_{e1}$	0	-1
A'''_3	$B^2_{o1}, B^2_{o1}, B^{2\prime}_{o1}, B^{2\prime}_{o1} \dots, B^2_{o1}, B^2_{o1}, B^2_{o1}$ $B^2_{e1}, B^2_{e1}, B^{2\prime}_{e1}, B^{2\prime}_{e1} \dots, B^2_{e1}, B^2_{e1}, B^{2\prime}_{e1}$	0	0

Table 4.6. Vertex and edge of $P_k \odot L^3_{4t+2,4}$

Otherwise not singed product cordial .

Case (5): $n \equiv 3(mod4)$, then $P_k \odot L^3_{4t+3,m}$ is singed product cordial.

consider $n = 4t + 3$ and $m = 4s + j, \forall j = 0,1,2,3$ is singed product cordial. The next table (5.1) illustrate the labeling of the lemniscate.

j	labeling of lemniscate $L^3_{4t+3,4s+j}$	$Y_{(-1)}$	Y_1	B_1	$B_{(-1)}$
0	$B^3_{oo} = 1L_{8t-8}N_4(-1)_2; (-1)L_{8s}1_2$ $B^3_{oe} = L_{8t-8}N_4(-1)1(-1); (-1)L'_{8s-8}N'_4 1_2$ $B^3_{eo} = L'_{8t}(-1)_2 1; 1L_{8s}(-1)1$ $B^3_{ee} = 1L'_{8t}1(-1)_2; L'_{8s-8}N'_4 1_2(-1)$	$4t+4s+1$ $4t+4s-1$ $4t+4s+3$ $4t+4s+1$	$4t+4s+1$ $4t+4s-1$ $4t+4s+3$ $4t+4s+1$	$8t+16s$ $8t+8s+2$ $12t+16s+2$ $12t+8s+4$	$8t+16s-1$ $8t+8s+1$ $12t+16s+1$ $12t+8s+3$

1	$B^3_{oo} = L'_{8t-8}N'_4 1_2; (-1)_2 L'_{8s-8} 1_2$ $B^3_{oe} = 1(-1)_2 L'_{8t-8}N'_4; L_{8s}$ $B^3_{eo} = L'_{8t}(-1)_2 1; 1_2 L_{8s-8} 1(-1)$ $B^3_{ee} = (-1)L_{8t}(-1)_2; 1_2 L_{8s-8} N_4 1_2$	$4t+4s-3$ $4t+4s$ $4t+4s-1$ $4t+4s+1$	$4t+4s-2$ $4t+4s-1$ $4t+4s$ $4t+4s+2$	$8t+4s+1$ $8t+8s+3$ $12t+4s+3$ $16t+8s+1$	$8t+4s+1$ $8t+8s+3$ $12t+4s+3$ $16t+8s+1$
2	$B^3_{oo} = L'_{8t-8}N'_4 1_2(-1); (-1)L'_{8s-8}N'_4$ $B^3_{oe} = (-1)L'_{8t-8}N'_4 1_2; (-1)_2 L'_{8s-8}N'_4 1_2(-1)$ $B^3_{eo} = (-1)L'_{8t}(-1)_2; 1_2 L'_{8s-8} 1(-1)1$ $B^3_{ee} = (-1)L'_{8t}(-1)_2; 1_2 L_{8s-8} N_4 1_2(-1)$	$4t+4s-2$ $4t+4s$ $4t+4s$ $4t+4s+2$	$4t+4s-2$ $4t+4s$ $4t+4s$ $4t+4s+2$	$8t+4s+3$ $8t+12s+1$ $12t+8s+1$ $12t+12s+3$	$4t+4s+2$ $8t+12s$ $12t+8s$ $12t+12s+2$
3	$B^3_{oo} = 1L_{8t-8}N_4(-1)_2; L'_{8s-8}N'_4 1(-1)$ $B^3_{oe} = (-1)L'_{8t-8}N'_4 1_2; (-1)_2 L'_{8s}$ $B^3_{eo} = 1L'_{8t}(-1)_2; 1_2 L_{8s-8} N_4$ $B^3_{ee} = (-1)_2 1 L'_{8t}; L'_{8s} 1(-1)$	$4t+4s-1$ $4t+4s+1$ $4t+4s$ $4t+4s+3$	$4t+4s-2$ $4t+4s$ $4t+4s+1$ $4t+4s+2$	$8t+8s$ $8t+12s+2$ $12t+8s+2$ $12t+12s+4$	$8t+8s$ $8t+12s+2$ $12t+8s+2$ $12t+12s+4$

Table 5.1. Vertex and edge of a lemniscate $L^3_{4t+3,4s+j}$

By using table (2), we study the singed product cordial of $P_k \odot L^3_{4t+3,4s+j}$, when $k = 1,2,3$.

j	P_k	$L^3_{4t+3,4s+j}$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
0	P_1	B^3_{oo} B^3_{oe} B^3_{eo} B^3_{ee}	1	-1
1	p_1 p'_1 p_1 p_1	B^3_{oo} B^3_{oe} B^3_{eo} B^3_{ee}	0	1
2	P_1	B^3_{oo} B^3_{oe} B^3_{eo} B^3_{ee}	1	-1
3	P'_1	B^3_{oo} B^3_{oe} B^3_{eo} B^3_{ee}	0	1
0	p_2	B^3_{oo}, B^3_{oo} B^3_{oe}, B^3_{oe} B^3_{eo}, B^3_{eo} B^3_{ee}, B^3_{ee}	0	-1
1	P'_2	B^3_{oo}, B^3_{oo} B^3_{oe}, B^3_{oe} B^3_{eo}, B^3_{eo} B^3_{ee}, B^3_{ee}	0	1
2	P_2	B^3_{oo}, B^3_{oo} B^3_{oe}, B^3_{oe} B^3_{eo}, B^3_{eo} B^3_{ee}, B^3_{ee}	0	-1
3	p''_2	B^3_{oo}, B^3_{oo} B^3_{oe}, B^3_{oe} B^3_{eo}, B^3_{eo} B^3_{ee}, B^3_{ee}	0	1

0	p_3	$B^3_{oo}, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}$	1	-1
1	P'_3 P_3 P'_3 P'_3	$B^3_{oo}, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}$	0	1 -1 1 1
2	P_3	$B^3_{oo}, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}$	1	-1
3	P''_3	$B^3_{oo}, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}$	0	1

Table 5.2. Vertex and edge of $P_k \odot L^3_{4t+3,4s+j}$

Hence we study the singed product cordial of $P_k \odot L^3_{4t+3,m}$, where $k = i(mod4)\ \forall i = 0,1,2,3$.

j	P_k	$L^3_{4t+3,4s+j}$ $j = 0,1,2,3$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
0	A'_0	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots$	0	-1
1	A_0 A'_0 A_0 A_0	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots$	0	1
2	A'_0	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots$	0	-1
3	A''_0 A''_0 A_0 A''_0	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots$	0	1
0	A'_1	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}$	1	0
1	A_1 A'_1 A_1 A_1	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}$	0	1

2	A'_1	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}$	1	-1
3	A''_1 A''_1 A_1 A''_1	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}$	0	1
0	A'_2	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}, B^3_{ee}$	0	-1
1	A_2 A'_2 A_2 A_2	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}, B^3_{ee}$	0	1
2	A'_2	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}, B^3_{ee}$	0	-1
3	A'_2 A'_2 A_2 A'_2	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}, B^3_{ee}$	0	1
0	A'_3	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}, B^3_{oo}, B^3_{oo}$ $B^2_{oe}, B^2_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}, B^3_{oe}, B^3_{oe}$ $B^2_{eo}, B^2_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}, B^3_{ee}, B^3_{ee}$	-1	0
1	A_3 A'_3 A_3 A_3	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}, B^3_{ee}, B^3_{ee}$	0	1
2	A'_3	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}, B^3_{ee}, B^3_{ee}$	1	-1
3	A''_3 A''_3 A_3 A''_3	$B^3_{oo}, B^3_{oo}, B^3_{oo}, B^3_{oo}\dots, B^3_{oo}, B^3_{oo}, B^3_{oo}$ $B^3_{oe}, B^3_{oe}, B^3_{oe}, B^3_{oe}\dots, B^3_{oe}, B^3_{oe}, B^3_{oe}$ $B^3_{eo}, B^3_{eo}, B^3_{eo}, B^3_{eo}\dots, B^3_{eo}, B^3_{eo}, B^3_{eo}$ $B^3_{ee}, B^3_{ee}, B^3_{ee}, B^3_{ee}\dots, B^3_{ee}, B^3_{ee}, B^3_{ee}$	0	1

Table 5.3. Vertex and edge of $P_k \odot L^3_{4t+3,4s+j}$

For a special case suppose that $m = 4$. one can label the vertices of $L^3_{4t+3,4}$ by

labeling of lemniscate $L^3_{4t+3,4}$	$Y_{(-1)}$	Y_1	B_1	$B_{(-1)}$
$B^3_{o1} = 1L_{8t-8}N_4; (-1)1_2$	$4t+1$	$4t+1$	$8t+3$	$8t+2$

Table 5.4. Vertex and edge of $L^3_{4t+3,4}$

By using table (3), we study the singed product cordial of $P_k \odot L^3_{4t+3,4}$.

P_k	$L^3_{4t+3,4}$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
P_1	B^3_{o1}	1	0
P_2	B^3_{o1}, B^3_{o1}	0	1
P_3	$B^3_{o1}, B^3_{o1}, B^3_{o1}$	1	0

Table 5.5. Vertex and edge of $P_k \odot L^3_{4t+3,4}$

By using table (2), we study the singed product cordial of $P_k \odot L^3_{4t+3,4}$, where $k = i(mod4)\ \forall i = 0,1,2,3$.

P_k	$L^3_{4t+3,4}$	$v_{(-1)} - v_1$	$e_{(-1)} - e_1$
A_0'''	$B^3_{o1}, B^3_{o1}, B^3_{o1}, B^3_{o1} \dots$	0	-1
A_1'''	$B^3_{o1}, B^3_{o1}, B^3_{o1}, B^3_{o1} \dots, B^3_{o1}$	1	0
A_2'''	$B^3_{o1}, B^3_{o1}, B^3_{o1}, B^3_{o1} \dots, B^3_{o1}, B^3_{o1}$	0	-1
A_3'''	$B^3_{o1}, B^3_{o1}, B^3_{o1}, B^3_{o1} \dots, B^3_{o1}, B^3_{o1}, B^3_{o1}$	-1	0

Table 5.6. Vertex and edge of $P_k \odot L^3_{4t+3,4}$

Otherwise not singed product cordial .

As a consequence of all cases mentioned above we conclude that the corona product between paths and third power of lemniscate graph $\boldsymbol{P_k \odot L^3_{n,m}}$ is signed product-cordial for all k, n and m.

Theorem 3.1. The corona product between paths and third power of lemniscate graph $\boldsymbol{P_k \odot L^3_{n,m}}$ is signed product-cordial for all k, n and m.

4. Conclusion

In this work we have discussed and established necessary and sufficient conditions for which the corona product between paths and third power of lemniscate graph $\boldsymbol{P_k \odot L^3_{n,m}}$ are singed product cordial labeling concept on other families of graphs and finding the application of this labelling will be our future.

References

[1] Rosa, A. (1966, July). On certain valuations of the vertices of a graph. In Theory of Graphs (Internat. Symposium, Rome (pp. 349-355).
[2] Diab, A. T. (2010). On cordial labelings of the second power of paths with other graphs. ARS COMBINATORIA, 97, 327-343.
[3] Cahit, I. (1987). Cordial graphs-a weaker version of graceful and harmonious graphs. Ars combinatoria, 23, 201-207.
[4] Harary, F. (1953). On the notion of balance of a signed graph. Michigan Mathematical Journal, 2(2), 143-146.
[5] Gallian, J. A. (2018). A dynamic survey of graph labeling. Electronic Journal of combinatorics, 1(DynamicSurveys), DS6.
[6] Cahit, I. (1990). On cordial and 3-equitable labellings of graphs. Util. Math, 37, 189-198.
[7] Seoud, M. A., & Maqsoud, A. A. (1999). On cordial and balanced labelings of graphs. J. Egyptian Math. Soc, 7, 127-135.
[8] Devaraj, J., & Delphy, P. (2011). On signed cordial graph. Int. J. of Mathematical Sciences and Applications, 1(3), 1156-1167.

[9] Babujee, J. B., & Loganathan, S. (2011). On signed product cordial labeling. Applied Mathematics, 2(12), 1525-1530.
[10] Badr, E., Almotairi, S., Elrokh, A., Abdel-Hay, A., & Almutairi, B. (2020). An integer linear programming model for solving radio mean labeling problem. IEEE Access, 8, 162343-162349.
[11] ELrokh, A., Ali Al-Shamiri, M. M., Nada, S., & El-hay, A. A. (2022). Cordial and Total Cordial Labeling of Corona Product of Paths and Second Order of Lemniscate Graphs. Journal of Mathematics, 2022.
[12] Badr, E., Nada, S., Ali Al-Shamiri, M. M., Abdel-Hay, A., & ELrokh, A. (2022). A novel mathematical model for radio mean square labeling problem. Journal of Mathematics, 2022.
[13] ELrokh, A., Ali Al-Shamiri, M. M., & El-hay, A. (2022). A Novel Problem to Solve the Logically Labeling of Corona between Paths and Cycles. Journal of Mathematics, 2022.
[14] Badr, E., Abd El-hay, A., Ahmed, H., & Moussa, M. (2021). Polynomial, exponential and approximate algorithms for metric dimension problem. Mathematical Combinatorics, 2, 51-67.
[15] Nada, S., Elrokh, A., & Abd El-hay, A. (2022). On signed product cordial of lemniscate graph and its second power. Turkish Journal of Computer and Mathematics Education (TURCOMAT), 13(03), 905-913.
[16] Nada, S., Abd El-hay, A., & Elrokh, A. (2022). Total cordial labeling of corona product of paths and second power of fan graph. Turkish Journal of Computer and Mathematics Education (TURCOMAT), 13(03), 681-690.
[17] Nada, S., Elrokh, A., & Abd El-hay, A. (2022). On singed product cordial of cone graph and its second power. Turkish Journal of Computer and Mathematics Education (TURCOMAT), 13(03), 597-606.
[18] ELrokh, Ashraf, Mohammed M. Ali Al-Shamiri, and Atef Abd El-hay. 2023. "A Novel Radio Geometric Mean Algorithm for a Graph" Symmetry 15, no. 3: 570. https://doi.org/10.3390/sym15030570.
[19] ELrokh, Ashraf, Mohammed M. Ali Al-Shamiri, Mohammed M. A. Almazah, and Atef Abd El-hay. 2023. "A Novel Problem for Solving Permuted Cordial Labeling of Graphs" Symmetry 15, no. 4: 825.
[20] ELrokh, Ashraf, Rashad Ismail, Atef Abd El-hay, and Yasser Elmshtaye. 2023. "On Cubic Roots Cordial Labeling for Some Graphs" Symmetry 15, no. 5: 990. https://doi.org/10.3390/sym15050990
[21] Elrokh, A. & Elmshtaye, Y. & Abd El-hay, Atef. (2022). The Cordiality of Cone and Lemniscate Graphs. Applied Mathematics & Information Sciences. 16. 1027-1034. 10.18576/amis/160620.